How "TRIG SUBSTITUTION" is normal "u-substitution" or "θ-substitution" using an Inverse TRIG FUNCTION

On the following two pages, the problem in EXAMPLE 1 on page 2 of the handout "TRIG SUBSTITUTION PRINCIPLES" is worked using normal "θ-substitution" with $\theta = \sin^{-1}\left(\frac{x}{4}\right)$.

Here, on this page, the problem is worked with normal Trig Substitution: FIND $\int \frac{1}{(16-x^2)^{3/2}}\,dx$.

Sol'n: $\int \frac{1}{(16-x^2)^{3/2}}\,dx = \int \frac{1}{(\sqrt{16-x^2})^3}\,dx$

$a=4$

$\sqrt{16-x^2} = \sqrt{a^2-x^2}$

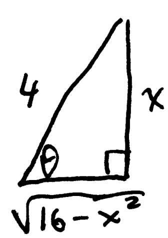
$\sqrt{16-x^2} = \sqrt{(4)^2-x^2}$

use $x = 4\sin\theta$

The Three special numbers are $a=4$, x, $\sqrt{16-x^2}$

$[\ a^2=16,\ x^2,\ 16-x^2\ ]$

$\Rightarrow$ The TRIANGLE!



$x = 4\sin\theta,\ dx = 4\cos\theta\,d\theta$

$\frac{\sqrt{16-x^2}}{4} = \cos\theta \Rightarrow \sqrt{16-x^2} = 4\cos\theta$, $\Rightarrow$

$\int \frac{1}{(\sqrt{16-x^2})^3}\,dx = \int \frac{1}{(4\cos\theta)^3}(4\cos\theta\,d\theta)$

$= \frac{1}{16}\int \frac{1}{\cos^2\theta}\,d\theta = \frac{1}{16}\int \sec^2\theta\,d\theta = \frac{1}{16}\tan\theta + C = \boxed{\frac{1}{16}\left(\frac{x}{\sqrt{16-x^2}}\right) + C}$

FACT 1: $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$.

Fact 2: When $u = f(x)$, then $\frac{d}{dx}(\sin^{-1}(u)) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$.

Fact 3: $\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{4}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}} \cdot \left(\frac{1}{4}\right) = \frac{1}{4}\left(\frac{1}{\sqrt{\frac{16-x^2}{16}}}\right)$

$$= \frac{1}{4}\left(\frac{1}{\frac{1}{4}\sqrt{16-x^2}}\right) = \frac{1}{\sqrt{16-x^2}}$$

FIND $\int \frac{1}{(\sqrt{16-x^2})^3}\,dx$ by using "θ-substitution", $\theta = \sin^{-1}\left(\frac{x}{a}\right)$;

Here, $\theta = \sin^{-1}\left(\frac{x}{4}\right)$.

So, $d\theta = \left(\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{4}\right)\right)\right) \cdot dx = \left(\frac{1}{\sqrt{16-x^2}}\right)dx$ by FACT 3.

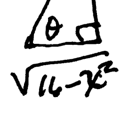
Setting up the integral for the substitution:

$$\int \frac{1}{(\sqrt{16-x^2})^3}\,dx = \int \underbrace{\left(\frac{1}{(\sqrt{16-x^2})^2}\right)}_{?} \underbrace{\left(\frac{1}{\sqrt{16-x^2}}\right)dx}_{d\theta}$$

To get an expression involving $\sqrt{16-x^2}$, we simplify $\cos\left(\sin^{-1}\left(\frac{x}{4}\right)\right) = \cos\theta$

$\cos\theta = \cos\left(\sin^{-1}\left(\frac{x}{4}\right)\right)$.

Since $\sin\theta = \frac{x}{4} = \frac{opp}{hyp}$, θ appears in



$\Rightarrow$ FACT 4: $\boxed{\cos\left(\sin^{-1}\left(\frac{x}{4}\right)\right) = \frac{\sqrt{16-x^2}}{4} = \cos\theta}$

To know what function of θ will substitute

in $\left(\frac{1}{(\sqrt{16-x^2})^2}\right)$, we solve for that expression in

terms of θ: From FACT 4, $\sqrt{16-x^2} = 4\cos\theta$

so $(\sqrt{16-x^2})^2 = 16\cos^2\theta$.

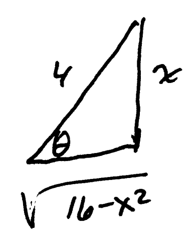
$$\boxed{\therefore \left(\frac{1}{(\sqrt{16-x^2})^2}\right) = \frac{1}{16\cos^2\theta} = \frac{1}{16}\left(\frac{1}{\cos^2\theta}\right).} \leftarrow \underline{\text{FACT 5}}$$

$\therefore$ By FACT 5, $\displaystyle\int \left(\frac{1}{(\sqrt{16-x^2})^2}\right)\left(\frac{1}{\sqrt{16-x^2}}\right)dx = \int \frac{1}{16}\cdot\frac{1}{\cos^2\theta}\,d\theta$

$$= \frac{1}{16}\int \sec^2\theta\,d\theta = \frac{1}{16}\tan\theta + C$$

$\theta = \sin^{-1}\left(\frac{x}{4}\right)$

$\sin\theta = \frac{x}{4} = \frac{\text{opp}}{\text{hyp}}$



$$= \frac{1}{16}\tan\left(\sin^{-1}\left(\frac{x}{4}\right)\right) + C$$

$$= \frac{1}{16}\frac{\text{opp}}{\text{adj}} + C = \frac{1}{16}\frac{x}{\sqrt{16-x^2}} + C$$

$$= \frac{x}{16\sqrt{16-x^2}} + C$$