Homework #5

Section 4.6 Solutions M325K SPRING 2024

#4 To Prove: $3\sqrt{2} - 7$ is irrational.

Proof: (By Contradiction)

Suppose, by way of contradiction, that $3\sqrt{2} - 7$ is RATIONAL.

Let $r = 3\sqrt{2} - 7$. Then, r is rational by assumption.

$\therefore r + 7 = 3\sqrt{2}$; $\therefore 3\sqrt{2} = r + 7$

$\therefore \sqrt{2} = \frac{r+7}{3}$, which is a rational number since the sum and defined quotients of rational numbers are rational numbers.

But, this contradicts the previously established fact (Theorem 4.6.1) that $\sqrt{2}$ is IRRATIONAL.

$\therefore 3\sqrt{2} - 7$ is IRRATIONAL, by proof-by-contradiction.

QED

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#8. The statement "The difference of any two irrational numbers is irrational" is FALSE.

By Theorem 4.6.1, $\sqrt{2}$ is irrational.
By Theorem 4.5.3, $\sqrt{2}+1$ is irrational.

But, $(\sqrt{2}+1)-\sqrt{2} = 1$ and 1 is rational.
$\therefore$ The statement is false by proof-by-counterexample. ∎

#10 The Statement "If r is any rational number and s is any irrational number, then r/s is irrational." is FALSE.

Let $r = 0$ and $s = \sqrt{2}$.
By Theorem 4.6.1, $\sqrt{2}$ is irrational, 0 is rational, but

$r/s = 0/\sqrt{2} = 0$ is rational, not irrational.

$\therefore$ The statement is false by proof-by-counter-example. ∎

Section 4.6, #19

To Prove: $\sqrt{5}$ is irrational.

Proof: Suppose (by way of contradiction) that $\sqrt{5}$ is a rational number.

Then, $\sqrt{5} = a/b$ for some integers a and b such that $b \neq 0$, and we can assume that a and b have no common prime factor.

Now, $5 = (\sqrt{5})^2$.

$\therefore 5 = (a/b)^2 = a^2/b^2$

$\therefore 5b^2 = a^2$, by Rules of Algebra.

$\therefore 5 \mid a^2$, by definition of "divides".

$\therefore$ Since 5 is prime and $5 \mid a^2$, $5 \mid a$, by Theorem (N18) 3.

$\therefore a = 5k$ for some integer k.

[We show that 5 also divides b.]

Recall that $5b^2 = a^2$. $\therefore$ By Substitution, $5b^2 = (5k)^2$.

$\therefore 5b^2 = 25k^2$.

$\therefore b^2 = 5k^2$. $\therefore 5 \mid b^2$.

Since 5 is prime and $5 \mid b^2$, $5 \mid b$, by Theorem (N18) 3.

$\therefore 5 \mid a$ and $5 \mid b$, which contradicts the fact that a and b have no common prime factor.

$\therefore \sqrt{5}$ is irrational, by proof-by-contradiction.

Q.E.D.

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#23 To Prove: $\sqrt{2}+\sqrt{3}$ is irrational.

Proof: Suppose, by way of contradiction, that $\sqrt{2}+\sqrt{3}$ is rational.

Let $r = \sqrt{2}+\sqrt{3}$.
Then, r is rational by assumption.

$$r^2 = (\sqrt{2}+\sqrt{3})^2$$
$$= \sqrt{2}^2 + 2\sqrt{2}\sqrt{3} + \sqrt{3}^2$$
$$= 2 + 2\sqrt{6} + 3$$
$\therefore r^2 = 2\sqrt{6} + 5$ by rules of algebra.

$\therefore r^2 - 5 = 2\sqrt{6}$

$\therefore 2\sqrt{6} = r^2 - 5$, which rational since r^2 and 5 are rational

$\therefore \sqrt{6} = \frac{1}{2}(r^2-5)$, which rational since the product of rationals is rat'l.

$\therefore \sqrt{6}$ is rational.

Now $0^2=0$, $1^2=1$, $2^2=4$, $3^2=9$, and every other perfect square is greater than 6.
$\therefore$ 6 is not a perfect square.
$\therefore$ By problem #22, $\sqrt{6}$ is irrational, which contradicts the fact that $\sqrt{6}$ is rational, just established.
$\therefore r = \sqrt{2}+\sqrt{3}$ is IRRATIONAL, by proof-by-contradiction.
QED.