A Separable First-Order Differential Equation Application

A full 200-gallon tank has brine (salt water) with 25 lbs of salt in the tank. Starting at time $\mathbf{t} = 0$ min, a brine solution with concentration 0.05 lbs / gal enters the tank at the constant rate of 10 gal/min, and the well-stirred solution is drained from the tank at the same rate, 10 gal/min.

(a) Determine the amount of salt in the tank as a function of time **t**.

(b) When does the amount of salt in the tank become 15 lbs ?

Solution: Let $y(t)$ = # of lbs of salt in the tank at time t minutes. The Initial Condition is $y(0) = 25$.

We Discover a Differential EQUATION in function y which has the function $y(t)$ as a solution.

$$\frac{dy}{dt} = \text{RATE}_{\text{IN}} - \text{RATE}_{\text{OUT}} \quad \text{in terms of} \ \text{lbs of SALT}/\text{MIN}.$$

BRINE CONCENTRATION × RATE OF FLOW

$$\text{RATE}_{\text{IN}} = (0.05 \text{ lbs/gal}) \times (10 \text{ gal/min})$$
$$= 0.5 \text{ lbs of salt/min}.$$

BRINE CONCENTRATION × RATE OF FLOW

$$\text{RATE}_{\text{OUT}} = \left(\frac{y(t) \text{ lbs}}{200 \text{ gal}}\right) \times (10 \text{ gal/min})$$
$$= \frac{y}{20} \text{ lbs of salt/min}$$

$$\frac{dy}{dt} = 0.5 - \frac{y}{20} = \frac{10}{20} - \frac{y}{20} = \frac{1}{20}(10 - y)$$

The Initial Condition is $y(0) = 25$.

So, the I.V.P. to solve is:

$$\frac{dy}{dt} = \frac{1}{20}(10-y) \text{ and } y(0) = 25.$$

$$\frac{1}{10-y}\,dy = \frac{1}{20}\,dt$$

$$\int \frac{1}{10-y}\,dy = \int \frac{1}{20}\,dt = \frac{1}{20}t + C_1$$

$$= -\int \frac{-1}{10-y}\,dy = -\ln|10-y| + C_2$$

$$-\ln|10-y| = \frac{1}{20}t + C_3 \qquad (C_3 = C_1 - C_2)$$

$$\ln|10-y| = C - \frac{1}{20}t \qquad C = -C_3$$

[TAKING EXPONENTIALS]

$$|10-y| = e^{C} \cdot e^{-\frac{1}{20}t}$$

[we solve for e^C] so at $t=0$, $y = 25$ lbs of salt, so

$$|10-25| = e^{C} \cdot e^{-0/20} = e^{C} \cdot e^{0} = e^{C}$$

$(|-15| = 15)$

$$15 = e^{C}$$

$(|10-y| = |y-10|)$

$$\text{So, } |y-10| = 15e^{-\frac{1}{20}t}$$

$$y - 10 = \pm 15e^{-t/20}$$

$$y = 10 \pm 15e^{-t/20}.$$

Since $y(0) = 25 > 0$, use $+$, not $-$.

Part (a) SOLUTION: $y = 10 + 15e^{-t/20}$ lbs of salt in the tank at time t_{min}

Part (b) sol'n: Set $y = 15$ and solve for t.

$$15 = 10 + 15e^{-t/20} \Rightarrow 5 = 15e^{-t/20} \Rightarrow e^{-t/20} = \frac{1}{3} \Rightarrow -\frac{t}{20} = \ln\left(\frac{1}{3}\right)$$

$$\Rightarrow -t/20 = -\ln 3 \Rightarrow \boxed{t = 20\ln 3 \approx 21.97 \text{ minutes}}$$