HW #7B SOLUTIONS: Section 5.4: #2, 6

SEC. 5.4, #2,

SEQUENCE DEFINITION: SEQUENCE (b_n) is defined as follows:

$b_1 = 4$, $b_2 = 12$;

$b_t = b_{t-2} + b_{t-1}$, for all integers $t \geq 3$.

To Prove: For all integers $n \geq 1$, b_n is divisible by 4.

Proof: [By Strong Mathematical Induction.]

[Basis Step: 2 initial cases suffice]

Let $n = 1$. $b_n = b_1 = 4 = 4 \times 1$. $\therefore 4 \mid b_n$, for $n = 1$.

Let $n = 2$. $b_n = b_2 = 12 = 4 \times 3$. $\therefore 4 \mid b_n$, for $n = 2$.

[END OF BASIS STEP]

[INDUCTIVE STEP]

Let k be any integer such that $k \geq 2$.

Suppose that, for every integer m such that $1 \leq m \leq k$, b_m is divisible by 4. [INDUCTIVE HYPOTHESIS]

[NEED TO SHOW: b_{k+1} is divisible by 4.]

Since $k \geq 2$, $k+1 \geq 3$.

Since $k+1 \geq 3$, $b_{k+1} = b_{k-1} + b_k$.

(Sec 5.4, #2 cont.)

Since $2 \le k$, $1 \le k-1 \le k$.

Since $1 \le k-1 \le k$, b_{k-1} is divisible by 4, by the INDUCTIVE HYPOTHESIS.

Since $1 \le k \le k$, b_k is divisible by 4, by the Inductive Hypothesis.

$\therefore$ There exist integers l and s such that
$b_{k-1} = 4l$ and $b_k = 4s$.

$$\therefore b_{k+1} = b_{k-1} + b_k$$
$$= 4l + 4s$$

$\therefore b_{k+1} = 4(l+s)$ and $(l+s)$ is an integer.

$\therefore b_{k+1}$ is divisible by 4.

$\therefore$ For all integers $k \ge 2$, if b_m is divisible by 4 for all integers m such that $1 \le m \le k$, then b_{k+1} is divisible by 4, by Direct Proof.
[END OF INDUCTIVE STEP]

$\therefore$ For all integers $n \ge 1$, b_n is divisible by 4, by the Principle of Strong Mathematical Induction.

QED

SECTION 5.4, #6

SEQUENCE (f_n) is defined as follows:

$f_0 = 5$, $f_1 = 16$; $f_t = 7f_{t-1} - 10f_{t-2}$, for all integers $t \geq 2$.

To Prove: $f_n = 3 \cdot 2^n + 2 \cdot 5^n$, for all integers $n \geq 0$.

Proof: [By Strong Mathematical Induction]

[BASIS STEP]

Let $n = 0$. $f_n = f_0 = 5$.

$3 \cdot 2^n + 2 \cdot 5^n = 3 \cdot 2^0 + 2 \cdot 5^0 = 3 + 2 = 5$

$\therefore$ For $n = 0$, $f_n = 3 \cdot 2^n + 2 \cdot 5^n$, by Subst.

Let $n = 1$. $f_n = f_1 = 16$.

$3 \cdot 2^n + 2 \cdot 5^n = 3 \cdot 2^1 + 2 \times 5^1 = 6 + 10 = 16$.

$\therefore$ For $n = 1$, $f_n = 3 \cdot 2^n + 2 \cdot 5^n$, by subst.

[END of BASIS STEP]

[INDUCTIVE STEP]

Let k be any integer such that $k \geq 1$.

[INDUCTIVE HYPOTHESIS] Suppose that $f_m = 3 \cdot 2^m + 2 \cdot 5^m$, for all integers m such that $0 \leq m \leq k$.

[NTS: $f_{k+1} = 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1}$.]

Since $k \geq 1$, $k+1 \geq 2$.

$\therefore$ Since $k+1 \geq 2$, $f_{k+1} = 7f_k - 10f_{k-1}$.

(Sec 5.4, #6 cont.)

Since $k \geq 1$, $k-1 \geq 0$, and so $0 \leq k-1 \leq k$.

Since $0 \leq k-1 \leq k$, $f_{k-1} = 3 \cdot 2^{(k-1)} + 2 \cdot 5^{(k-1)}$,
by the Inductive Hypothesis.

Since $0 \leq k \leq k$, $f_k = 3 \cdot 2^k + 2 \cdot 5^k$, by the Inductive Hypothesis.

$$\therefore f_{k+1} = 7f_k - 10f_{k-1}$$

$$= 7(3 \cdot 2^k + 2 \cdot 5^k) - 10(3 \cdot 2^{(k-1)} + 2 \cdot 5^{(k-1)})$$

$$= 21 \times 2^k + 14 \times 5^k - 30 \cdot 2^{(k-1)} - 20 \times 5^{(k-1)}$$

$$= 21 \times 2^k + 14 \times 5^k - 15 \times 2^k - 4 \times 5^k$$

$$= 6 \times 2^k + 10 \times 5^k$$

$$\therefore f_{k+1} = 3 \times 2 \times 2^k + 2 \times 5 \times 5^k$$

$$\therefore f_{k+1} = 3 \times 2^{k+1} + 2 \times 5^{k+1}.$$

$\therefore$ For every integer $k \geq 1$, if $f_m = 3 \times 2^m + 2 \times 5^m$ for all integers m such that $0 \leq m \leq k$, then $f_{k+1} = 3 \times 2^{(k+1)} + 2 \times 5^{(k+1)}$, by Direct Proof.

[END OF INDUCTIVE STEP]

$\therefore$ For all integers $n \geq 0$, $f_n = 3 \cdot 2^n + 2 \cdot 5^n$, by STRONG MATHEMATICAL INDUCTION.

QED