FIRST NOTES ON POWER SERIES

__Definition:__ A __power series__ is a series (with variable x) of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

where $c_0, c_1, c_2, \ldots$ are constants (called the coefficients of the power series).

[Also, starting at $n=0$ is optional and $x^0 = 1$.]

__EXAMPLE:__ $$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots$$

Here, $c_n = \dfrac{1}{2^n}$.

The power series is the formula or recipe for a process.

It alone does not converge or diverge.

But...

__When x is given a value__, it converges or it diverges for that value of x, depending on what x is for that particular series.

For the power series $\sum_{n=0}^{\infty} \frac{1}{2^n} x^n$,

when $x = 3$,

$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot 3^n = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n.$$

This is a Geometric Series with $r = \frac{3}{2} > 1$, so the power series diverges when $x = 3$.

When $x = 1$,

$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot 1^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

This is a Geometric Series with $r = \frac{1}{2} < 1$, so the power series converges with $x = 1$.

The set of values of x for which the power series converges is called

The INTERVAL OF CONVERGENCE (IOC)

For which values of x does $\sum_{n=0}^{\infty} \frac{1}{2^n} x^n$ converge?

What is the IOC for this series?

$$\sum_{n=0}^{\infty} \frac{1}{2^n} x^n = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n \text{, which is a}$$

GEOMETRIC SERIES WITH COMMON RATIO

$$r = \frac{x}{2}.$$

A geometric series with common ratio r is convergent if and only if $-1 < r < 1$, and here $r = \frac{x}{2}$.

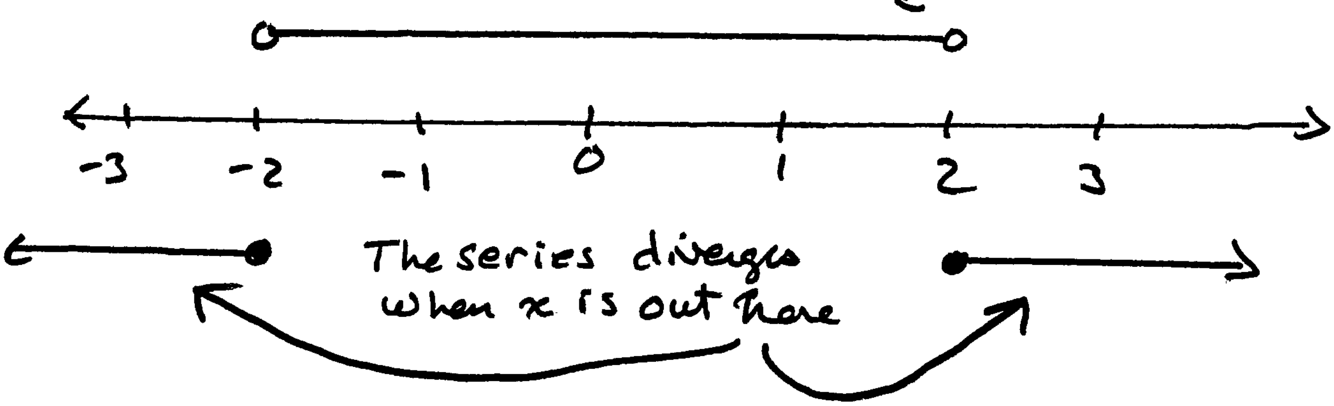
So, the power series above converges if and only if

$$-1 < \frac{x}{2} < 1. \qquad \text{[Now, multiply by 2.]}$$

$$-2 < x < 2.$$

The I.O.C. for $\sum_{n=0}^{\infty} \frac{1}{2^n} x^n$ is $\{x \mid -2 < x < 2\} = (-2, 2)$.

The series converges when x is in here.



The <u>Center</u> of this IOC is $x = 0$.
The <u>endpoints</u> of this IOC is $x = 2$ and $x = -2$.

THE DISTANCE BETWEEN THE CENTER AND EACH ENDPOINT (HERE IT is 2) IS CALLED R = the <u>RADIUS OF CONVERGENCE</u>.

<u>DEFINITION</u>: Let a be a fixed number, $a \in \mathbb{R}$.

Define the
"<u>power series about a</u>"
OR
"<u>power series centered at a</u>"
OR
"<u>power series in $(x-a)$</u>"

to be a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

<u>EXAMPLE</u>: $$\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n = (x-3) + \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3 + \dots$$

Here, $c_n = \frac{1}{n}$ and the series is centered at $a=3$.

We will find the interval of convergence for this power series.

[Use the ratio test or root test usually for this. Here, we use the ratio test.]

Here, $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n = \sum_{n=1}^{\infty} a_n$.

So, $|a_n| = \left|\frac{1}{n}(x-3)^n\right| = \left|\frac{1}{n}\right| \cdot \left|(x-3)^n\right|$

$= \frac{1}{n} \cdot |(x-3)|^n$

$|a_n| = \dfrac{|x-3|^n}{n}$.

So, $\dfrac{|a_{n+1}|}{|a_n|} = \left(\dfrac{|x-3|^{n+1}}{n+1}\right)\left(\dfrac{n}{|x-3|^n}\right)$

$= \left(\dfrac{n}{n+1}\right)\left(\dfrac{|x-3|^{n+1}}{|x-3|^n}\right)$

$= \left(\dfrac{\cancel{n} \cdot 1}{\cancel{n} \cdot (1 + 1/n)}\right) \dfrac{|x-3|}{1}$

$\lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = \dfrac{1}{1} \cdot \dfrac{|x-3|}{1} = |x-3| = L$

For the test to guarantee convergence, we need to have $L < 1$.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$ converges absolutely when $|x-3| < 1$;

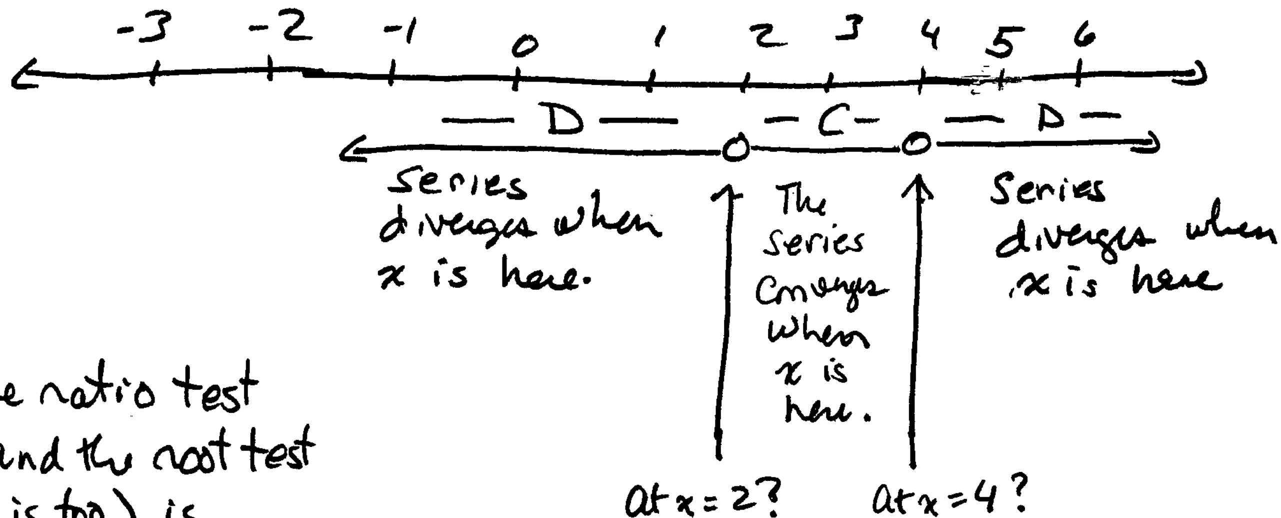
That is, for $-1 < (x-3) < 1$; That is, $\underline{\underline{2 < x < 4}}$.

We also know that the series diverges when

$$L = |x-3| > 1 \text{ ; that is,}$$

$$x-3 < -1 \quad \text{OR} \quad 1 < x-3 \text{ ; That is,}$$

$$x < 2 \quad \text{or} \quad 4 < x$$



The ratio test (and the root test is too) is INCONCLUSIVE when x is at an endpoint of the IOC, here at $x=2$ and at $x=4$.

We must use a different test to

CHECK FOR CONVERGENCE AT THE ENDPOINTS.

<u>When $x=2$</u>, $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n = \sum_{n=1}^{\infty} \frac{1}{n}(2-3)^n = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$.

This is an alternating series and can be seen to be <u>convergent</u> BY THE ALTERNATING SERIES TEST.

When $x=4$, $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n = \sum_{n=1}^{\infty} \frac{1}{n}(4-3)^n = \sum_{n=1}^{\infty} \frac{1}{n} \cdot 1^n = \sum_{n=1}^{\infty} \frac{1}{n}$.

This is the Harmonic Series which is <u>Divergent</u>.

So, for $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$, the Interval of Convergence is $IOC = \{x \mid 2 \le x < 4\} = [2,4)$.

The Center of the IOC is $x = 3$.
The RADIUS OF Convergence is $R = 1$

When the IOC is a **bounded** interval, there are four possibilities depending on the convergence or divergence at the endpoints:
$[c,d]$ OR $[c,d)$ OR $(c,d]$ or (c,d).

In general,
There are three overall possibilities for the interval of Convergence.
See the BLUE Box on p. 749 (THEOREM $\boxed{4}$) to see what these are.

See also the table (FIGURE 3) AT THE BOTTOM OF PAGE 749 for a list of power series illustrating all three possibilities.

Note: The power series $\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n (x-0)^n$

is really a "power series centered at 0"

Problem: Find the IOC for $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

Sol'n: Here, $c_n = \frac{1}{n!}$ and $|a_n| = \frac{1}{n!} |x|^n$

We Apply the RATIO TEST

$$\frac{|a_{n+1}|}{|a_n|} = \left(\frac{|x|^{n+1}}{(n+1)!}\right)\left(\frac{n!}{|x|^n}\right) = \left(\frac{n!}{(n+1)n!}\right)\left(\frac{|x|^n|x|}{|x|^n}\right)$$

$$= \frac{|x|}{n+1}.$$

So, $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$ because x is a fixed number divided by an increasingly BIG $n+1$.

Thus, $L = 0$ for every value that x might be given.

Thus, the series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges absolutely for all $x \in \mathbb{R}$.

The IOC $= (-\infty, \infty)$ and the

Radius of Convergence is $R = \infty$.

[Note: When the radius of convergence is $R = \infty$, the question of convergence or divergence at the endpoints of the IOC is not an issue.]

PROBLEM: Find the IOC for

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{3^n n^2} = \sum_{n=1}^{\infty} a_n$$

$$\frac{|a_{n+1}|}{|a_n|} = \left(\frac{|x-5|^{n+1}}{3^{n+1}(n+1)^2}\right)\left(\frac{3^n n^2}{|x-5|^n}\right)$$

$$= \frac{3^n \; |x-5|^{n+1} \; n^2}{3^{n+1} \; |x-5|^n \; (n+1)^2}$$

$$= \frac{|x-5|}{3}\left(\frac{n}{n+1}\right)^2 \xrightarrow[\substack{\text{as} \\ n\to\infty}]{} L = \frac{|x-5|}{3} \times 1^2$$

Thus, $L = \dfrac{|x-5|}{3} = \frac{1}{3}|x-5|$.

To start finding the IOC, set $L < 1$ and solve the inequality. $L = \frac{1}{3}|x-5| < 1$ [multiply by 3]

$|x-5| < 3$ [INTERPRET]

$-3 < x-5 < 3$ [Add 5]

$2 < x < 8$ CENTER $= 5$, $R = 3$

Test for convergence at the ENDPOINTS $x = 2$ and $x = 8$

At $x=2$: $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-3)^n}{3^n n^2} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$.

Note: $2-5=-3$

This series is seen to be absolutely convergent by the p-series test, but also convergent by the Alternating Series test.

Again, WORKING WITH $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(x-5)^n}{3^n n^2}$,

At $x=8$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

NOTE : $8-5=3$

which is seen to be convergent by the p-series test.

Therefore, the IOC is $[2, 8] = \{x \mid 2 \leq x \leq 8\}$

The Center is $a=5$ and the R.O.C. is $R=3$.

Here are calculations showing how we could have worked this last problem using the Root Test.

$$|a_n| = \frac{|(x-5)|^n}{3^n n^2}.$$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{|x-5|^n}{3^n n^2}} = \frac{\sqrt[n]{|x-5|^n}}{\sqrt[n]{3^n}\sqrt[n]{n^2}}$$

$$\sqrt[n]{|a_n|} = \frac{|x-5|}{3(\sqrt[n]{n})^2}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x-5|}{3\sqrt[n]{n}^{\,2}} = \frac{|x-5|}{3 \times 1^2} = \frac{|x-5|}{3}.$$

So, As before, $L = \frac{|x-5|}{3}$ and the solution continues as before.