Differentiation and Integration with Inverse Trig Functions

The Inverse Sine Function : $y = \sin^{-1}(x)$

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1 - x^2}} \quad \textbf{and} \quad \frac{d}{dx}\left(\sin^{-1}(u)\right) = \left(\frac{1}{\sqrt{1 - u^2}}\right)\left(\frac{du}{dx}\right)$$

For any positive constant $a > 0$:

$$\int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1}(x) + C \quad \textbf{and} \quad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

The Inverse Tangent Function : $y = \tan^{-1}(x)$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1 + x^2} \quad \textbf{and} \quad \frac{d}{dx}\left(\tan^{-1}(u)\right) = \left(\frac{1}{1 + u^2}\right)\left(\frac{du}{dx}\right)$$

For any non-zero constant a :

$$\int \frac{1}{1 + x^2}\,dx = \tan^{-1}(x) + C \quad \textbf{and} \quad \int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

The Inverse Secant Function : $y = \sec^{-1}(x)$

$$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{x\sqrt{x^2 - 1}} \quad \textbf{and} \quad \frac{d}{dx}\left(\sec^{-1}(u)\right) = \left(\frac{1}{u\sqrt{u^2 - 1}}\right)\left(\frac{du}{dx}\right)$$

For any positive constant $a > 0$:

$$\int \frac{1}{x\sqrt{x^2 - 1}}\,dx = \sec^{-1}(x) + C \quad \textbf{and} \quad \int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

Using the formulas relating the other three inverse trig functions with these three, the derivatives of the other three functions are easily calculated:

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1 - x^2}} = -\frac{d}{dx}\left(\sin^{-1}(x)\right) \;;$$

$$\frac{d}{dx}\left(\cot^{-1}(x)\right) = \frac{-1}{1 + x^2} = -\frac{d}{dx}\left(\tan^{-1}(x)\right) \;;$$

$$\frac{d}{dx}\left(\csc^{-1}(x)\right) = \frac{-1}{x\sqrt{x^2 - 1}} = -\frac{d}{dx}\left(\sec\ (x)\right)$$

Derivation of derivatives for Inverse Trig functions

Derivation #1 : Proof that $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1 - x^2}}$ **:**

Write $u = \sin^{-1}(x)$ **.** **We seek** $\frac{du}{dx} = \frac{d}{dx}\left(\sin^{-1}(x)\right)$ **.**

Then, $\sin(u) = \sin(\sin^{-1}(x)) = x$; **so,** $\frac{d}{dx}(\sin(u)) = \frac{d}{dx}(x) = 1$;

so, $\cos(u)\frac{du}{dx} = 1$ **, and thus** $\frac{du}{dx} = \frac{1}{\cos(u)} = \frac{1}{\cos(\sin^{-1}(x))}$.

Claim: $\cos(\sin^{-1}(x)) = \sqrt{1 - x^2}$ *for all* $x \in [-1,1]$.

Proof of Claim:

The method of drawing $u = \sin^{-1}(x)$ **presented above shows that**

$$\cos(\sin^{-1}(x)) = \sqrt{1 - x^2} \quad \textit{for all } x \in [-1,1] \textit{ such that } x > 0 .$$

If $x < 0$, *then* $x = -|x|$, *and so* $\sin^{-1}(x) = \sin^{-1}(-|x|) = -\sin^{-1}(|x|)$ **. This last equality is a direct consequence of the fact that** $\sin(-z) = -\sin(z)$ *for all* z **.**

Thus, $\cos(\sin^{-1}(x)) = \cos\left(\sin^{-1}(-|x|)\right) = \cos\left(-\sin^{-1}(|x|)\right) = \cos\left(\sin^{-1}(|x|)\right)$,

and so, $\cos(\sin^{-1}(x)) = \cos\left(\sin^{-1}(|x|)\right) = \sqrt{1 - |x|^2} = \sqrt{1 - x^2}$.

When x = 0, the claim is true because $\sin^{-1}(0) = 0$ **and** $\cos(0) = 1$ **, and the claim is proven.**

Finally, $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{du}{dx} = \frac{1}{\cos(u)} = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1 - x^2}}$ **, and the proof is complete.**

Derivation #2 : Proof that $\frac{d}{dx}\left(\tan^{-1}(x) \right) = \frac{1}{1 + x^2}$ **:**

Write $u = \tan^{-1}(x)$ **.** **We seek** $\frac{du}{dx} = \frac{d}{dx}\left(\tan^{-1}(x) \right)$ **.**

Then, $\tan(u) = \tan(\tan^{-1}(x)) = x$; **so,** $\frac{d}{dx}\left(\tan(u) \right) = \frac{d}{dx}(x) = 1$;

so, $\sec^2(u)\frac{du}{dx} = 1$, **and thus** $\frac{du}{dx} = \frac{1}{\sec^2(u)} = \frac{1}{\sec^2(\tan^{-1}(x))}$.

Claim: $\sec(\tan^{-1}(x)) = \sqrt{1 + x^2}$ *for all* $x \in (-\infty , \infty)$.

Proof of Claim:

The method of drawing $u = \tan^{-1}(x)$ **presented above shows that**

$$\sec(\tan^{-1}(x)) = \sqrt{1 + x^2} \quad \textit{for all } x \textit{ such that } x > 0 .$$

If $x < 0$, *then* $x = -| x |$, *and so* $\tan^{-1}(x) = \tan^{-1}(-|x|) = -\tan^{-1}(|x|)$ **. This last equality is a direct consequence of the fact that** $\tan(-z) = -\tan(z)$ *for all* z **.**

Thus, $\sec(\tan^{-1}(x)) = \sec\left(\tan^{-1}(-|x|) \right) = \sec\left(-\tan^{-1}(|x|) \right) = \sec\left(\tan^{-1}(|x|) \right)$ **,**

and so, $\sec(\tan^{-1}(x)) = \sec\left(\tan^{-1}(|x|) \right) = \sqrt{1 + |x|^2} = \sqrt{1 + x^2}$.

When x = 0, the claim is true because $\tan^{-1}(0) = 0$ **and** $\sec(0) = 1$ **, and the claim is proven.**

Finally,

$$\frac{d}{dx}\left(\tan^{-1}(x) \right) = \frac{du}{dx} = \frac{1}{\sec^2(u)} = \frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{\left(\sqrt{1 + x^2}\right)^2} = \frac{1}{1 + x^2} ,$$

and the proof is complete.

Derivation #3 :

Proof that $\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{x\sqrt{x^2 - 1}}$ *for all* $x \in (-\infty, -1) \cup (1, \infty)$ **:**

Write $u = \sec^{-1}(x)$ **.** **We seek** $\frac{du}{dx} = \frac{d}{dx}\left(\sec^{-1}(x)\right)$ **.**

Then, $\sec(u) = \sec(\sec^{-1}(x)) = x$; **so,** $\frac{d}{dx}(\sec(u)) = \frac{d}{dx}(x) = 1$;

so, $\sec(u)\tan(u)\frac{du}{dx} = 1$ **, and thus**

$$\frac{du}{dx} = \frac{1}{\sec(u)\tan(u)} = \frac{1}{\sec(\sec^{-1}(x))\ \tan(\sec^{-1}(x))} = \frac{1}{x\ \tan(\sec^{-1}(x))}$$

Claim:

$$\tan(\sec^{-1}(x)) = \left\{ \begin{array}{l} \sqrt{x^2 - 1} \ \textit{for all } x \in [1, \infty) \\ \\ \sqrt{x^2 - 1} \ \textit{for all } x \in (-\infty, -1] \end{array} \right\}$$

Proof of Claim:

If x > 1 , then method of drawing $u = \sec^{-1}(x)$ **presented above shows that**

$$\tan(\sec^{-1}(x)) = \sqrt{x^2 - 1} \ .$$

If $x < -1$, *then* $x = -|x|$, *and so* $\sec^{-1}(x) = \pi + \sec^{-1}(|x|)$ **.**

This last equality is evident from the definition of $u = \sec^{-1}(x)$ **presented above.**

Thus, $\tan(\sec^{-1}(x)) = \tan\left(\pi + \sec^{-1}(|x|)\right) = \tan\left(\sin^{-1}(|x|)\right)$ **and so,**

$\tan(\sec^{-1}(x)) = \tan\left(\sec^{-1}(|x|)\right) = \sqrt{|x|^2 - 1} = \sqrt{x^2 - 1}$.

When x = 1, the claim is true because $\sec^{-1}(1) = 0$ **and** $\tan(0) = 0$ **;**
when x = -1, the claim is true because $\sec^{-1}(-1) = \pi$ **and** $\tan(\pi) = 0$ **, and the claim is proven.**

Thus, when x > +1 or x < -1 ,

$$\frac{du}{dx} = \frac{1}{x\tan(\sec^{-1}(x))} = \frac{1}{x\sqrt{x^2 - 1}} \ ,$$

and the proof is complete.