Four Possible Methods of PSR Discovery

In all of these methods, the Radius R of convergence of the final PSR is the same as the Radius R of other PSR's considered during the use of the method.

Given function $f(x)$, these methods *might* be successful for finding a PSR for $f(x)$.

METHOD 1: For a positive integer k, if $f(x) = x^k g(x)$ where $g(x)$ has a PSR, $g(x) = \sum_{n=0}^{\infty} c_n x^n$.

then $f(x) = x^k g(x) = \sum_{n=0}^{\infty} c_n x^{n+k}$

METHOD 2: If $f(x)$ can be put into the form

$$f(x) = \ell \cdot \frac{1}{1 - t x^k}, \quad \text{Then } f(x) = \sum_{n=0}^{\infty} \ell t^n x^{kn}.$$

METHOD 3: If an antiderivative $G(x)$, where $G'(x) = f(x)$, has a PSR, $G(x) = \sum_{n=0}^{\infty} c_n x^n$,

then differentiate the PSR for $G(x)$ term-by-term

to get a PSR for $G'(x) = f(x)$, so that

$$f(x) = G'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n-1) c_{n+1} x^n.$$

METHOD 4

If $f(x)$ has a derivative $f'(x)$ which has a PSR, then perform the following steps:

① Find a PSR for $f'(x)$: $f'(x) = \sum_{n=0}^{\infty} c_n x^n$.

② Integrate the PSR for $f'(x)$ term-by-term to get a PSR for an antiderivative $G(x)$ of $f'(x)$, i.e., $G'(x) = f'(x)$, and

$$\int f'(x)\,dx = G(x) + C.$$

Thus, $G(x) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$.

③ Form the indefinite integral

$$\int f'(x) = G(x) + C = \left[\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}\right] + C$$

④ Evaluate $f(x)$ and $G(x)$ in its series form at the same value $x = x_1$ (often $x_1 = 0$) and set $f(x_1) = G(x_1) + C$; solve for the correct value of C, $C = C_1$, so that $f(x) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1} + C_1$.

⑤ Then, after adjusting n appropriately,

$$f(x) = C_1 + \sum_{n=1}^{\infty} \frac{c_{n-1}}{n} x^n.$$

In the following pages are examples of finding a PSR for a given function f(x) using each method.

Remember, for some given functions, none of these methods work to find PRS's for them.

Example using Method 2: Find a PSR for $f(x) = \frac{1}{5+4x^2}$.

Sol'n: $f(x) = \frac{1}{5+4x^2} = \frac{1}{5(1+\frac{4}{5}x^2)} = \frac{1}{5}\frac{1}{1+\frac{4}{5}x^2} = \frac{1}{5}\frac{1}{1-r}$

where $r = \left(-\frac{4}{5}x^2\right)$.

$$f(x) = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{4}{5}x^2\right)^n = \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\frac{4^n x^{2n}}{5^n} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{5}\frac{4^n x^{2n}}{5^n}$$

$$\boxed{\therefore,\ f(x) = \sum_{n=0}^{\infty}(-1)^n\frac{4^n x^{2n}}{5^{n+1}}}$$

For Radius R, $|r| < 1$ so $|-\frac{4}{5}x^2| < 1 \Rightarrow \frac{4}{5}x^2 < 1 \Rightarrow x^2 < \frac{5}{4} \Rightarrow |x| < \frac{\sqrt{5}}{2}$

$\boxed{R = \sqrt{5}/2}$

EXAMPLE USING Method 1:

Find a PSR for $f(x) = \frac{3x^7}{5+4x^2}$.

Sol'n: Consider $g(x) = \frac{3}{5+4x^2}$. Note: $f(x) = x^7 \cdot g(x)$.

We can get a geometric power series for g(x), as in the previous example:

$$g(x) = \sum_{n=0}^{\infty}(-1)^n\frac{3\cdot 4^n x^{2n}}{5^{n+1}}.$$

$R = \sqrt{5}/2$

So, $\boxed{f(x) = \sum_{n=0}^{\infty}(-1)^n\frac{3\cdot 4^n x^{2n+7}}{5^{n+1}},\quad R = \sqrt{5}/2}$

EXAMPLE USING METHOD 3:

Find a PSR for $f(x) = \dfrac{1}{(3+2x)^2} = (3+2x)^{-2}$.

Solution: Considering the form of $f(x)$ as being $(3+2x)^{-2}$,

an antiderivative of $f(x)$ will be related to

$$H(x) = (3+2x)^{-1} = \frac{1}{3+2x} = \frac{1}{3}\frac{1}{(1+\frac{2}{3}x)} = \frac{1}{3}\left(\frac{1}{1-r}\right)$$

where $r = -\frac{2}{3}x$, so an antiderivative of $f(x)$ has a PSR we can find!

To determine an antiderivative of $f(x) = \frac{1}{(3+2x)^2}$, we compute its indefinite integral using u-substitution.

$$\int f(x)dx = \int (3+2x)^{-2}dx = \frac{1}{2}\int u^{-2}du = \frac{-1}{2u}+C$$

$$\text{Set } u = 3+2x \text{ ; } du = 2dx \text{ ; } dx = \tfrac{1}{2}du$$

$$\int f(x)\,dx = -\frac{1}{2u}+C = \frac{-1}{2(3+2x)}+C$$

So, we take $G(x) = \dfrac{-1}{2(3+2x)}$ and we have $G'(x) = f(x)$.

Determining a PSR for $G(x)$:

$$G(x) = -\frac{1}{2(3+2x)} = -\frac{1}{2}\left(\frac{1}{3(1+\frac{2}{3}x)}\right) = \frac{-1}{2\times 3}\left(\frac{1}{1+\frac{2}{3}x}\right) = \frac{-1}{2\times 3}\left(\frac{1}{1-r}\right)$$

where $r = -\frac{2}{3}x$. Note $|r| < 1$ means $\left|-\frac{2}{3}x\right| < 1$;

$\therefore$ $\frac{2}{3}|x| < 1 \Rightarrow |x| < \frac{3}{2}$, so the Radius of Convergence is $R = \frac{3}{2}$.

EXAMPLE USING Method 3 (continued)

$$G(x) = -\frac{1}{2\times 3}\left(\frac{1}{1+\frac{2}{3}x}\right) = -\frac{1}{2\times 3}\left(\frac{1}{1-\left(-\frac{2}{3}\right)x}\right)$$

so, $$G(x) = \frac{-1}{2\times 3}\sum_{n=0}^{\infty}\left(-\frac{2}{3}x\right)^n = (-1)\left(\frac{1}{2\times 3}\right)\sum_{n=0}^{\infty}(-1)^n\frac{2^n x^n}{3^n}$$

So, $$G(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{2}\times\frac{1}{3}\times\frac{2^n x^n}{3^n}$$

so $$G(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^{n-1}}{3^{n+1}}x^n.$$ This is the Power Series Representation of $G(x)$.

and $\underline{R = 3/2}$

To determine the PSR for $f(x)$,
we differentiate the PSR for $G(x)$ term-by-term.

$$G'(x) = f(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\, n\,\frac{2^{n-1}}{3^{n+1}}x^{n-1}, \quad R = 3/2.$$

This is a form of a PSR for $f(x)$, but it is not the final form.

We drop the "$n=0$" term since that term equals 0. Thus,

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\, n\,\frac{2^{n-1}}{3^{n+1}}x^{n-1}.$$ Now, Replace $n-1$ with n, so Replace $\underset{n}{+1}$ with $\underset{n+1}{+1}$.

$$f(x) = \sum_{\substack{n+1=1\\ -1\quad -1\\ n=0}}^{\infty}(-1)^{(n+1+1)}(n+1)\frac{2^{n+1-1}}{3^{n+1+1}}x^{n+1-1} = \sum_{n=0}^{\infty}(-1)^{n+2}(n+1)\frac{2^n x^n}{3^{n+2}}$$

(Note: $(-1)^{n+2} = (-1)^n$ since $(-1)^2 = 1$.

A PSR for $f(x)$ in its final form is: $$\boxed{f(x) = \sum_{n=0}^{\infty}(-1)^n(n+1)\frac{2^n x^n}{3^{n+2}}, \quad R = \frac{3}{2}.}$$

EXAMPLE USING METHOD 4:

Find a PSR for $f(x) = \tan^{-1}(x) = \arctan(x)$.

Solution: For $f(x) = \tan^{-1}(x)$, its derivative $f'(x)$ is $f'(x) = \frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} = \frac{1}{1-r}$

where $r = -x^2$, so $f'(x)$ has a PSR that we will be able to determine. This tells us that method 4 is the method to use.

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

and $|r| < 1 \Rightarrow |-x^2| < 1 \Rightarrow x^2 < 1 \Rightarrow |x| < 1 \Rightarrow R = 1$

So $f'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ with Radius of Convergence $R = 1$.

We get a PSR for $\int f'(x)\,dx$ by integrating this last PSR term-by-term.

$$\int f'(x)\,dx = \left[\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}\right] + C = G(x) + C$$

where $G(x)$ is the antiderivative of $f'(x)$ given by the summation part.

Since $f(x) = \tan^{-1}(x)$ is another antiderivative of $f'(x) = \frac{1}{1+x^2}$,

$\tan^{-1}(x) = G(x) + C_1$ for the correct value of C, $C = C_1$.

We evaluate both $\tan^{-1}(x)$ and $G(x)$ at $x_1 = 0$ and we solve for $C = C_1$.

Example using method 4 (continued)

at $x = x_1 = 0$,

$\tan^{-1}(0) = 0$ and $\tan^{-1}(x) = \underbrace{\left[\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}\right]}_{G(x).} + C$

for the correct value of $C = C_1$

Evaluating $G(x_1) = G(0)$, we point out that

$$G(x) = \frac{1}{1}x^1 + (-1)\frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$$

so that, when $x = 0$, $G(0) = 0 + 0 + 0 + \cdots$ which sums to $S = 0$.

Thus, since $\tan^{-1}(x) = G(x) + C$ and

$0 = \tan^{-1}(0) = G(0) + C$, this means we can

solve for the correct $C = C_1$ value as follows

$\tan^{-1}(0) = 0$ and $G(0) = 0$, so $0 = 0 + C$

and so $C = 0$, that is, the correct $C_1 = C_1$ is $C_1 = 0$.

and so $\tan^{-1}(x) = G(x) + 0$

$$\boxed{\therefore \tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \text{ and } R = 1.}$$