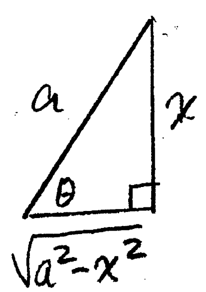
TRIG SUBSTUTION Principles

Let $a>0$ be a constant

Use this method when the Integrand involves one of the following expressions:

$\sqrt{a^2-x^2}$

DRAW:



$\frac{x}{a}=\sin\theta$

Substitutions:

$x=a\sin\theta$

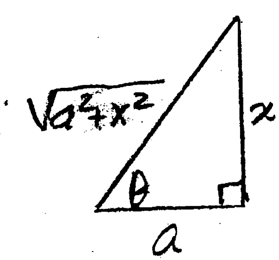
$dx=a\cos\theta\,d\theta$

$\sqrt{a^2-x^2}=a\cos\theta$

$\theta=\sin^{-1}\left(\frac{x}{a}\right)$

$\sqrt{a^2+x^2}$

DRAW:



$\frac{x}{a}=\tan\theta$

Substitutions:

$x=a\tan\theta$

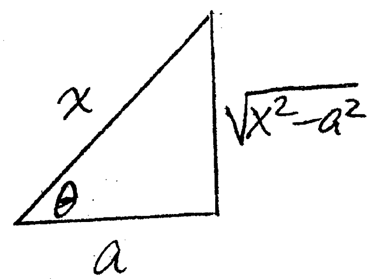
$dx=a\sec^2\theta\,d\theta$

$\sqrt{a^2+x^2}=a\sec\theta$

$\theta=\tan^{-1}\left(\frac{x}{a}\right)$

$\sqrt{x^2-a^2}$

DRAW:



$\frac{x}{a}=\sec\theta$

Substitutions:

$x=a\sec\theta$

$dx=a\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-a^2}=a\tan\theta$

$\theta=\sec^{-1}\left(\frac{x}{a}\right)$

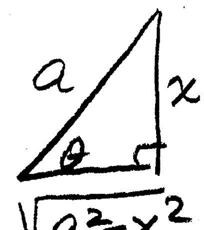
When possible, put x on the side opposite θ. Otherwise, put a on the adjacent side.

TRIG SUBSTITUTION EXAMPLES

EXAMPLE 1: FIND $\displaystyle\int \frac{1}{(16-x^2)^{3/2}}\,dx$

$$\int \frac{1}{(16-x^2)^{3/2}}\,dx = \int \frac{1}{\left(\sqrt{16-x^2}\right)^3}\,dx$$

$a = 4,\quad \sqrt{16-x^2} = \sqrt{a^2-x^2}$

Draw 

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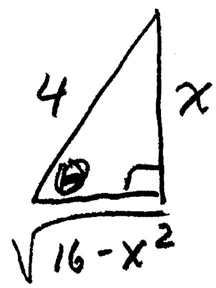

$\dfrac{x}{4} = \sin\theta \Rightarrow x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$\dfrac{\sqrt{16-x^2}}{4} = \dfrac{\text{Adj}}{\text{Hyp}} = \cos\theta \Rightarrow \sqrt{16-x^2} = 4\cos\theta$

$$\int \frac{1}{\left(\sqrt{16-x^2}\right)^3}\,dx = \int \frac{1}{(4\cos\theta)^3}(4\cos\theta)\,d\theta$$

$$= \int \frac{4\cos\theta}{64\cos^3\theta}\,d\theta = \frac{1}{16}\int \frac{1}{\cos^2\theta}\,d\theta =$$

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Ex 1 (continued)

$$= \frac{1}{16}\int \sec^2\theta \, d\theta = \frac{1}{16}\tan\theta + C$$

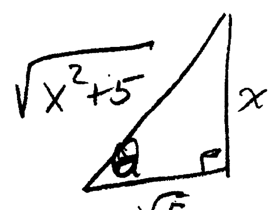
(Referring back to the triangle drawn earlier)

$$= \frac{1}{16}\,\frac{x}{\sqrt{16-x^2}} + C = \frac{x}{16\sqrt{16-x^2}} + C$$

EXAMPLE 2

$$\int\left(\frac{1}{\sqrt{x^2+5}}\right)^3 dx = D$$

$a = \sqrt{5}$



$\frac{x}{\sqrt{5}} = \tan\theta$

$x = \sqrt{5}\tan\theta$

$dx = \sqrt{5}\sec^2\theta \, d\theta$

$$\frac{\sqrt{x^2+5}}{\sqrt{5}} = \sec\theta \Rightarrow \sqrt{x^2+5} = \sqrt{5}\sec\theta$$

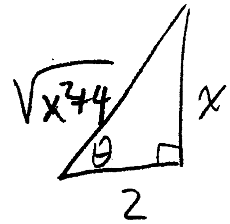
$$D = \int\left(\frac{1}{\sqrt{5}\sec\theta}\right)^3(\sqrt{5}\sec^2\theta \, d\theta) = \int \frac{\sqrt{5}\sec^2\theta}{5\sqrt{5}\sec^3\theta}\, d\theta$$

$$= \frac{1}{5}\int \frac{1}{\sec\theta}\, d\theta = \frac{1}{5}\int \cos\theta \, d\theta = \frac{1}{5}\sin\theta + C$$

$$= \boxed{\frac{1}{5}\cdot\frac{x}{\sqrt{x^2+5}} + C}$$

EXAMPLE 3

$$\int \frac{x}{\sqrt{x^2+4}}\,dx$$



$a=2$

$\frac{x}{2}=\tan\theta \Rightarrow x=2\tan\theta \Rightarrow dx=2\sec^2\theta\,d\theta$

$\frac{\sqrt{x^2+4}}{2}=\sec\theta \Rightarrow \sqrt{x^2+4}=2\sec\theta$

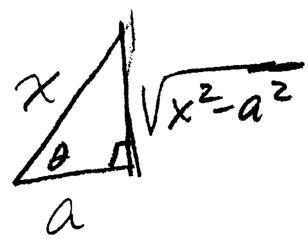
$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \int \frac{2\tan\theta}{2\sec\theta}\,2\sec^2\theta\,d\theta = 2\int \sec\theta\tan\theta\,d\theta$$

$$= 2\sec\theta + C = 2\cdot\frac{\sqrt{x^2+4}}{2}+C = \sqrt{x^2+4}+C$$

Note: EXAMPLE 3 could be solved more easily using straight u-substitution.

EXAMPLE 4

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = E$$



$\frac{x}{a}=\sec\theta \quad x=a\sec\theta \qquad dx=a\sec\theta\tan\theta\,d\theta$

$\frac{\sqrt{x^2-a^2}}{a}=\tan\theta \Rightarrow \sqrt{x^2-a^2}=a\tan\theta$

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EXAMPLE 4 (continued)

$$(E) = \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \int \frac{1}{a\tan\theta}\, a\sec\theta\tan\theta\, d\theta$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

$$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C$$

$$= \ln|x+\sqrt{x^2-a^2}| - \ln a + C$$

$$= \boxed{\ln|x+\sqrt{x^2-a^2}| + C}$$