THE "LINEAR FIRST-ORDER DIFFERENTIAL EQUATION INITIAL VALUE PROBLEM PRESENTED IN CLASS"

PROBLEM: SOLVE THE FOLLOWING I.V.P.:

$$\frac{dx}{dt} = 4.5 - \frac{x}{300+2t}, \quad x(0) = 0, \; t \geq 0$$

SOLUTION: CONVERTING this D.E. INTO the STANDARD FORM OF A LINEAR FIRST-ORDER D.E.:

$$\frac{dx}{dt} + \left(\frac{1}{300+2t}\right)x = 4.5, \quad x(0) = 0.$$

$$\frac{dx}{dt} + \left(\frac{1}{300+2t}\right)x = \frac{9}{2}.$$

FORM: $\frac{dx}{dt} + P(t)\cdot x = Q(t)$ is a LINEAR FIRST-ORDER DIFFERENTIAL EQUATION with Integrating FACTOR $I(t) = e^{\int P(t)\,dt}$.

$$\int P(t)\,dt = \int \frac{1}{300+2t}\,dt = \frac{1}{2}\int \frac{1}{u}\,du \quad \begin{pmatrix} u = 300+2t \\ du = 2\,dt \end{pmatrix}$$

$$= \frac{1}{2}\ln|u| = \frac{1}{2}\ln(300+2t) = \ln\left((300+2t)^{\frac{1}{2}}\right)$$

(Note: since $t \geq 0$, $|300+2t| = (300+2t)$.)

$$\text{So, } I(t) = e^{\int P(t)dt} = e^{\int \frac{1}{300+2t}dt} = e^{\ln\left((300+2t)^{1/2}\right)} = (300+2t)^{\frac{1}{2}} = I(t).$$

To : $\frac{dx}{dt} + \left(\frac{1}{300+2t} \right) x = \frac{9}{2}$, we multiply

to both sides the integrating factor $(300+2t)^{1/2}$,

to get : $(300+2t)^{1/2} \frac{dx}{dt} + (300+2t)^{1/2} (300+2t)^{-1} x = \frac{9}{2}(300+2t)^{1/2}$

$$(300+2t)^{1/2} \frac{dx}{dt} + (300+2t)^{-1/2} x = \frac{9}{2}(300+2t)^{1/2}$$

$$\therefore \left[(300+2t)^{\frac{1}{2}} \cdot x \right]' = \frac{9}{2}(300+2t)^{1/2}$$

$$\therefore (300+2t)^{1/2} \cdot x = \int \left[(300+2t)^{\frac{1}{2}} \cdot x \right]' dx = \int \frac{9}{2}(300+2t)^{1/2} dt$$

$$= \left(\frac{9}{2}\right)\left(\frac{1}{2}\right) \int u^{1/2} du \quad \left(\text{where } u = (300+2t) \text{ and } du = 2dt \right)$$

$$= \frac{9}{4} \times \frac{2}{3} u^{3/2} + C$$

$$(300+2t)^{1/2} \cdot x = \frac{3}{2}(300+2t)^{3/2} + C$$

(multiplying both sides by $(300+2t)^{-1/2}$):

$$x = \frac{3}{2}(300+2t) + C(300+2t)^{-1/2} = \underline{450 + 3t + C(300+2t)^{-1/2}}$$

when $t=0$, $x=0$ and so, $0 = 450 + 3\times0 + C(300)^{-1/2}$

$$0 = 450 + \frac{C}{\sqrt{300}} \Rightarrow \frac{C}{10\sqrt{3}} = -450$$

$$\boxed{C = -4500\sqrt{3}}$$

So, $\boxed{x(t) = 450 + 3t - 4500\sqrt{3}\,(300+2t)^{-1/2}}$