The Derivation of the TRIG INTEGRAL REDUCTION FORMULAS

Here, we provide derivations of the following FORMULAS:
(k and t represent positive integers where k is even and t is odd and $t > 1$.)

$$\text{I.} \quad \int \tan^k x \, dx = \frac{1}{k-1} \tan^{k-1} x - \int \tan^{k-2} x \, dx$$

and

$$\text{II.} \quad \int \sec^t x \, dx = \frac{1}{t-1} \tan x \sec^{t-2} x + \frac{t-2}{t-1} \int \sec^{t-2} x \, dx.$$

Derivation of I: Recall that k is an even positive integer. Thus, $k \geq 2$ and $k-2 \geq 0$.

First note that $\int \tan^{k-2} x \sec^2 x \, dx = \int u^{k-2} du$

where $u = \tan x$ and $du = \sec^2 x \, dx$.

* Thus, $\int \tan^{k-2} x \sec^2 x \, dx = \int u^{k-2} du = \frac{1}{k-1} u^{k-1} + C = \frac{1}{k-1} \tan^{k-1} x + C$.

Using the fact that $\tan^2 x = (\sec^2 x - 1)$, we have that

$$\int \tan^k x \, dx = \int (\tan^{k-2} x)(\tan^2 x) dx = \int (\tan^{k-2} x)(\sec^2 x - 1) dx$$

$$= \int (\tan^{k-2} x)(\sec^2 x) dx - \int \tan^{k-2} x \, dx$$

$$= \frac{1}{k-1} \tan^{k-1} x - \int \tan^{k-2} x \, dx$$

Therefore, $\int \tan^k x \, dx = \frac{1}{k-1} \tan^{k-1} x - \int \tan^{k-2} x \, dx$ when k is an even positive integer.

Derivation of II:

$$\int \sec^t x\,dx = \frac{1}{t-1}(\tan x)(\sec^{t-2} x) + \frac{t-2}{t-1}\int \sec^{t-2} x\,dx$$

Recall that t is an odd integer such that $t > 1$. Since $t \geq 3$, $t-2 \geq 1$.

FIRST NOTE THAT $\int \sec^t x\,dx = \int (\sec^{t-2} x)(\sec^2 x)\,dx$.
We apply integration-by-parts with

$$u = \sec^{t-2} x, \qquad dv = (\sec^2 x)\,dx,$$

$du = (t-2)(\sec^{t-3} x)(\sec x)(\tan x)\,dx$, and $v = \tan x$.

$$\text{Thus, } \int \sec^t x\,dx = (\sec^{t-2} x)(\tan x) - (t-2)\int (\tan^2 x)(\sec^{t-2} x)\,dx$$

$$= (\tan x)(\sec^{t-2} x) - (t-2)\int (\sec^2 x - 1)(\sec^{t-2} x)\,dx$$

$$= (\tan x)(\sec^{t-2} x) - (t-2)\left[\int \sec^t x\,dx - \int (\sec^{t-2} x)\right].$$

$$\text{Thus, } \int \sec^t x\,dx = (\tan x)(\sec^{t-2} x) - (t-2)\int \sec^t x\,dx + (t-2)\int \sec^{t-2} x\,dx.$$

We solve for $\int \sec^t x\,dx$ algebraically, first by adding $(t-2)\int \sec^t x\,dx$ to both sides of this equation. Note that $1 + (t-2) = (t-1)$.

$$\text{Thus, } (t-1)\int \sec^t x\,dx = (\tan x)(\sec^{t-2} x) + (t-2)\int \sec^{t-2} x\,dx.$$

Dividing both sides of this equation by $(t-1)$, we have

$$\int \sec^t x\,dx = \frac{1}{t-1}(\tan x)(\sec^{t-2} x) + \frac{t-2}{t-1}\int \sec^{t-2} x\,dx,$$

and Reduction Formula II has been derived.