HW #5, Section 4.5 Solutions M325K SPRING 2024

Sec. 4.5, #4.

<u>To Prove</u>: For all integers m,
$7m+4$ is not divisible by 7.

<u>Proof</u>: Suppose, by way of contradiction, that there exists an integer m such that $7m+4$ is divisible by 7.

$\therefore$ By definition of "is divisible by",

$7m+4 = 7k$, for some integer k.

$\therefore 4 = 7k - 7m$

$\therefore 4 = 7(k-m)$, and $(k-m)$ is an integer since the difference of integers is an integer.

$\therefore 7 \mid 4$ by definition of "divides".

$\therefore$ By Theorem 4.3.1, $7 \le 4$, which contradicts the fact that $4 < 7$.

$\therefore$ For all integers m, $(7m+4)$ is not divisible by 7, by proof-by-contradiction.

QED

A SOLUTION FOR Sec 4.5, #13, That uses Theorem (NIB) 3, But NOT DIVISION INTO CASES.

<u>To Prove</u>: For any integer n, n^2-2 is not divisible by 4.

<u>Proof</u>: Suppose, by way of contradiction, that there exists an integer n such that n^2-2 is divisible by 4.

$\therefore$ $n^2-2 = 4k$, for some integer k. [EQUATION (*)]

$\therefore$ $n^2 = 2+4k$
$= 2+2(2k)$
$= 2(1+2k)$.

$\therefore$ $n^2 = 2l$, where $l = 1+2k$, which is an integer.

$\therefore$ $2 \mid n^2$, by definition of divides

Since 2 is a prime number and $2 \mid n^2$, $2 \mid n$ by Theorem (NIB) 3.

$\therefore$ $n = 2t$, for some integer t.

$\therefore$ $(2t)^2 - 2 = 4k$, by substituting $2t$ for n in EQUATION (*) Above.

$\therefore$ By rules of algebra, $4t^2 = 4k+2$,

So, $4t^2-4k = 2$.

Thus, $4(t^2-k) = 2$, and t^2-k is an integer.

$\therefore$ $4 \mid 2$, and so, by Theorem 4.3.1, $4 \leq 2$, which contradicts the fact that $4 > 2$.

$\therefore$ For any integer n, n^2-2 is not divisible by 4, by Proof-by-Contradiction. QED.

A Solution For Sec 4.5, #13, that uses DIVISION INTO CASES, But not Theorem (NIB) 3.

To Prove: For any integer n, n^2-2 is not divisible by 4.

Proof: Let n be any integer.

By the Parity Corollary, n is even OR n is odd.

CASE 1 (n is even):

Suppose n is even.

Then, there exists an integer k such that $n = 2k$.

Suppose, by way of contradiction, that n^2-2 is divisible by 4.

Then $n^2-2 = 4t$, for some integer t.

$\therefore$ $(2k)^2-2 = 4t$, by substitution.

$\therefore$ $4k^2-2 = 4t$, and so, $4k^2-4t = 2$.

$\therefore$ $2 = 4(k^2-t)$, and k^2-t is an integer.

$\therefore$ $4 \mid 2$

By Theorem 4.3.1, $4 \leq 2$, which contradicts the fact that $4 > 2$.

$\therefore$ n^2-2 is not divisible by 4, in the case that n is even, by proof-by-contradiction.

[END of CASE 1]

[The proof continues with CASE 2 on the next page]

[Solution for Sec 4.5; #13 (Continued)]

CASE 2 (n is odd):

Suppose n is odd.

Then, there exists an integer k such that

$$n = 2k+1.$$

Suppose, by way of contradiction, that n^2-2 is divisible by 4.

Then $n^2-2 = 4t$ for some integer t.

$\therefore$ $(2k+1)^2-2 = 4t$, by substitution.

$\therefore$ By rules of algebra, $(4k^2+4k+1)-2 = 4t$.

$\therefore$ $4k^2+4k-1 = 4t$, and so, $4k^2+4k-4t = 1$.

$\therefore$ $4(k^2+k-t) = 1$, and (k^2+k-t) is an integer.

$\therefore$ $4 \mid 1$.

$\therefore$ By Theorem 4.3.1, $4 \leq 1$, which contradicts the fact that $4 > 1$.

$\therefore$ n^2-2 is not divisible by 4, in the case that n is odd, by proof-by-contradiction. [END of CASE 2].

$\therefore$ n^2-2 is not divisible by 4, in General!

$\therefore$ For any integer n, n^2-2 is not divisible by 4, by Direct Proof.

Q.E.D.

Sec. 4.5, #20

To Prove: If the sum of two real numbers is less than 50, then at least one of the numbers is less than 25.

[To Prove: $\forall r, s \in \mathbb{R}$, if $r+s<50$, then $r<25$ OR $s<25$.]

Proof (by Contraposition)

Let r and s be real numbers.
Suppose $r \geq 25$ and $s \geq 25$.

$\therefore r+s \geq 25+s$ [Adding s to both sides of "$r \geq 25$"]

$\therefore 25+s \geq 25+25 = 50$ [Adding 25 to both sides $s \geq 25$]

$\therefore r+s \geq 50$ by Transitivity of "$\geq$".

$\therefore$ If $r+s<50$, then $r<25$ or $s<25$ by contraposition.

$\therefore$ QED, by Direct Proof.