SECTION 11.9 — SAMPLE SOLUTIONS

Problem 1: FIND A POWER SERIES REPRESENTATION (PSR)

for $f(x) = \dfrac{1}{(7-x)^2} = (7-x)^{-2}$

Sol'n: NOTE: $f(x) = F'(x)$ where
$F(x) = (7-x)^{-1} = \dfrac{1}{7-x}$

$\boxed{\text{Read Thm 2 on p. 754}}$ GET A SERIES FOR $F(x)$ AND THEN DIFFERENTATE IT TERM-BY-TERM!

$$F(x) = \frac{1}{7-x} = \frac{1}{7(1-\frac{x}{7})} = \frac{1}{7}\left(\frac{1}{(1-\frac{x}{7})}\right)$$

$$F(x) = \frac{1}{7}\sum_{n=0}^{\infty}\left(\frac{x}{7}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{7^{n+1}}, \quad R=7$$

GEOMETRIC SERIES
$r = \frac{x}{7}$
$\left|\frac{x}{7}\right| < 1$
$|x| < 7$

$\underline{R=7}$

$$\frac{1}{(7-x)} = F(x) = \frac{1}{7} + \frac{1}{7^2}x + \frac{1}{7^3}x^2 + \frac{1}{7^4}x^3 + \cdots$$

$$\frac{1}{(7-x)^2} = F'(x) = 0 + \frac{1}{7^2} + \frac{1}{7^3}(2x) + \frac{1}{7^4}(3x^2) + \cdots \quad \underline{R=7}$$

$$\frac{1}{(7-x)^2} = \sum_{n=0}^{\infty}\frac{1}{7^{n+1}}(nx^{n-1})$$

Currently, we have:

$$\frac{1}{(7-x)^2} = \sum_{n=0}^{\infty} \frac{1}{7^{n+1}} n x^{n-1}, \quad R = 7.$$

Drop the $n=0$ term.

① GOAL: DROP all initial terms which equal to 0.

$$\frac{1}{(7-x)^2} = \sum_{n=1}^{\infty} \frac{1}{7^{n+1}} n x^{n-1} = \sum_{n=1}^{\infty} a_n$$

② GOAL: WHEN POSSIBLE;

MAKE THE FORMULA for a_n HAVE x^n (instead of x^{n-1})

SOLUTION: Replace $n-1$ with n, [Next, Solve for n.]

$$\boxed{\text{Replace } \overset{+1}{\overline{n}} \text{ with } \overset{+1}{\overline{n+1}} \text{ everywhere.}}$$

$$\frac{1}{(7-x)^2} = \sum_{(n+1)=1}^{\infty} \frac{1}{7^{(n+1)+1}} (n+1) x^{(n+1)-1}$$

$$\frac{(n+1) = 1}{\begin{matrix} -1 & -1 \end{matrix}} \qquad n = 0$$

Don't forget to reset the starting value of n.

$$\frac{1}{(7-x)^2} = \sum_{n=0}^{\infty} \frac{1}{7^{n+2}} (n+1) x^n, \quad R = 7.$$

PROBLEM 1 SOLUTION REVIEW:

$$f(x) = \frac{1}{(7-x)^2} = \sum_{n=0}^{\infty} \ ? \quad , \ R = ?$$

$$f(x) = F'(x) \text{ for } F(x) = \frac{1}{7-x}$$

$$F(x) = \frac{1}{7-x} = \sum_{n=0}^{\infty} \left(\frac{1}{7^{n+1}}\right) x^n, \quad R = 7$$

$$F'(x) = f(x) = \frac{1}{(7-x)^2} = \sum_{n=0}^{\infty} \left(\frac{1}{7^{n+1}}\right)\left(n x^{n-1}\right), R = 7$$

[Drop the $n=0$ TERM!]

$$= \sum_{n=1}^{\infty} \left(\frac{1}{7^{n+1}}\right)\left(n x^{n-1}\right), \ R = 7$$

[Replace n with $n+1$]

$$\frac{1}{(7-x)^2} = \sum_{n=0}^{\infty} \frac{1}{7^{n+2}}(n+1) x^n, \ R = 7$$

$$\left(\begin{array}{r} n+1 = 1 \\ -1 \quad -1 \\ \hline n = 0 \end{array}\right)$$

PROBLEM 2: USE THE RESULT FROM PROBLEM 1 TO FIND A PSR for

$$f(x) = \frac{x^5}{(7-x)^2} = \frac{x^5}{1}\left(\frac{1}{(7-x)^2}\right), \quad R=7$$

Sol'n

$$f(x) = x^5\left(\frac{1}{(7-x)^2}\right) = x^5\left(\sum_{n=0}^{\infty} \frac{(n+1)}{7^{n+2}} x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)}{7^{n+2}} (x^n)(x^5)$$

$$= \sum_{n=0}^{\infty} \frac{n+1}{7^{n+2}} x^{n+5}, \quad R=7$$

[(n-5)+5 = n]

[Replace n with n-5]

$$= \sum_{n=5}^{\infty} \frac{n-4}{7^{n-3}} x^n, \quad R=7$$

PROBLEM 3: FIND a PSR for $\tan^{-1}(x) = f(x)$

Sol'n:

$$f(x) = \tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx = \int \left[\frac{1}{1-(-x^2)} \right] dx$$

Geometric Series: $r = (-x^2)$

$$\tan^{-1}(x) = \int \left[\sum_{n=0}^{\infty} (-x^2)^n \right] dx, \ R=1$$

$$= \int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n} \right] dx$$

$a_n = (-1)^n x^{2n}$

$\int a_n\,dx = (-1)^n \frac{1}{2n+1} x^{2n+1}$

$$\tan^{-1}(x) = \left(\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1}\right) x^{2n+1} \right) + C \quad , R=1$$

$C = \_\_?$, Set $x=0$, $\tan^{-1}(0) = 0$

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} (0)^{2n+1} + C = 0 + C = C$$

$\therefore C = 0$,
$$\boxed{\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1}\right) x^{2n+1}, \quad R=1}$$

Problem 3A: Use the PSR for $\tan^{-1}(x)$ to express π as the sum of a series.

$$\pi/6 = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \left(\frac{1}{\sqrt{3}}\right)^{2n+1} =$$

PROBLEM 3A (Continued)

$$\left[\frac{1}{\sqrt{3}} \doteq 0.577 < 1 \text{, so the series converges} \right]$$

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1}\right)\left(\frac{1}{(\sqrt{3})^{2n+1}}\right),$$

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1}\right)\left(\frac{1}{(\sqrt{3})(3^n)}\right),$$

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1}\right)\left(\frac{1}{3^n}\right),$$

$$\pi = \frac{6}{\sqrt{3}} \left(\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1}\right)\left(\frac{1}{3^n}\right)\right),$$

PROBLEM 4: USE THE RESULT OF PROBLEM 3 TO FIND A PSR of $\int \tan^{-1}(x^3)\,dx$

Sol'n: $$\int \tan^{-1}(x^3)\,dx = \int \left[\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1}\right)(x^3)^{2n+1} \right] dx$$

$$\int \tan^{-1}(x^3)\,dx = \int \left[\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1}\right) x^{6n+3} \right] dx$$

For $|x^3| < 1$
For $|x| < 1$
$R = 1$

$$\int \tan^{-1}(x^3)\,dx = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{(2n+1)(6n+4)}\right) x^{6n+4} + C$$

$R = 1$

Here:

$$a_n = (-1)^n \frac{1}{2n+1} x^{6n+3}; \quad \int a_n\,dx = (-1)^n \left(\frac{1}{2n+1}\right)\left(\frac{1}{6n+4}\right) x^{6n+4}$$

PROBLEM 5: FIND an approximation of $\int_0^{0.3} \frac{1}{1+x^6}\,dx$ correct to six decimal places.

Sol'n: "Six places" means $|\text{Error}| < 0.0000005$.

$$\int \frac{1}{1+x^6}\,dx = \int \frac{1}{1-(-x^6)}\,dx = \int \left[\sum_{n=0}^{\infty} (-x^6)^n\right] dx$$

For $|-x^6| < 1$
$|x^6| < 1$
$|x| < 1$
$R = 1$

$$= \int \left[\sum_{n=0}^{\infty} (-1)^n x^{6n}\right] dx \quad , R=1$$

$$\int \frac{1}{1+x^6}\,dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{1}{6n+1} x^{6n+1}\right) + C \quad , R=1$$

$0.3 < 1$, so The next series converges!

$$\int_0^{0.3} \frac{1}{1+x^6}\,dx = \left[\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{6n+1}\right) x^{6n+1}\right]_0^{0.3} \quad (-) \quad , R=1$$

$$= \underbrace{\left(\sum_{n=0}^{\infty} (-1)^n \frac{1}{6n+1} (0.3)^{6n+1}\right)}\ - 0$$

$$\int_0^{0.3} \frac{1}{1+x^6}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{6n+1} (0.3)^{6n+1} = (0.3) - \frac{1}{7}(0.3)^7 + \frac{1}{13}(0.3)^{13} - \cdots$$

PROBLEM 5 (Continued)

$$\int_0^{0.3} \frac{1}{1+x^6}\,dx = (0.3) - \frac{1}{7}(0.3)^7 + \frac{1}{13}(0.3)^{13} - \frac{1}{19}(0.3)^{19} + \cdots$$

$$= b_0 - b_1 + b_2 - b_3 + b_4 - \cdots$$

The Series is ALTERNATING AND CONVERGENT.

The ERROR in $S \approx S_n$ is not greater than b_{n+1}

$\therefore$ $|\text{ERROR}| < b_{n+1}$

$$b_1 = \frac{1}{7}(0.3)^2 = 0.000031243$$

$$b_2 = \frac{1}{13}(0.3)^{13} = 0.000000012$$

$$< 0.0000005$$

So $S_1 = b_0 - b_1$ has error small enough.

$$S_1 = (0.3) - \frac{1}{7}(0.3)^7 = 0.299968757$$

$$\therefore \int_0^{0.3} \frac{1}{1+x^6}\,dx = 0.299969, \text{ correct}$$

to six decimal places.