Sec. 8.2

HW # 9 SOLUTIONS, M325K SPRING 2024

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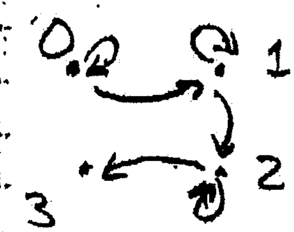
#2 $A = \{0, 1, 2, 3\}$

$R_2 = \{ (0,0), (0,1), (1,1), (1,2), (2,2), (2,3) \}$

a) The Directed GRAPH of R_2:



b) R_2 is not reflexive since $(3,3) \notin R_2$ and so $3 \not R_2 3$

c) R_2 is not symmetric because, for example, whereas $(1,2) \in R_2$, $(2,1) \notin R_2$, so $1 R_2 2$ but $2 \not R_2 1$.

d) R_2 is not transitive:

$1 R_2 2$ and $2 R_2 3$ since $(1,2) \in R_2$ and $(2,3) \in R_2$

But $1 \not R_2 3$ since $(1,3) \notin R_2$.

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#10. Def'n: Define relation C on $\mathbb{R}$ as follows:
For all $x, y \in \mathbb{R}$, $x C y \Leftrightarrow x^2 + y^2 = 1$.

C is not reflexive:
Proof: Let $x = 2$. [For example]
$2^2 + 2^2 = 8 \neq 1$. $\therefore 2 \not C 2$.
$\therefore$ C is not reflexive, by proof-by-counterexample.
QED.

C is symmetric
Proof: [NTS: For all $x, y \in \mathbb{R}$, if xCy, then yCx.]

Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$ be given.
Suppose xCy.
$\therefore$ $x^2 + y^2 = 1$, by def'n of relation C.
$\therefore$ $y^2 + x^2 = 1$, by the Commutative Property of Addition.
$\therefore$ yCx, by def'n of relation C.
$\therefore$ C is symmetric, by direct proof. QED

C is not transitive
Proof: Let $x = 0$, $y = 1$ and $z = 0$. [For example]
$0^2 + 1^2 = 1$. $\therefore 0C1$, by def'n of C.
$1^2 + 0^2 = 1$. $\therefore 1C0$, by def'n of C.
But, $0^2 + 0^2 = 0 \neq 1$. $\therefore 0 \not C 0$.
$\therefore$ C is not transitive, by proof-by-counterexample.
QED

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#13 Def'n: Define relation F on $\mathbb{Z}$ as follows:
For all $m, n \in \mathbb{Z}$, $mFn \Leftrightarrow 5 \mid (m-n)$.
The relation F is called the "congruence modulo 5" relation.

F is reflexive.

Proof: [NTS: For all $n \in \mathbb{Z}$, nFn.]
Let $n \in \mathbb{Z}$ be given.
$n - n = 0$ and $0 = 0 \cdot 5$.
$\therefore 5 \mid (n-n)$
$\therefore nFn$ by def'n of F.
$\therefore$ F is reflexive, by Direct Proof
QED.

F is symmetric.

Proof: [NTS: For all $m, n \in \mathbb{Z}$,
if mFn, then nFm.]
Let $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ be given.
Suppose mFn.
$\therefore 5 \mid (m-n)$ by def'n of F.
$\therefore m - n = 5t$ for some integer t.
$\therefore -(m-n) = -5t$
$\therefore n - m = 5 \cdot (-t)$, and $-t$ is an integer.
$\therefore 5 \mid (n-m)$
$\therefore nFm$ by def'n of F. $\therefore$ F is symmetric, by Direct Proof.
QED

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#13 (cont.)

F is transitive

Proof:

[WTS: For all $m, n, p \in \mathbb{Z}$,
if mFn and nFp, then mFp.]

Let m, n and p be any elements in $\mathbb{Z}$ (i.e., integers).
Suppose mFn and nFp.
Then, $5 \mid (m-n)$ and $5 \mid (n-p)$ by def'n of F.

$\therefore$ $m-n = 5k$ and $n-p = 5l$ for some integers k and l.

$\therefore (m-n) + (n-p) = 5k + 5l$.

$\therefore m + (n-n) - p = 5(k+l)$.

$\therefore m + 0 - p = 5(k+l)$

$\therefore m - p = 5(k+l)$ and $k+l$ is an integer.

$\therefore 5 \mid (m-p)$

$\therefore$ mFp by def'n of F.

$\therefore$ F is transitive, by Direct Proof.
QED.

#26 The Relation R on $A = \{$all strings of 0's, 1's and 2's of length 4$\}$ is defined as follows: For all $s, t \in A$,
$sRt \Leftrightarrow$ the sum of the characters in s = the sum of the characters in t.

R is reflexive, symmetric and transitive

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$\boxed{\#30}$ $A = \mathbb{R} \times \mathbb{R} - \{(0,0)\}$

For all $p_1, p_2 \in A$,

$p_1 R p_2 \Leftrightarrow p_1$ and p_2 are on the same half-line emanating from $(0,0)$.

R is reflexive: For any $p_1 \in A$, p_1 and p_1 are (is?) on the same half-line emanating from $(0,0)$.

R is symmetric: Suppose $p_1 R p_2$, then p_1 and p_2 are on the same half-line, so p_2 and p_1 are on the same half-line so $p_2 R p_1$.

R is transitive: Given any $p_1, p_2, p_3 \in A$, there is only one unique half-line emanating from $(0,0)$ which passes through p_2. If $p_1 R p_2$ and $p_2 R p_3$, then p_1, p_2 and p_3 are on this same half-line, so p_1 and p_3 are on the same half-line, so $p_1 R p_3$.

$\boxed{\#33}$ $A = \{$all lines in the plane $\mathbb{R} \times \mathbb{R}\}$

$l_1 R l_2 \Leftrightarrow l_1 \perp l_2$.

R is not reflexive: No line l is perpendicular to itself.

R is symmetric: If $l_1 \perp l_2$, then $l_2 \perp l_1$, so if $l_1 R l_2$, then $l_2 R l_1$.

R is not transitive: Let $l_1 = x$-axis, $l_2 = y$-axis, $l_3 = x$-axis. So $l_1 = l_3$; $l_1 R l_2$, $l_2 R l_3$, $l_1 \not R l_3$.