HW #6, Section 5.2 Solutions M325K SPRING 2024

SEC. 5.2 #12 To Prove: For all integers $n \geq 1$,

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Proof: [By Mathematical Induction]

[BASIS STEP] Let $n = 1$.

Then, $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} = \frac{1}{2}$, by Convention.

Also, $\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$.

$\therefore$ For $n = 1$, $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

by Substitution. [END OF BASIS STEP]

[INDUCTIVE STEP]

Let k be any integer such that $k \geq 1$.

Suppose $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$. [INDUCTIVE HYPOTHESIS]

[N.T.S.: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$.]

$$\left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)}\right) + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$ by The Inductive Hypothesis and Substitution

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

$$\therefore \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}.$$

$\therefore$ By Direct Proof, For all integers $k \geq 1$,

If $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$,

THEN $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$.

[END of Induction STEP]

$\therefore$ For all integers n,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1},$$

by the Principle of Mathematical Induction.

QED.

Sec 5.2

#14 : To Prove : For all integers $n \geq 0$,

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2.$$

Proof : [By MATHEMATICAL INDUCTION]

[BASIS STEP] Let $n = 0$.

$$\sum_{i=1}^{n+1} i \cdot 2^i = \sum_{i=1}^{1} i \cdot 2^i = 1 \cdot 2^1 = 2$$

Also, $n \cdot 2^{n+2} + 2 = 0 \cdot 2^{0+2} + 2 = 0 + 2 = 2$.

$\therefore$ For $n = 0$, $\sum_{i=1}^{n+1} i\, 2^i = n \cdot 2^{n+2} + 2$, by sub.

[INDUCTIVE STEP]

Let k be any integer such that $k \geq 0$,

[INDUCTIVE Hypothesis] Suppose that $\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2$.

[NTS : $\sum_{i=1}^{k+2} i \cdot 2^i = (k+1)\, 2^{(k+1)+2} + 2$.]

5.2, #14 (continued)

$i = k+2$

$$\sum_{i=1}^{k+2} i\cdot 2^i = \left(\sum_{i=1}^{k+1} i\cdot 2^i\right) + (k+2)\cdot 2^{(k+2)}$$

$$= \left(k\cdot 2^{k+2} + 2\right) + (k+2)\,2^{k+2},$$

by the Inductive Hypothesis,

$$= 2^{k+2}\cdot(k + k+2) + 2$$

$$= 2^{k+2}\cdot(2k+2) + 2$$

$$= 2^{k+2}\cdot 2(k+1) + 2$$

$$= (k+1)\,2^{(k+3)} + 2$$

$$\therefore \sum_{i=1}^{k+2} i\cdot 2^i = (k+1)\cdot 2^{(k+1)+2} + 2.$$

$\therefore$ For all integers $k \geq 0$,

if $\displaystyle\sum_{i=1}^{k+1} i\cdot 2^i = k\,2^{k+2} + 2$,

then $\displaystyle\sum_{i=1}^{(k+1)+1} i\cdot 2^i = (k+1)\,2^{(k+1)+2} + 2$, by Direct Proof.

[END OF INDUCTIVE STEP]

$\therefore$ By the Principle of Mathematical Induction, for all integers $n \geq 0$, $\displaystyle\sum_{i=1}^{n+1} i\cdot 2^i = n\cdot 2^{n+2} + 2$.

QED