HW #6, Section 5.3 SOLUTIONS M325K SPRING 2024

Sec 5.3, #10:

TO PROVE: For each integer $n \geq 0$, $n^3 - 7n + 3$ is divisible by 3.

Proof: [By Mathematical Induction]

[BASIS STEP]

Let $n = 0$. Then, $n^3 - 7n + 3 = 0 - 0 + 3 = 3$
and 3 is divisible by 3 since $3 = 3 \cdot 1$.

$\therefore$ For $n = 0$, $n^3 - 7n + 3$ is divisible by 3, by substitution.

[END of BASIS STEP]

[INDUCTIVE STEP]

Let k be any integer with $k \geq 0$.

[Induction Hypothesis] Suppose $k^3 - 7k + 3$ is divisible by 3.

[NTS: $(k+1)^3 - 7(k+1) + 3$ is divisible by 3]

By the Induction Hypothesis, there exists an integer t such that $k^3 - 7k + 3 = 3t$

$\therefore$ $k^3 = 3t + 7k - 3$ by rules of algebra.

Now, $(k+1)^3 - 7(k+1) + 3 = (k^3 + 3k^2 + 3k + 1) - 7k - 7 + 3$

$= k^3 + 3k^2 - 4k - 3$

$= (3t + 7k - 3) + 3k^2 - 4k - 3$ by substitution,

$= 3t + 3k^2 + 3k - 6$

$= 3(t + k^2 + k - 2)$

$= 3s$, where $s = t + k^2 + k - 2$, which is an integer.

(Sec 5.3, #10 cont.)

$\therefore$ $(k+1)^3 - 7(k+1) + 3$ is divisible by 3.

$\therefore$ For all integers $k \geq 0$, if $k^3 - 7k + 3$ is divisible by 3, then $(k+1)^3 - 7(k+1) + 3$ is divisible by 3, by Direct Proof.

[END OF INDUCTIVE STEP]

$\therefore$ For all integers $n \geq 0$, $n^3 - 7n + 3$ is divisible by 3, by the Principle of Mathematical Induction.

QED.

Sec 5.3

#12 To Prove: For any integer $n \geq 0$,

$7^n - 2^n$ is divisible by 5.

Proof: [By mathematical Induction].

[BASIS STEP] Let $n = 0$.

$7^n - 2^n = 7^0 - 2^0 = 1 - 1 = 0 = 5 \times 0$

$\therefore$ For $n = 0$, $7^n - 2^n$ is divisible by 5.

[INDUCTIVE STEP]

~~Let~~ k be any integer such that $k \geq 0$.

Suppose that $7^k - 2^k$ is divisible by 5. [INDUCTIVE HYPOTHESIS]

[NTS: $7^{k+1} - 2^{k+1}$ is divisible by 5.]

By the Inductive Hypothesis, there exists an integer t such that $7^k - 2^k = 5t$.

$\therefore 7^k = 5t + 2^k$.

$\therefore 7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k$

$= 7(5t + 2^k) - 2 \cdot 2^k$ by substitution,

$$\therefore 7^{k+1} - 2^{k+1} = 35t + 7\cdot 2^k - 2\cdot 2^k$$
$$= 35t + (7-2)2^k$$
$$= 35t + 5\cdot 2^k$$
$$= 5(7t + 2^k).$$

$$\therefore 7^{k+1} - 2^{k+1} = 5(7t+2^k).$$

Since $7t+2^k$ is an integer, we conclude, by the definition of "divisible" that $7^{k+1} - 2^{k+1}$ is divisible by 5.

$\therefore$ For all integers $k \geq 0$,

if $7^k - 2^k$ is divisible by 5,

Then $7^{(k+1)} - 2^{(k+1)}$ is divisible by 5. by Direct proof.

[END OF INDUCTIVE STEP]

$\therefore$ For all integers $n \geq 0$,

$7^n - 5^n$ is divisible by 5 by the Principle of Mathematical Induction.

QED

Section 5.3

#19 To Prove: $n^2 < 2^n$, for all integers $n \geq 5$.

Proof: [By Mathematical Induction]

[Basis Step] Let $n = 5$.
Then, $n^2 = 25$ and $2^n = 2^5 = 32$ and $25 < 32$.
$\therefore$ For $n = 5$, $n^2 < 2^n$ by substitution. [END of Basis Step]

[INDUCTIVE STEP]
Let k be any integer such that $k \geq 5$.

Suppose $k^2 < 2^k$. [The Inductive Hypothesis]

[We need to show that $(k+1)^2 < 2^{(k+1)}$.]

$(k+1)^2 = k^2 + 2k + 1$.

Now, $k \geq 5 > 3$, so $k \geq 3$, which means that Proposition 5.3.2 applies here.

$\therefore$ By Proposition 5.3.2, $2k+1 < 2^k$.

$\therefore$ Since $k^2 < 2^k$, $k^2 + 2k + 1 < 2^k + 2k + 1$.
$\therefore$ Since $2k+1 < 2^k$, $2^k + 2k + 1 < 2^k + 2^k = 2 \cdot 2^k$

$\therefore$ $k^2 + 2k + 1 < 2 \cdot 2^k = 2^{(k+1)}$ by transitivity of "<".

$\therefore$ $(k+1)^2 < 2^{(k+1)}$ by substitution

Section 5.3, # 19 (Continued)

$\therefore$ For every integer $k \geq 5$,
if $k^2 < 2^k$, then $(k+1)^2 < 2^{(k+1)}$, by Direct Proof.
[END of Induction Step].

$\therefore$ For all integers $n \geq 5$, $n^2 < 2^n$, by the Principle of Mathematical Induction.
QED

Sec 5.3, #25

The sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$, for all integers $k \geq 1$.

To Prove: For all integers $n \geq 0$, $b_n > 4n$.

Proof: [BASIS STEP].
Let $n = 0$. $\therefore b_n = b_0 = 5$, by definition of b_0, and $4n = 4 \cdot 0 = 0$. Also $5 > 0$.
$\therefore$ For $n = 0$, $b_n > 4n$ by substitution. [END of BASIS STEP]

[INDUCTIVE STEP]
Let k be any integer such that $k \geq 0$.

[INDUCTIVE HYPOTHESIS:] Suppose $b_k > 4k$.
[NTS: $b_{k+1} > 4(k+1)$] Since $k \geq 0$, $k+1 \geq 1$.
$\therefore b_{k+1} = 4 + b_k$, by definition of b_m when $m \geq 1$.

[Adding 4 to both sides] By the Inductive Hypothesis, $4 + b_k > 4k + 4$.
Since $b_{k+1} = 4 + b_k$ and since $4k + 4 = 4(k+1)$,
$b_{k+1} > 4(k+1)$, by substitution.

$\therefore$ For all integers $k \geq 0$, if $b_k > 4k$, then $b_{k+1} > 4(k+1)$, by Direct Proof.
[END OF INDUCTIVE STEP]

$\therefore$ By Mathematical Induction $b_n > 4n$, for all integers $n \geq 0$.
QED