HW #8, Section 6.2 Solutions: #14, #15, #17 and #34 (TWO SOLUTIONS)

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SECTION 6.2, #14

To Prove: For all sets A, B, C, if $A \subseteq B$, then $(A \cup C) \subseteq (B \cup C)$.

Proof: Let A, B, and C be sets.

Suppose $A \subseteq B$. [NTS: $(A \cup C) \subseteq (B \cup C)$]

Let $x \in (A \cup C)$ be given. [NTS: $x \in (B \cup C)$]

$\therefore$ $x \in A$ or $x \in C$ by def'n of "union".

CASE 1: $(x \in A)$

Suppose $x \in A$.

Since $A \subseteq B$, $x \in B$ by Universal Modus Ponens.

$\therefore$ $x \in B$ or $x \in C$ by generalization.

$\therefore$ $x \in B$ or $x \in C$ in the case that $x \in A$.

[or just "$x \in B$ or $x \in C$ in CASE 1"].

[END of Case 1].

CASE 2: $(x \in C)$.

Suppose $x \in C$.

$\therefore$ $x \in B$ or $x \in C$ by generalization.

$\therefore$ $x \in B$ or $x \in C$ in the case that $x \in C$.

[END of CASE 2]

$\therefore$ $x \in B$ or $x \in C$ in general.

$\therefore$ $x \in (B \cup C)$.

$\therefore$ $(A \cup C) \subseteq (B \cup C)$ by Direct Proof.

$\therefore$ FOR ALL sets A, B, and C, if $A \subseteq B$, then $(A \cup C) \subseteq (B \cup C)$, by Direct Proof.

QED.

Section 6.2, #15

To Prove: For all sets A and B, if $A \subseteq B$, then $B^c \subseteq A^c$.

Proof (TWO VERSIONS!)

Let A and B be sets.

Suppose $A \subseteq B$. [NTS: $B^c \subseteq A^c$].

Let $x \in B^c$ be given. [NTS: $x \in A^c$]

[Version 1: using Universal Modus Tollens]

Since $x \in B^c$, $x \notin B$ by definition of "complement".

[Since $A \subseteq B$, for all $y \in U$, if $y \in A$, then $y \in B$, by defin of "subset of".]

$\therefore x \notin A$ by Universal Modus Tollens.

$\therefore x \in A^c$ by def'n of "complement".

[Version 2: Using Proof-by-Contradiction]

Suppose, by way of contradiction, that $x \notin A^c$. It is false that $x \in A^c$.

It is false that $x \notin A$ by def'n of "complement".

$\therefore x \in A$

$\therefore x \in B$, since $A \subseteq B$ and Universal Modus Ponens.

Since $x \in B^c$, $x \notin B$ by def'n of "complement."

$\therefore x \in B$ and $x \notin B$, which is a contradiction.

$\therefore x \in A^c$ by proof-by-contradiction.

$\therefore B^c \subseteq A^c$ by direct proof.

QED, by Direct Proof.

[or ": For all sets A, B, if $A \subseteq B$, then $B^c \subseteq A^c$ by Direct Proof. QED"]

SECTION 6.2, #17 [Recall: U = The Universal Set]

To Prove: For all sets A, B, and C, if $A \subseteq C$ and $B \subseteq C$, then $(A \cup B) \subseteq C$.

Proof: Let A, B, and C be any sets.
Suppose $A \subseteq C$ and $B \subseteq C$.

Let $x \in (A \cup B)$ be given.

$\therefore$ $x \in A$ OR $x \in B$ by definition of "UNION".

CASE 1 ($x \in A$):

Suppose $x \in A$.
$\therefore$ $x \in C$ by Universal Modus Ponens since $A \subseteq C$.
$\therefore$ $x \in C$ in Case 1. [END of CASE 1.]

CASE 2 ($x \in B$):

Suppose $x \in B$.
$\therefore$ $x \in C$ by Universal Modus Ponens since $B \subseteq C$.
$\therefore$ $x \in C$ in Case 2. [END of CASE 2]

$\therefore$ $x \in C$, IN GENERAL.

[$\therefore$ For all $x \in U$, if $x \in (A \cup B)$, then $x \in C$]

$\therefore$ $(A \cup B) \subseteq C$, by Direct PROOF.

$\therefore$ For all sets A, B, and C, if $A \subseteq C$ and $B \subseteq C$, then $(A \cup B) \subseteq C$, by Direct Proof.

QED.

Section 6.2, #34, Solution #1. (NOT in Full Detail)

To Prove: For all sets A, B and C,

if $(B \cap C) \subseteq A$, then $(C-A) \cap (B-A) = \emptyset$.

Proof: Let A, B, and C be any sets such that

$B \cap C \subseteq A$.

[NTS: $(C-A) \cap (B-A) = \emptyset$]

Suppose, by way of contradiction, that $(C-A) \cap (B-A) \neq \emptyset$.

$\therefore$ There exists an element $x \in U$ such that

$x \in (C-A) \cap (B-A)$, by definition of "non-empty set."

$\therefore$ $x \in (C-A)$ AND $x \in (B-A)$, by definition of "Intersection."

Since $x \in C-A$, $x \in C$ and $x \notin A$, by definition of "Set Difference."

Similarly, since $x \in B-A$, $x \in B$ and $x \notin A$.

$\therefore$ $x \in B \cap C$, since $x \in B$ and $x \in C$.

$\therefore$ $x \in A$, since $B \cap C \subseteq A$.

So, $x \in A$, but this contradicts the fact that $x \notin A$.

$\therefore$ $(C-A) \cap (B-A) = \emptyset$, by proof-by-contradiction.

$\therefore$ For all sets A, B and C,

if $(B \cap C) \subseteq A$, then $(C-A) \cap (B-A) = \emptyset$,

by Direct Proof

QED.

Section 6.2, #34, Solution #2 (NOT IN FULL Detail)

To Prove: For all sets A, B and C,
if $(B \cap C) \subseteq A$, then $(C-A) \cap (B-A) = \emptyset$.

Proof: Suppose, by way of contradiction, that there exist sets A, B and C such that
$(B \cap C) \subseteq A$ AND $(C-A) \cap (B-A) \neq \emptyset$.

Since $(C-A) \cap (B-A) \neq \emptyset$, there exists an element $x \in U$ such that $x \in (C-A) \cap (B-A)$.

$\therefore$ $x \in (C-A)$ AND $x \in (B-A)$ by definition of "Intersection".

Since $x \in C-A$, $x \in C$ and $x \notin A$, by def'n of "Set Difference".

Similarly, since $x \in B-A$, $x \in B$ and $x \notin A$.

$\therefore$ $x \in B \cap C$, since $x \in B$ and $x \in C$.

$\therefore$ $x \in A$, since $B \cap C \subseteq A$.

So, $x \in A$, but this contradicts the fact that $x \notin A$.

$\therefore$ For all sets A, B and C,
if $B \cap C \subseteq A$, then $(C-A) \cap (B-A) = \emptyset$,
by proof-by-contradiction.