THREE TAYLOR'S INEQUALITY PROBLEMS PRESENTED IN CLASS

Problem #1:

(a) Find T_3, the $n = 3$ Taylor Polynomial at $a = 1$ of $f(x) = \ln x$.

(b) Determine $T_3(1.1)$.

(c) Show that $T_3(1.1) \approx \ln(1.1)$ is correct to 4 decimal places.

(d) Show that $T_3(x) \approx \ln(x)$ is correct to 4 decimal places for all x in $[0.9, 1.1]$.

Solutions:

(a) $f(x) = \ln(x)$; $a = 1$, $n = 3$; $f(1) = 0$

$f'(x) = \frac{1}{x}$ $\qquad f'(1) = 1$

$f''(x) = -1/x^2$ $\qquad f''(1) = -1$

$f^{(3)}(x) = 2/x^3$ $\qquad f^{(3)}(1) = 2$

$f^{(4)}(x) = -6/x^4$

$$T_3(x) = \frac{0}{0!} + \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$\boxed{T_3(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3}$$

(b) $T_3(1.1) = \_\_\_ ?$

$$T_3(1.1) = (1.1-1) - \tfrac{1}{2}(1.1-1)^2 + \tfrac{1}{3}(1.1-1)^3$$

$$T_3(1.1) = (0.1) - \tfrac{1}{2}(0.1)^2 + \tfrac{1}{3}(0.1)^3$$

$$\boxed{T_3(1.1) = 0.0953333\ldots}$$

Problem #1 (cont.)

(c) Show that $T_3(1.1) \approx \ln(1.1)$ is correct to 4 decimal places.

Sol'n: $R_3(1.1)$ = The error in the Approximation $T_3(1.1) \approx \ln(1.1)$.

Since $1.1 - 1.0 = 0.1$, we can apply Taylor's inequality with $a = 1$ and $d = 0.1$ because $x = 1.1$ is in the interval $|x-1| \leq 0.1$. ($|x-a| \leq d$)

By Taylor's Inequality,

$$R_3(1.1) \leq \frac{M}{4!}|1.1 - 1.0|^4 = \frac{M}{4!}(0.1)^4$$

where $|f^{(4)}(x)| \leq M$ for all x with $|x-1| \leq 0.1$,

that, is $|f^{(4)}(x)| \leq M$ on the interval $[0.9, 1.1]$.

We can use $M = MAX\,|f^{(4)}(x)|$ on $[0.9, 1.1]$.

let $h(x) = |f^{(4)}(x)|$, $\quad f^{(4)}(x) = -6/x^4$,

so $h(x) = \frac{6}{x^4} = 6x^{-4}$, and $M = MAX\ h(x)$ on $[0.9, 1.1]$

$h'(x) = -24/x^5$. Since $h'(x) < 0$ on $[0.9, 1.1]$, $h(x)$ is decreasing ($\searrow$) on $[0.9, 1.1]$.

So $M = MAX\,|f^{(4)}(x)| = h(0.9) = \frac{6}{(0.9)^4}$

$M = 9.144947\ldots$

Applying Taylor's Inequality,

$$|R_3(1.1)| \le \frac{M}{4!}|1.1-1|^4 = \frac{9.144947\ldots}{24}(0.1)^4$$

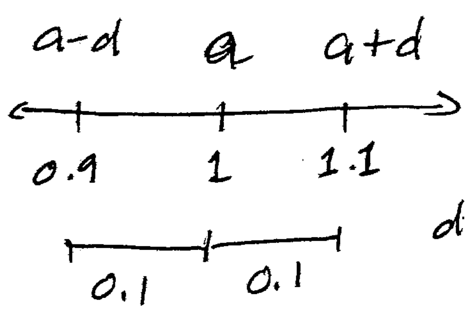
$$|R_3(1.1)| \le 0.000038104$$

so, $|R_3(1.1)| \le 0.00005$.

so, $T_3(1.1) \approx \ln(1.1)$ is correct to 4 decimal places.

(d) Show that $T_3(x) \approx \ln(x)$ is correct to 4 decimal places for all x in $[0.9, 1.1]$.

<u>Sol'n</u>:



$d = 0.1$

$[0.9, 1.1]$ is the interval $|x-1| \le 0.1$

If x is in $[0.9, 1.1]$,

Then $|x-1| \le |1.1-1.0| = 0.1 = d$

By Taylor's Inequality, with $M = 9.144947\ldots$ (as in (c))

$$|R_3(x)| \le \frac{M}{4!}|x-1|^4 \le \frac{M}{4!}(0.1)^4 = \frac{M}{4!}d^4$$

So $|R_3(x)| \le \frac{M}{4!}(0.1)^4 = 0.000038104$
≤ 0.00005.

DONE!

Problem #2:

(a) Approximate $f(x) = x^{-3}$ by a Taylor Polynomial of "degree 2"

centered at the number $a = 1$.

(b) Use Taylor's Inequality to estimate the Level of Error of the approx.

$T_2(x) \approx f(x) = x^{-3}$ when x lies in the interval $0.9 \leq x \leq 1.1$.

Solutions: $n = 2$, $a = 1$, $f(x) = x^{-3} = \frac{1}{x^3}$

(a)

$f(x) = \frac{1}{x^3} = x^{-3}$ $\qquad f(1) = 1$

$f'(x) = -3x^{-4} = \frac{-3}{x^4}$ $\qquad f'(1) = -3$

$f''(x) = \frac{12}{x^5} = 12x^{-5}$ $\qquad f''(1) = 12$

$f^{(3)}(x) = -60/x^6$

$$T_2(x) = \frac{1}{0!} + \frac{-3}{1!}(x-1) + \frac{12}{2!}(x-1)^2$$

$$\boxed{T_2(x) = 1 - 3(x-1) + 6(x-1)^2}$$

(b) We seek a level of error t so that

$|R_2(x)| = \left|T_2(x) - \frac{1}{x^3}\right| \leq t$ for all x, $0.9 \leq x \leq 1.1$,

That is, for all x with $|x-1| \leq 0.1 = d$

By Taylor's inequality,

$$|R_2(x)| \leq \frac{M}{3!}|x-1|^3 \leq \frac{M}{3!}(0.1)^3$$

Problem #2 (cont.):

We let $M = \text{MAX}\, |f^{(3)}(x)|$ for x in $[0.9, 1.1]$.

Let $h(x) = |f^{(3)}(x)| = \left| \frac{-60}{x^6} \right| = 60/x^6$.

$M = \text{MAX}\, h(x)$.

$$h(x) = 60 \cdot x^{-6}$$

$$h'(x) = \frac{-360}{x^7} \text{ and } h'(x) < 0 \text{ on } [0.9, 1.1].$$

So, $h(x)$ is decreasing ($\searrow$) on $[0.9, 1.1]$.

So, $M = \text{MAX}\, h(x) = h(0.9)$

$$M = |f^{(3)}(0.9)| = 60/(0.9)^6 = 112.9005\ldots$$

With $M = 112.9005\ldots$,

$$|R_2(x)| \le \frac{M}{3!}(0.1)^3 = 0.018816764\ldots$$

A level of Error for $T_2(x) \approx \frac{1}{x^3}$ on $|x-1| \le 0.1$,

That is, on $[0.9, 1.1]$ is

$$\text{Level of Error} = 0.018817 = \frac{112.90}{3!}(0.1)^3$$

Alternative method to find M:

Assuming $|f^{(n)}(x)|$ has no critical points inside the interval $[a-d, a+d]$, which is the same as $|x-a| \le d$,

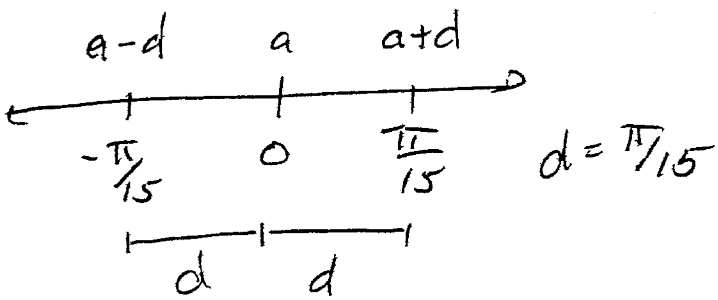
you can let M be the larger of the two evaluations: $|f^{(n)}(a-d)|$ and $|f^{(n)}(a+d)|$, i.e. $|f^{(n)}(x)|$ at the two endpts.

Problem #3:

Using a Taylor Polynomial for $f(x) = \sin x$ centered at $a = 0$,

approximate $\sin\left(\frac{\pi}{15}\right)$ correct to 5 decimal places.

Solution:



We use Taylor's Inequality applied to the interval centered at $a=0$ with $\frac{\pi}{15}$ as the RIGHT-HAND ENDPOINT: $-\frac{\pi}{15} \le x \le \frac{\pi}{15} \iff |x| \le \frac{\pi}{15} \iff |x-0| \le d = \frac{\pi}{15}$.

SINCE $f(x) = \sin x$ and all derivatives of $\sin x$ are either one of $\pm \cos x$ or one of $\pm \sin x$, we have $|f^{(k)}(x)| \le 1$ for all k and for all x. THUS, we can let $M = 1$ for use in Taylor's Inequality.

With Trial and Error, I discovered that, it is first with $n=4$ and $n+1=5$, that $\frac{M}{(n+1)!} d^{(n+1)} = \frac{1}{(n+1)!}\left(\frac{\pi}{15}\right)^{n+1} = \frac{1}{5!}\left(\frac{\pi}{15}\right)^5 \le 0.000005$.

Problem #3 (Cont.)

In fact, $\frac{1}{5!}\left(\frac{\pi}{15}\right)^5 = \frac{1}{120}(0.00402989)$

$= 0.000003358.$

So, $\left|R_4\left(\frac{\pi}{15}\right)\right| \leq 0.000003358.$

From the MacLaurin Series on p. 762,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

So, the $n=4$ Taylor Polynomial for $\sin x$ at $a=0$ is

$$T_4(x) = 0\cdot x^0 + 1\cdot x + 0\cdot x^2 - \frac{1}{6}x^3 + 0\cdot x^4,$$

That is, $T_4(x) = x - \frac{1}{6}x^3$

So, $T_4(\pi/15) = \frac{\pi}{15} - \frac{1}{6}\left(\frac{\pi}{15}\right)^3 = 0.2079833\ldots$

$T_4\left(\frac{\pi}{15}\right) = 0.20791$ Rounded to 5 decimal places.

So, $T_4(\pi/15) \approx \sin(\pi/15)$ correct to 5 places.

Since $\frac{\pi}{15}$ RADIANS $= 12^\circ$, this problem could have originally be stated "Approximate $\sin(12^\circ)$".

If it had, we could have converted the 12° to $\frac{\pi}{15}$ Radians and worked that problem exactly the same way.