HW 7A Solutions

SEC 5.3, #21

HW #7A, PART I SOLUTION: Sec 5.3, #21

M325K SPRING 2024

TO PROVE: For all integers $n \geq 2$, $\sqrt{n} < \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right)$.

Proof: [By Mathematical Induction]

[BASIS STEP] Let $n = 2$. $\therefore \sqrt{n} = \sqrt{2}$ and

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}.$$

Now, $1 < 2$.

$\therefore \sqrt{1} < \sqrt{2}$ since $y = \sqrt{x}$ is an increasing function.

$\therefore 1 < \sqrt{2}$. $\therefore 1 + 1 < 1 + \sqrt{2}$. $\therefore 2 < 1 + \sqrt{2}$

$\therefore (\sqrt{2})(\sqrt{2}) < 1 + \sqrt{2}$

$\therefore$ [By dividing through by $\sqrt{2}$] $\sqrt{2} < \frac{1+\sqrt{2}}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}}$

$\therefore \sqrt{2} < 1 + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$.

$\therefore$ FOR $n = 2$, $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$, by substitution.

[END OF BASIS STEP]

[INDUCTIVE STEP]

Let $k$ be any integer such that $k \geq 2$.

Suppose $\sqrt{k} < \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}}\right)$. [Inductive Hypothesis]

$\therefore$ By The Induction Hypothesis,

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} < \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}}\right) + \frac{1}{\sqrt{k+1}} \qquad (*)$$

$$= \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right).$$

[We N.T.S. $\sqrt{k+1} < \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right)$.

We first show that $\sqrt{k+1} < \left(\sqrt{k} + \frac{1}{\sqrt{k+1}}\right)$ and then combine this with inequality $(*)$ above and use the transitivity of "less than" to conclude the inequality we need.]

Now, $k < k+1$

$\therefore$ $\sqrt{k} < \sqrt{k+1}$ since $y = \sqrt{x}$ is an increasing function.

$\therefore$ [Multiplying through by $\sqrt{k}$] $\sqrt{k}\sqrt{k} < \sqrt{k}\sqrt{k+1}$.

$\therefore$ Since $\sqrt{k}\sqrt{k} = k$, $k < \sqrt{k}\sqrt{k+1}$ by substitution.

$\therefore$ [Adding 1 to both sides] $k+1 < (\sqrt{k}\sqrt{k+1} + 1)$

$\therefore$ [Dividing through by $\sqrt{k+1}$] $\dfrac{k+1}{\sqrt{k+1}} < \dfrac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}}$.

$\therefore$ Since $\dfrac{k+1}{\sqrt{k+1}} = \sqrt{k+1}$ and $\dfrac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} = \sqrt{k} + \dfrac{1}{\sqrt{k+1}}$,

$\sqrt{k+1} < \left(\sqrt{k} + \dfrac{1}{\sqrt{k+1}}\right)$, by substitution. (**).

Combining statements (**) and (*), we conclude that
$\sqrt{k+1} < \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{k+1}}$ by the Transitive property of "less than".

$\therefore$ For all integers $k \geq 2$, if $\sqrt{k} < \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{k}}$,

then $\sqrt{k+1} < \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{k}} + \dfrac{1}{\sqrt{k+1}}$,

by Direct Proof.

[END of Induction Step]

$\therefore$ For all integers $n \geq 2$, $\sqrt{n} < \left(\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}}\right)$,

by the Principle of Mathematical Induction

QED.

Section 5.4, #9 HW# 7A, PART II SOLUTION:

SECTION 5.4, #9

Sequence $(a_n)$ is defined as follows:

$a_1 = 1$, $a_2 = 3$, $a_t = a_{t-1} + a_{t-2}$ for all integers $t \geq 3$.

To Prove: For all integers $n \geq 1$, $a_n \leq \left(\frac{7}{4}\right)^n$.

Proof: [By STRONG MATHEMATICAL INDUCTION]

Let $n = 1$. $a_n = a_1 = 1$. $\left(\frac{7}{4}\right)^n = \left(\frac{7}{4}\right)^1 = \frac{7}{4}$

$1 \leq \frac{7}{4}$. $\therefore$ For $n = 1$, $a_n \leq \left(\frac{7}{4}\right)^n$, by substitution.

Let $n = 2$. $a_n = a_2 = 3$. $\left(\frac{7}{4}\right)^n = \left(\frac{7}{4}\right)^2 = \frac{49}{16}$.

$3 = \frac{48}{16} < \frac{49}{16}$. $\therefore$ For $n = 2$, $a_n \leq \left(\frac{7}{4}\right)^n$, by substitution.

[END of Basis Step]

Let $k$ be any integer such that $k \geq 2$.

[INDUCTIVE HYPOTHESIS] Suppose $a_m < \left(\frac{7}{4}\right)^m$, for all integers $m$ such that $1 \leq m \leq k$.

[NTS: $a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}$ ]

Since $k \geq 2$, $k+1 \geq 3$, so, $a_{k+1} = a_k + a_{k-1}$.

Since $k \geq 2$, $k-1 \geq 1$. $\therefore$ $1 \leq k-1 \leq k$.

Since $1 \leq k-1 \leq k$, $a_{k-1} \leq \left(\frac{7}{4}\right)^{k-1}$, by the INDUCTIVE Hypothesis.

(Sec 5.4, #9 cont.)

Since $1 \le k \le k$, $a_k \le \left(\frac{7}{4}\right)^k$, by the Induction Hypothesis.

$\therefore a_{k+1} = a_k + a_{k-1} \le \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{(k-1)}$

Also, $\left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{(k-1)} = \left(\frac{7}{4}\right)\left(\frac{7}{4}\right)^{(k-1)} + \left(\frac{7}{4}\right)^{(k-1)}$

$= \left(\frac{7}{4}+1\right)\left(\frac{7}{4}\right)^{(k-1)} = \frac{11}{4}\left(\frac{7}{4}\right)^{(k-1)} = \frac{44}{16}\left(\frac{7}{4}\right)^{(k-1)}$.

$\therefore a_{k+1} \le \frac{44}{16}\left(\frac{7}{4}\right)^{(k-1)}$, by substitution.

Now, $\left(\frac{7}{4}\right)^{k+1} = \left(\frac{7}{4}\right)^2\left(\frac{7}{4}\right)^{(k-1)} = \frac{49}{16}\left(\frac{7}{4}\right)^{(k-1)}$.

Since $\frac{44}{16} \le \frac{49}{16}$, $\frac{44}{16}\left(\frac{7}{4}\right)^{(k-1)} \le \frac{49}{16}\left(\frac{7}{4}\right)^{(k-1)} = \left(\frac{7}{4}\right)^{(k+1)}$.

$\therefore$ By Transitivity of "Less than or Equals",

$a_{k+1} \le \left(\frac{7}{4}\right)^{(k+1)}$.

$\therefore$ For all integers $k \ge 2$ if $a_m \le \left(\frac{7}{4}\right)^m$ for all integers such that $1 \le m \le k$, then $a_{k+1} \le \left(\frac{7}{4}\right)^{k+1}$, by Direct Proof.

[END OF INDUCTIVE STEP]

$\therefore$ For all integers $n \ge 1$,

$a_n \le \left(\frac{7}{4}\right)^n$, by the Principle of Strong Mathematical Induction.

QED.

HW #7A, PART III SOLUTION: SEC 5.3, #12 with a proof using the Well-Ordering Principle of the Integers.

Section 5.3, #12 [USING THE Well-Ordering Principle]

To Prove: For any integer $n \geq 0$,

$$7^n - 2^n \text{ is divisible by } 5.$$

Proof: [Proof-by-Contradiction]

Suppose, by way of contradiction, that there exists an integer $N$, $N \geq 0$, such that

$$7^N - 2^N \text{ is not divisible by } 5.$$

Let set $S = \{$all integers $t \geq 0$ such that $7^t - 2^t$ is not divisible by $5\}$.

[We show that $S$ satisfies the conditions of the Well-Ordering Principle.]

Since $N \geq 0$ and $7^N - 2^N$ is not divisible by 5, $N$ is in $S$, and so, $S$ is non-empty.

By definition of set $S$, for all integers $t$ in $S$, $t \geq 0$.

$\therefore$ Set $S$ satisfies the conditions of the Well-Ordering Principle.

$\therefore$ By the Well-Ordering Principle, $S$ has a least element $m$.

(Sec 5.3, #12 cont.)

Since $m$ is in $S$, $m \geq 0$ and $7^m - 2^m$ is not divisible by 5, by definition of set $S$.

We need to show that $7^{(m-1)} - 2^{(m-1)}$ is divisible by 5. We do this by showing that $m-1 \geq 0$ and $m-1$ is not in set $S$. This allows us to conclude that $7^{(m-1)} - 2^{(m-1)}$ is divisible by 5.

Recall that $m \geq 0$ and $7^m - 2^m$ is not divisible by 5.

Since $7^0 - 2^0 = 1 - 1 = 0 = 5 \cdot 0$,
$7^0 - 2^0$ is divisible by 5.

$\therefore$ $m \neq 0$, because $7^m - 2^m$ is not divisible by 5.

Since $m \geq 0$ and $m \neq 0$, $m > 0$.

$\therefore$ $m \geq 1$, and so, $m - 1 \geq 0$.

Now, $0 \leq m-1 < m$, so $m-1$ is not in set $S$, because $m$ is the least element of $S$.

$\therefore$ $m-1$ is not in set S.

[We need to show that $7^{(m-1)} - 5^{(m-1)}$ is divisible by 5]

Suppose, by way of contradiction, that

$7^{(m-1)} - 5^{(m-1)}$ is not divisible by 5:

Since $m-1 \geq 0$ and $7^{(m-1)} - 5^{(m-1)}$ is not divisible by 5, we can conclude that,

by definition of Set S, $(m-1)$ is in Set S.

But, this contradicts the fact that $m-1$ is not in Set S.

$\therefore$ $7^{(m-1)} - 5^{(m-1)}$ is divisible by 5, by proof-by-contradiction.

$\therefore$ By definition of "divisibility", there exists an integer $l$ such that $7^{(m-1)} - 2^{(m-1)} = 5l$.

$$\therefore 7^m - 2^m = 7 \cdot 7^{(m-1)} - 2^m$$
$$= 7 \cdot (2^{(m-1)} + 5l) - 2^m, \text{ by subst.,}$$
$$= 7 \cdot 2^{(m-1)} + 35l - 2 \cdot 2^{(m-1)}$$
$$= (7-2) \cdot 2^{(m-1)} + 35l$$
$$= 5 \times 2^{(m-1)} + 5 \times 7l$$
$$\therefore 7^m - 2^m = 5(2^{(m-1)} + 7l),$$

and $(2^{(m-1)} + 7l)$ is an integer.

$\therefore 7^m - 2^m$ is divisible by 5, which contradicts the fact that $7^m - 2^m$ is not divisible by 5.

$\therefore$ For any integer $n \geq 0$, $7^n - 2^n$ is divisible by 5, by proof-by-contradiction.

QED

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