M325K SPRING 2024

HW #5, Solutions for Section 4.4

SEC 4.4, #19:

<u>To Prove:</u> For all integers n, n^2-n+3 is odd.

<u>Proof:</u>

Let integer n be given.

By the Parity Corollary, n is even or n is odd.

<u>CASE 1 (n is even)</u>

Suppose n is even

Then, for some integer k, $n=2k$.

$$\begin{aligned}
\therefore n^2-n+3 &= (2k)^2-(2k)+3, \text{ by substitution,} \\
&= 4k^2-2k+3 \\
&= 2(2k^2)-2k+2+1 \\
&= 2(2k^2-k+1)+1 \\
&= 2t+1,
\end{aligned}$$

where $t = 2k^2-2k+1$.

$\therefore n^2-n+3 = 2t+1$ and t is an integer.

$\therefore$ By definition of odd, n^2-n+3 is odd, in CASE 1.

[END of CASE 1]

<u>CASE 2 (n is odd)</u>

Suppose n is odd.

$\therefore n = 2k+1$ for some integer k.

$$\begin{aligned}
\therefore n^2-n+3 &= (2k+1)^2-(2k+1)+3, \text{ by subst;} \\
&= (4k^2+4k+1)-2k-1+3 \\
&= 4k^2+4k-2k+3
\end{aligned}$$

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$\therefore n^2-n+3 = 4k^2+2k+2+1$, by Rules of Algebra.

$\therefore n^2-n+3 = 2(2k^2+k+1)+1$
$= 2t+1$, where $t = 2k^2+k+1$

$\therefore n^2-n+3 = 2t+1$ and t is an integer.
$\therefore n^2-n+3$ is odd, in CASE 2. [END of CASE 2]

$\therefore n^2-n+3$ is odd, in general.

$\therefore$ For all integers n, n^2-n+3 is odd, by Direct Proof.

QED.

(NOT ASSIGNED) → Sec 4.4, #20. It is given that $a \bmod 7 = 4$.

So, from the Q-R Theorem, there is an integer q with $a = 7q+4$.

So, $5a = 5(7q+4) = 35q+20$

and, $5a = 35q+14+6$
So, $5a = 7(5q+2)+6$ and $5q+2$ is an integer and $0 \le 6 < 7$.

So, The remainder when $5a$ is divided by 7 is 6, and so,

$\boxed{5a \bmod 7 = 6}$.

Sec 4.4

#37. To Prove: The square of any integer has the form $4k$ or $4k+1$ for some integer k.

EQUIVALENTLY, For every integer n, there exists an integer k such that $n^2 = 4k$ or $n^2 = 4k+1$

Proof: Let integer n be given.

By the PARITY ~~corollary~~, n is even OR n is odd.

CASE 1: (n is even)

Suppose n is even.
Then, there exists an integer t such that $n = 2t$, by defn of "even".

$\therefore n^2 = (2t)^2 = 4t^2$. Let $k = t^2$, which is an integer. Thus, $n^2 = 4k$.
$\therefore n^2 = 4k$ OR $n^2 = 4k+1$ by generalization.
$\therefore$ There exists an integer k such that $n^2 = 4k$ OR $n^2 = 4k+1$ in CASE 1.

Case 2 (n is odd.)

Suppose n is odd.
Since n is odd, there exists an integer t such that $n = 2t+1$.

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#37, (Continued)

$$\therefore\ n^2 = (2t+1)^2$$
$$= 4t^2 + 4t + 1$$
$$= 4(t^2+t) + 1 \quad \text{by substitution and rules of algebra.}$$

Let $k = t^2 + t$, which is an integer.

$\therefore\ n^2 = 4k + 1$

$\therefore\ n^2 = 4k$ OR $n^2 = 4k+1$ by generalization.

$\therefore$ There exists an integer k such that $n^2 = 4k$ OR $n^2 = 4k+1$ in CASE 2.

Therefore, in general, there exists an integer k [illegible] such that $n = 4k$, or $n = 4k+1$. [illegible] exists an integer k such that $n^2 =$ [illegible]

$\therefore$ The square of any integer has the form $4k$ or $4k+1$ for some integer k, by Direct Proof.

QED