M 408 K Fall 2005 Inverse Trig Functions

Important Decimal Approximations and Useful Trig Identities

Decimal Approximations:

$0 = 0.000$

$\frac{1}{2} = 0.500$

$\frac{\sqrt{3}}{3} = 0.577 = \frac{1}{\sqrt{3}} = \left(\frac{1}{2}\right) \Big/ \left(\frac{\sqrt{3}}{2}\right)$

$\frac{\sqrt{2}}{2} = 0.707$; $\frac{\sqrt{3}}{2} = 0.866$

$1 = 1.000$

$\frac{2\sqrt{3}}{3} = 1.155 = 1 \Big/ \frac{\sqrt{3}}{2}$

$\sqrt{2} = 1.414 = 1 \Big/ \frac{\sqrt{2}}{2}$

$\sqrt{3} = 1.732 = \frac{\sqrt{3}}{1} = \left(\frac{\sqrt{3}}{2}\right) \Big/ \left(\frac{1}{2}\right)$

$0 = 0.000 = 0 \frac{\pi}{12}$

$\frac{\pi}{6} = 0.524 = 2 \frac{\pi}{12}$

$\frac{\pi}{4} = 0.785 = 3 \frac{\pi}{12}$

$\frac{\pi}{3} = 1.047 = 4 \frac{\pi}{12}$

$\frac{\pi}{2} = 1.571 = 6 \frac{\pi}{12}$

$\frac{2\pi}{3} = 2.094 = 8 \frac{\pi}{12}$

$\frac{3\pi}{4} = 2.356 = 9 \frac{\pi}{12}$

$\frac{5\pi}{6} = 2.618 = 10 \frac{\pi}{12}$

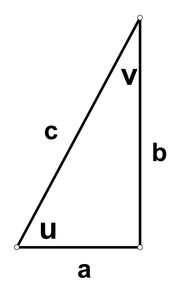
Trigonometric Identities:

When u + v = pi / 2 = 90 degrees,

sin (u) = b / c = cos (v) = cos (pi/2 - u) ,

tan (u) = b / a = cot (v) = cot (pi/2 - u) ,

sec (u) = c / a = csc (v) = csc (pi/2 - u) .



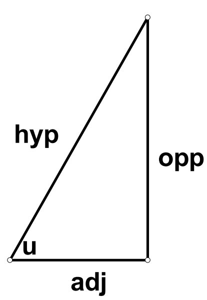
SOH-CAH-TOA

sin (u) = opp / hyp

tan (u) = opp / adj

sec (u) = hyp / adj

cos (u) = adj / hyp



sin (- x) = - sin (x)

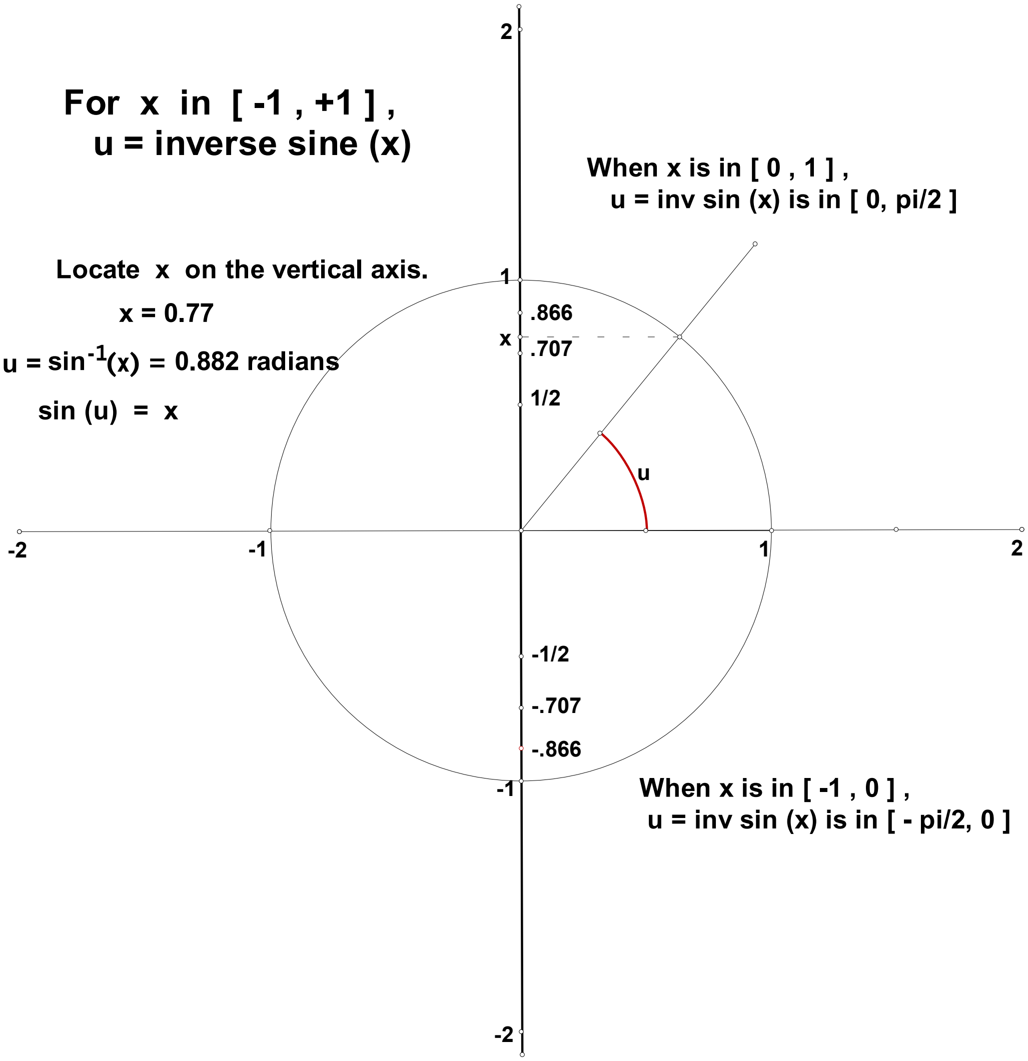
tan (- x) = - tan (x)

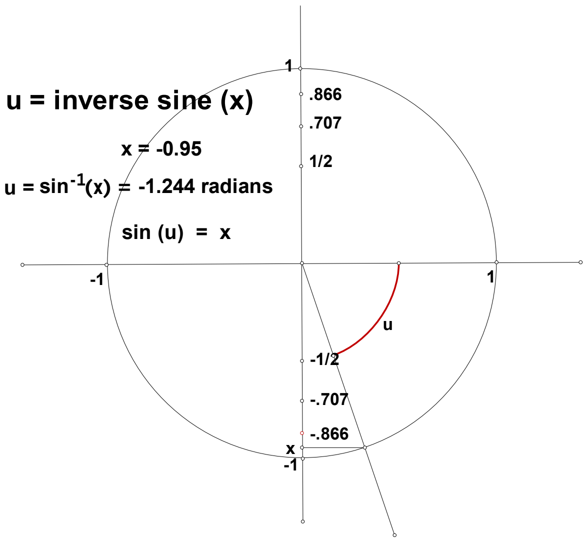
cos (- x) = cos (x)

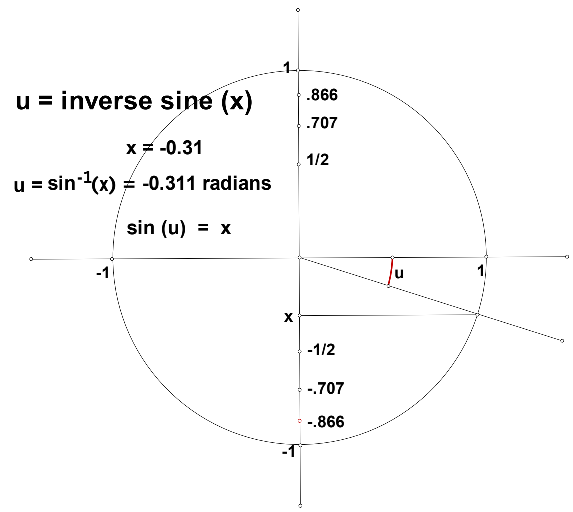
sec (- x) = sec (x)

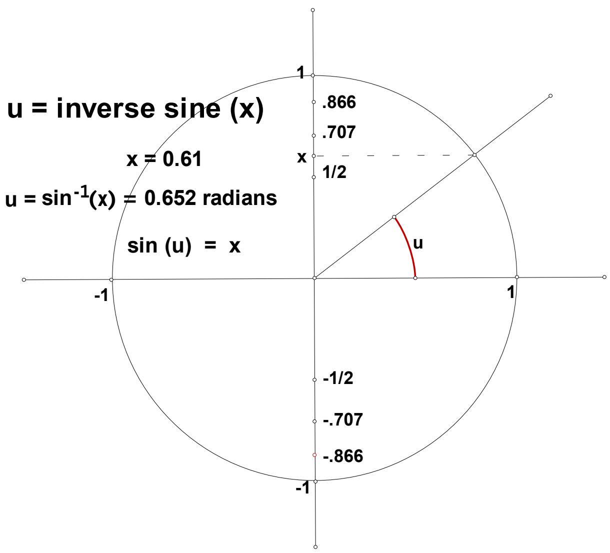
Inverse Sine Function: $u = \sin^{-1}(x)$, $x \in [-1, 1]$, $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

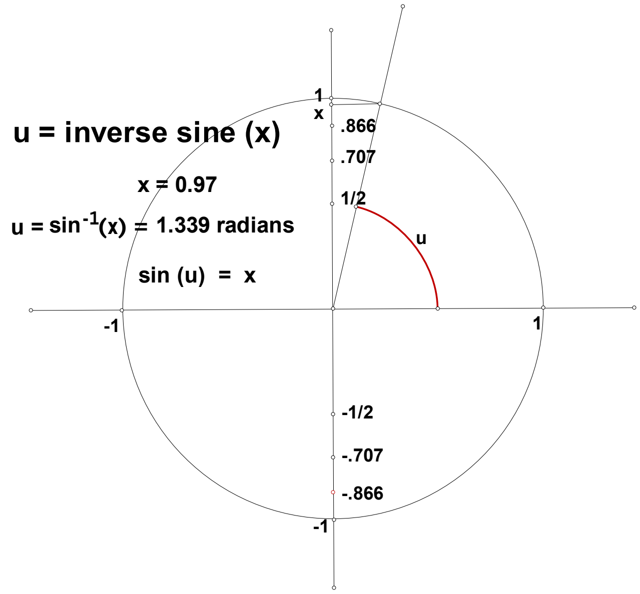
$u = \sin^{-1}(x)$ **when** $\sin(u) = x$ **and** $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.











Drawing the angle u = inv sin (x) when x > 0 :

u = inv sin (x)

sin (u) = x = x/1 = opp / hyp ,

Make opp = x and hyp = 1 .

cos (u) = adj / hyp

cos (inv sin (x)) = Sqrt(1 - x^2)

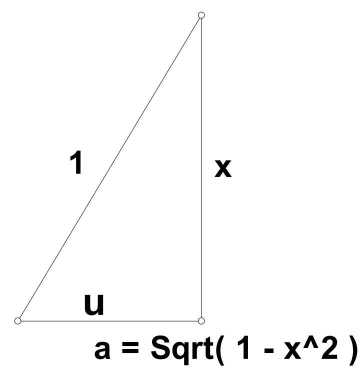
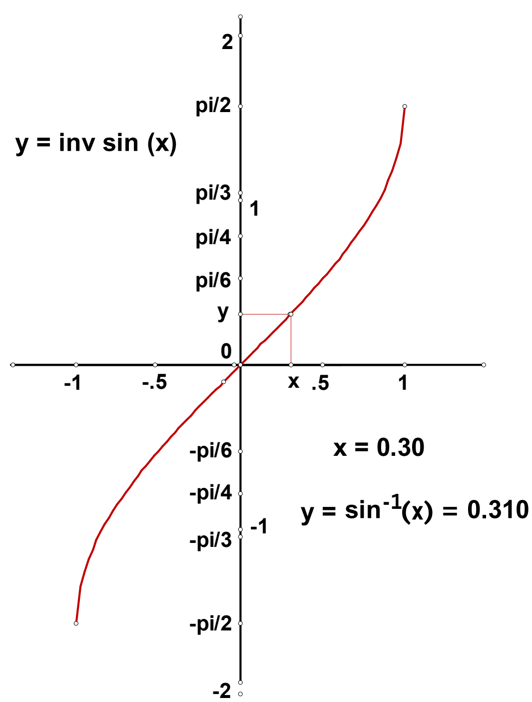


Table of values: $y = \sin^{-1}(x)$

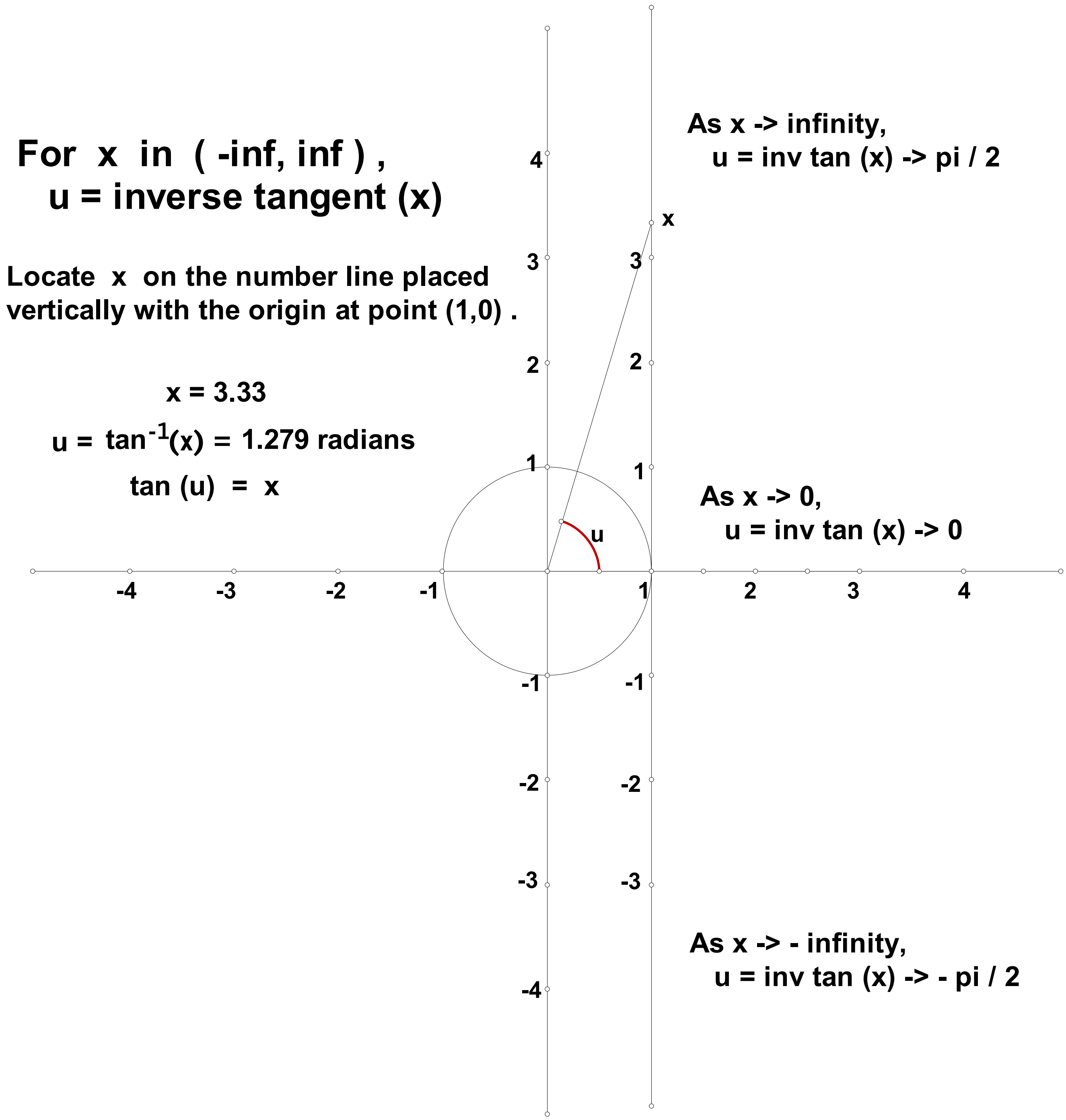
x	y = sin ^{1}(x)
- 1	- π / 2 = - 1.571
-√3 / 2 = -0.866	- π / 3 = - 1.047
-√2 / 2 = -0.707	- π / 4 = - 0.785
-1/2	- π / 6 = - 0.524
0	0
1/2	π / 6 = 0.524
√2 / 2 = 0.707	π / 4 = 0.785
√3 / 2 = 0.866	π / 3 = 1.047
1	π / 2 = 1.571

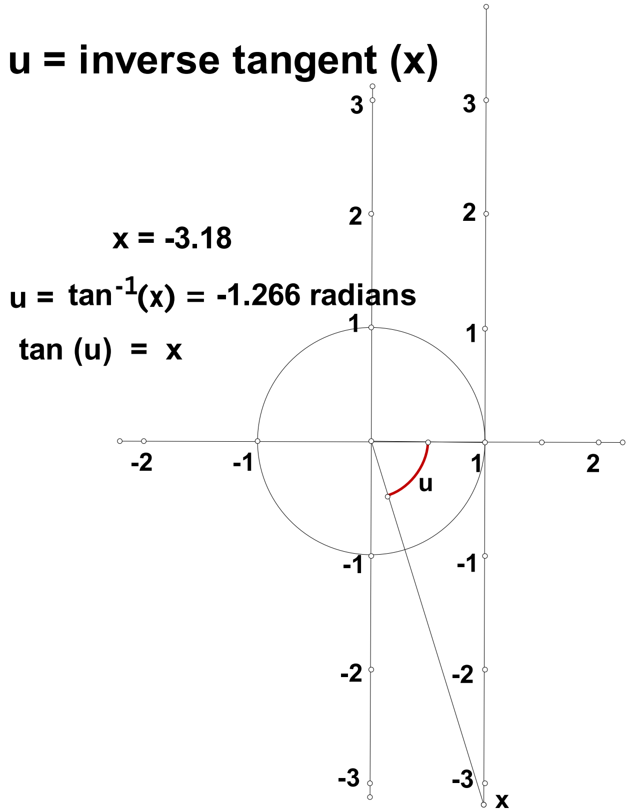


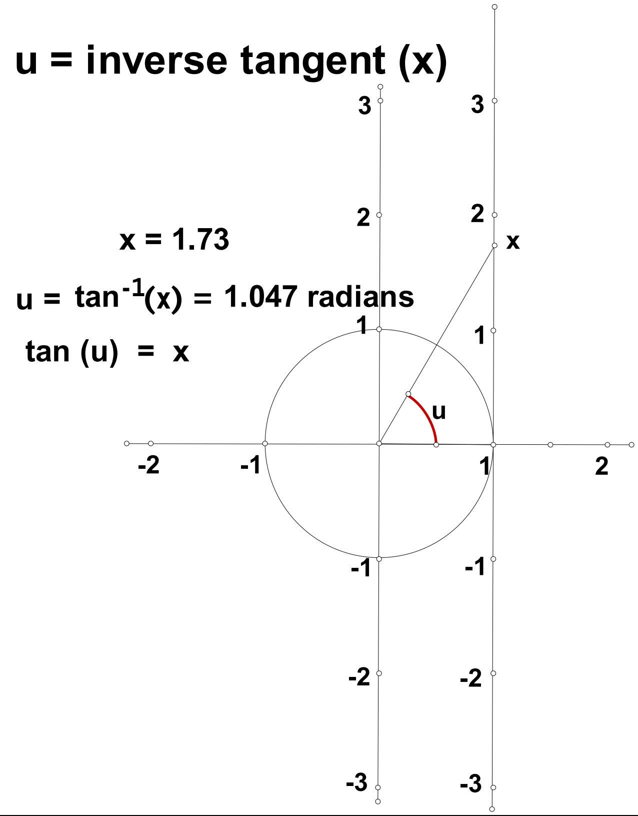
Inverse Tangent Function:

$$u = \tan^{-1}(x), \quad x \in (-\text{infinity}, \text{infinity}), \quad u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

$$u = \tan^{-1}(x) \quad \text{when} \quad \tan(u) = x \quad \text{and} \quad u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$







Drawing the angle u = inv tan (x) when x > 0 :

u = inv tan (x)

tan (u) = x = x/1 = opp / adj ,

Make opp = x and adj = 1 .

sec (u) = hyp / adj

sec (inv tan (x)) = Sqrt(1 + x^2)

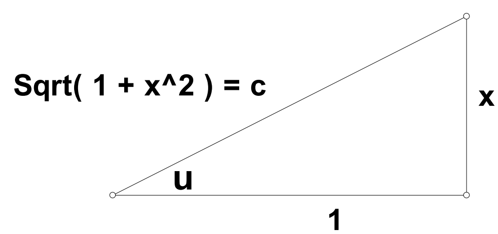
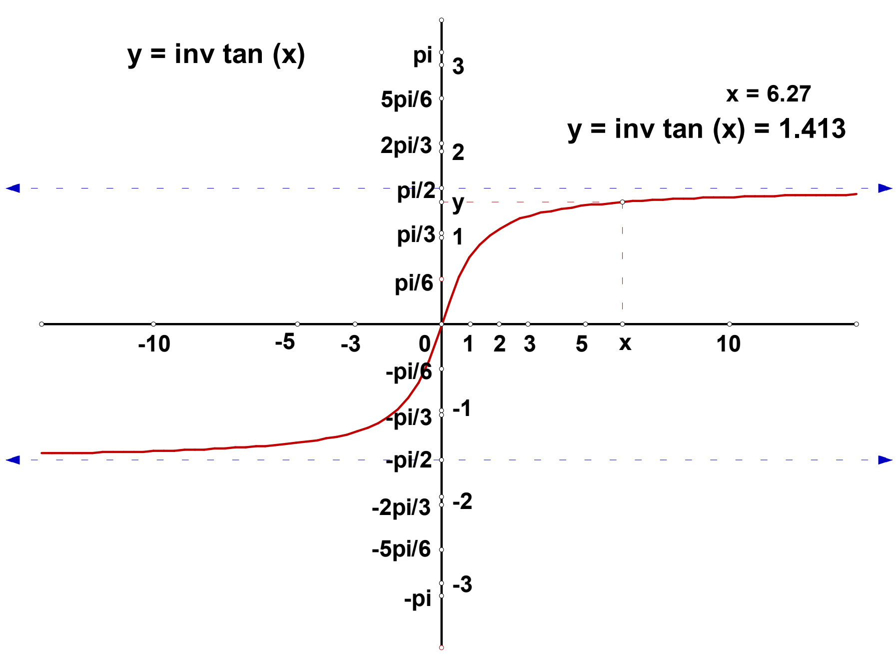


Table of values: $y = \tan^{-1}(x)$

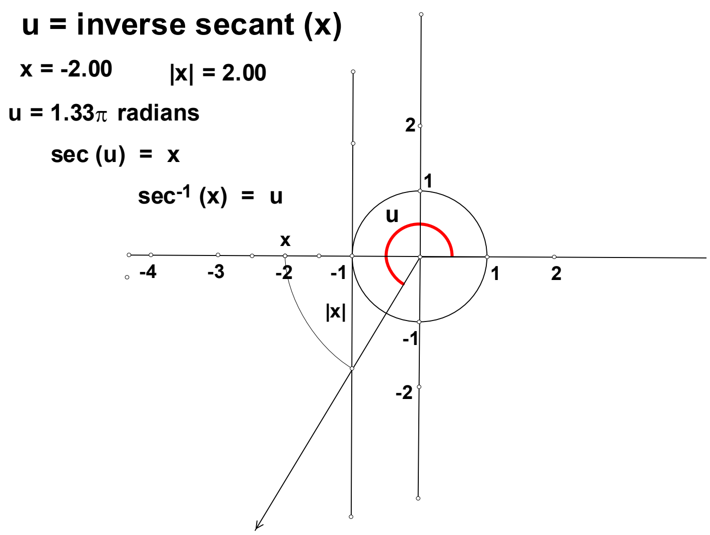
x	$y = \tan^{-1}(x)$
-10	- 1.471
-5	- 1.373
$-\sqrt{3}$ = -1.732	$-\pi/3$ = - 1.047
-1	$-\pi/4$ = - 0.785
$-\sqrt{3}/3$ = -0.577	$-\pi/6$ = - 0.524
0	0
$\sqrt{3}/3$ = 0.577	$\pi/6$ = 0.524
1	$\pi/4$ = 0.785
$\sqrt{3}$ = 1.732	$\pi/3$ = 1.047
5	1.373
10	1.471

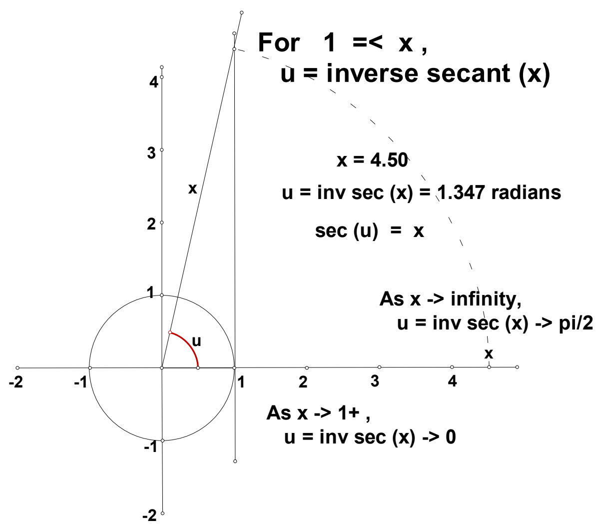


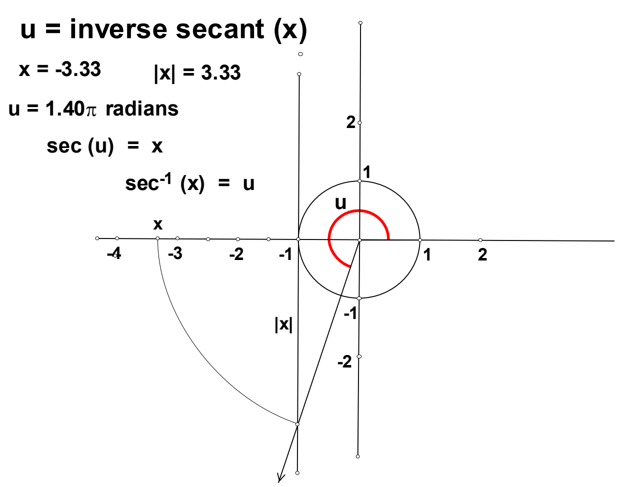
Inverse Secant Function:

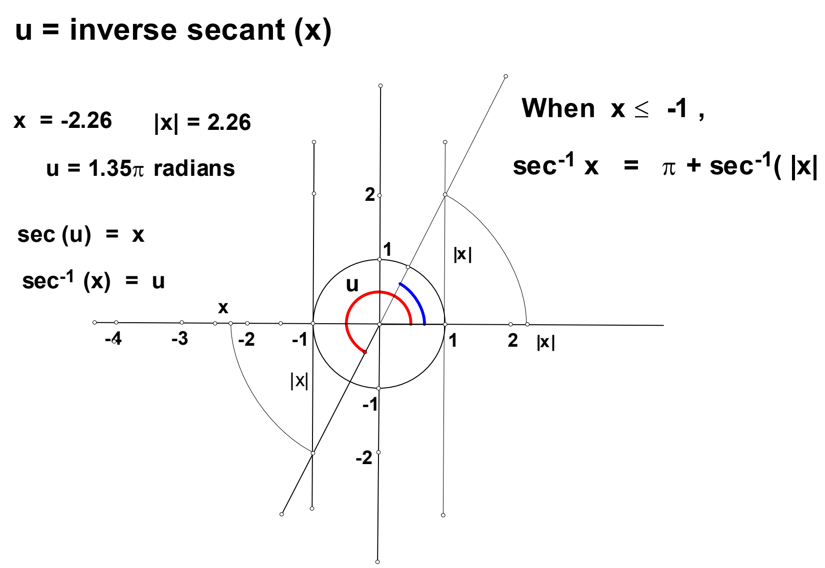
$$\mathbf{u} = \mathbf{sec}^{-1}(\mathbf{x}), \quad \mathbf{x} \in (-\infty, -1] \cup [1, \infty), \ \mathbf{u} \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right).$$

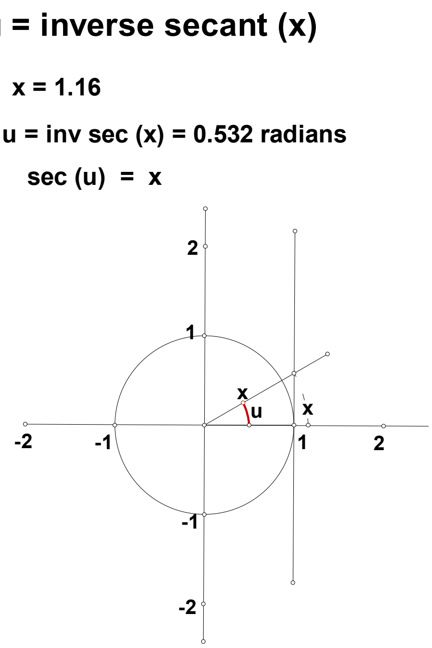
$$\mathbf{u} = \mathbf{sec}^{-1}(\mathbf{x}) \ \text{ when } \ \mathbf{sec}(\mathbf{u}) = \mathbf{x} \ \text{ and } \ \mathbf{u} \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right).$$

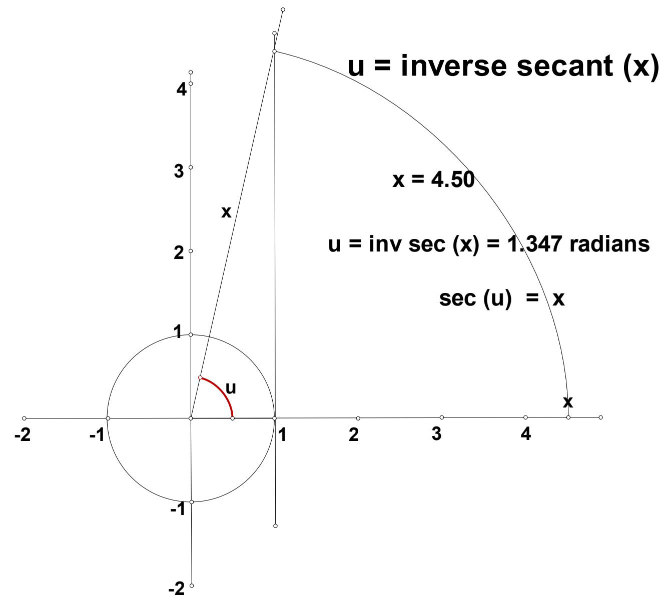










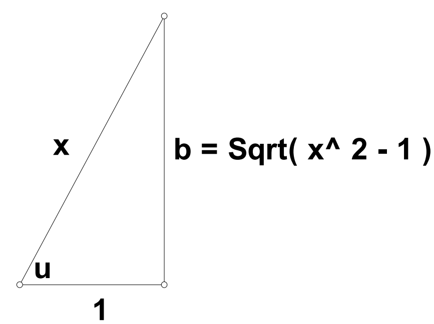


Drawing the angle u = inv sec (x) when x > 1 :

u = inv sec (x)

sec (u) = x = x/1 = hyp / adj

Make hyp = x and adj = 1 .

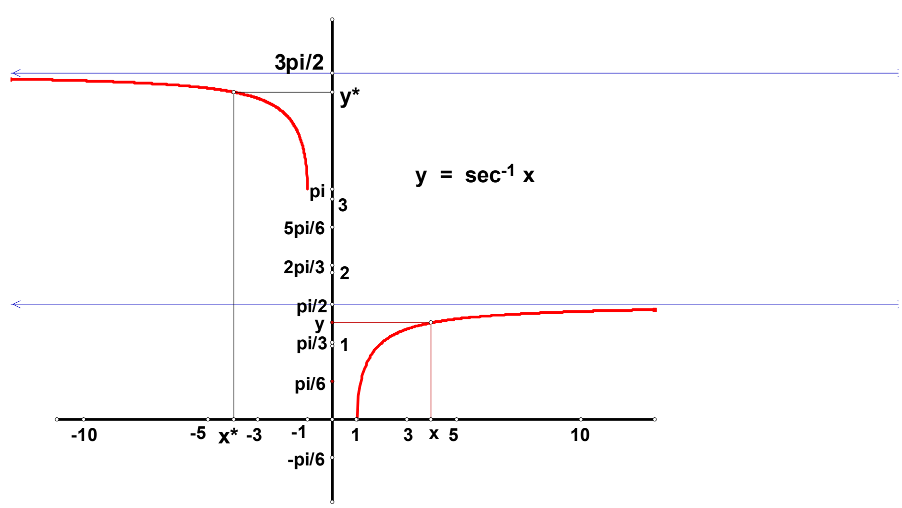


tan (u) = opp / adj

tan (inv sec (x)) = Sqrt(x^2 - 1)

Table of values: $y = \sec^{-1}(x)$

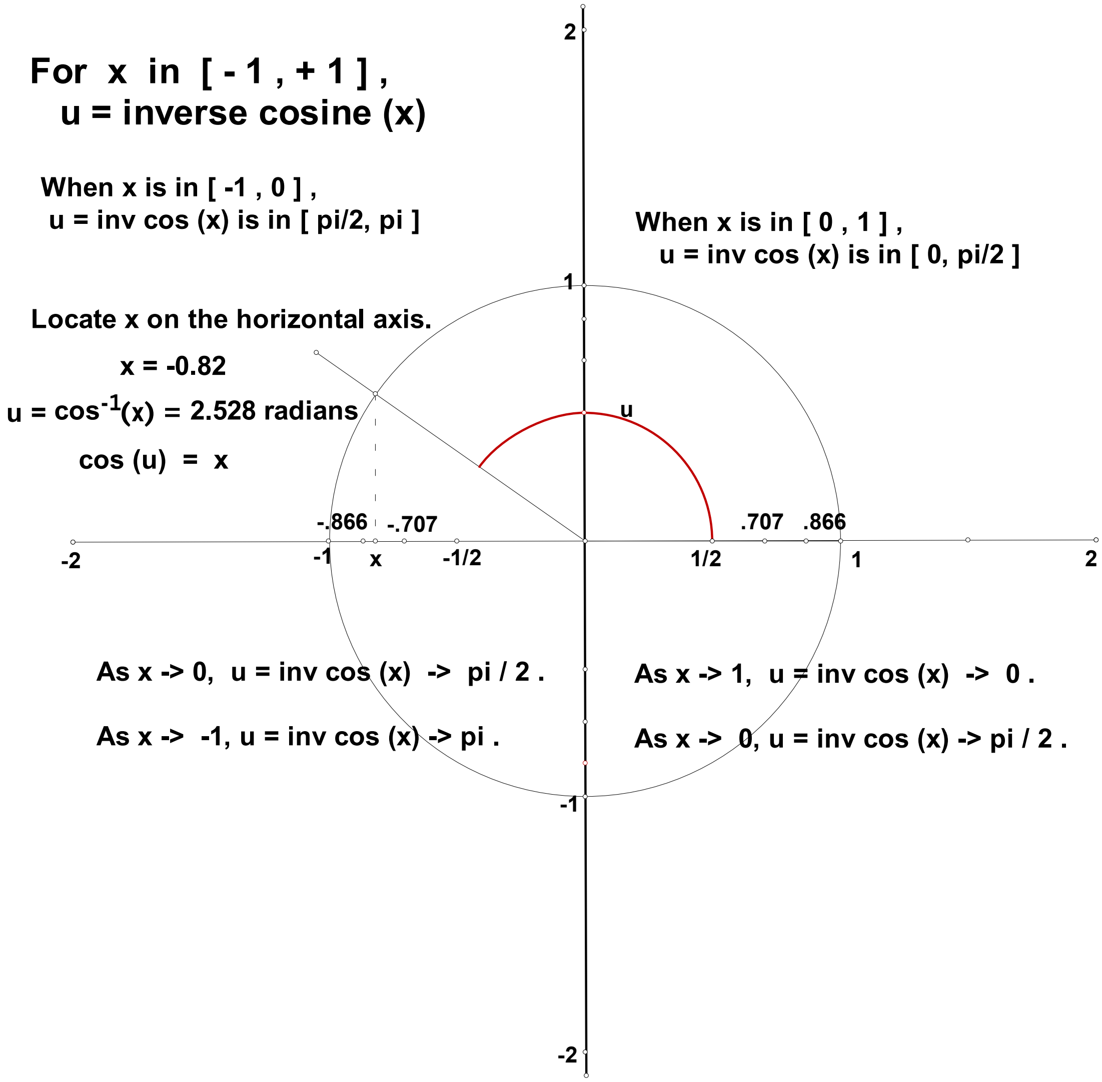
x	$y = \sec^{-1}(x)$
-10	4.612
- 5	4.511
-2	$4\pi/3 = 4.189$
$-\sqrt{2} = -1.414$	$5\pi/4 = 3.927$
$-2\sqrt{3}/3 = -1.155$	$7\pi/6 = 3.665$
-1	$\pi = 3.1416$
1	0
$2\sqrt{3}/3 = 1.155$	$\pi/6 = 0.524$
$\sqrt{2} = 1.414$	$\pi/4 = 0.785$
2	$\pi/3 = 1.047$
5	1.369
10	1.470

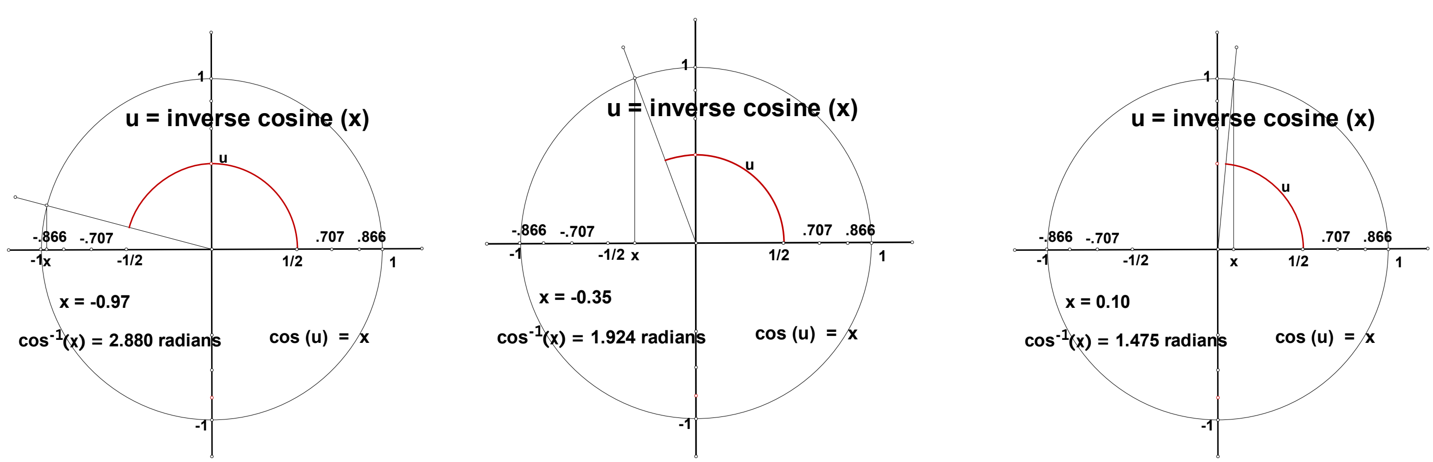


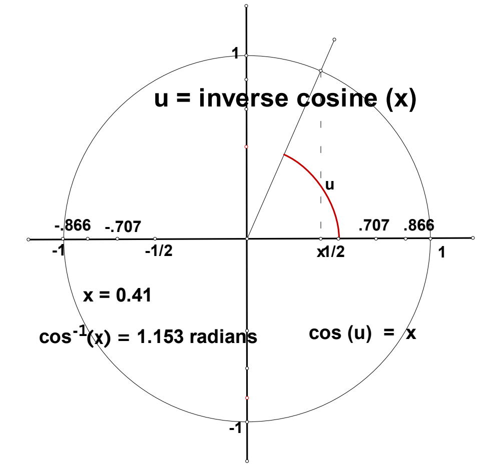
Inverse Cosine Function:

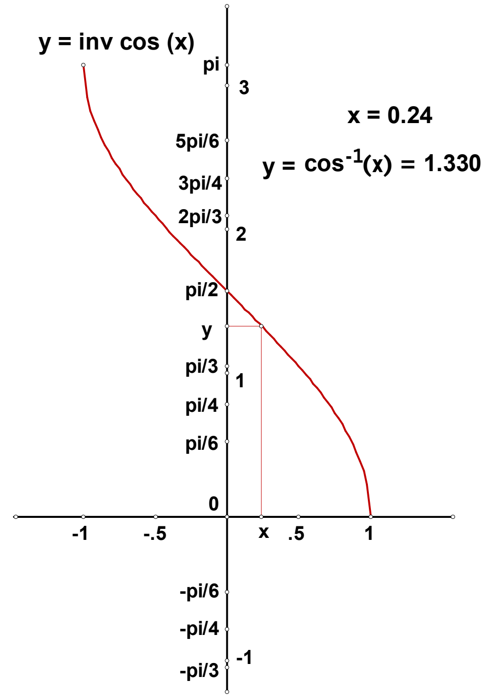
$$u = \cos^{-1}(x), \quad x \in [-1, 1], \quad u \in [0, \pi].$$

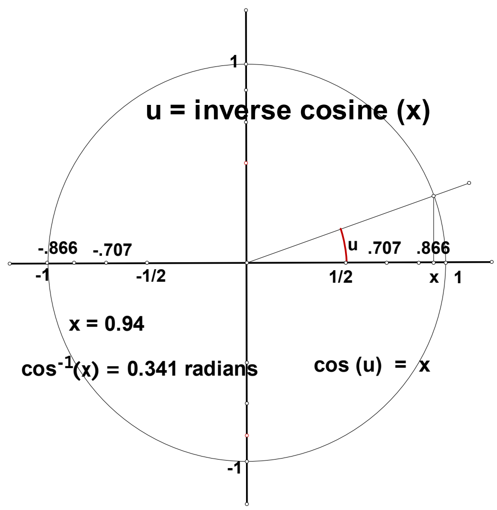
$$u = \cos^{-1}(x) \ \text{when} \ \cos(u) = x \ \text{and} \ u \in [0, \pi].$$









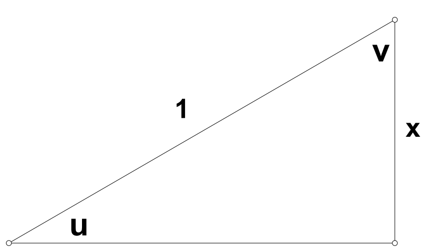


The figures below show that cos $^{-1}$ (x) and sin $^{-1}$ (x) are closely related:

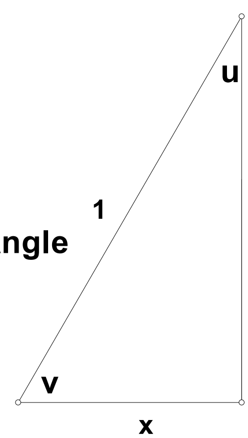
$$\cos^{-1}(x) \;=\; \frac{\pi}{2} - \sin^{-1}(x)$$. **Similar arguments show that:**

$$\cot^{-1}(x) \;=\; \frac{\pi}{2} - \tan^{-1}(x) \quad \textbf{and} \quad \csc^{-1}(x) \;=\; \frac{\pi}{2} - \sec^{-1}(x)$$.

By construction,
u = inv sin (x) .



Flipping the triangle around shows that angle v is v = inv cos (x) .



Since u + v = pi / 2 = 90 degrees,

inv sin (x) + inv cos (x) = pi / 2 .

Thus, inv cos (x) = pi / 2 - inv sin (x) .

Differentiation and Integration with Inverse Trig Functions

The Inverse Sine Function : $y = \sin^{-1}(x)$

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1 - x^2}} \quad \textbf{and} \quad \frac{d}{dx}\left(\sin^{-1}(u)\right) = \left(\frac{1}{\sqrt{1 - u^2}}\right)\left(\frac{du}{dx}\right)$$

For any positive constant $a > 0$:

$$\int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1}(x) + C \quad \textbf{and} \quad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

The Inverse Tangent Function : $y = \tan^{-1}(x)$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1 + x^2} \quad \textbf{and} \quad \frac{d}{dx}\left(\tan^{-1}(u)\right) = \left(\frac{1}{1 + u^2}\right)\left(\frac{du}{dx}\right)$$

For any non-zero constant a :

$$\int \frac{1}{1 + x^2}\,dx = \tan^{-1}(x) + C \quad \textbf{and} \quad \int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

The Inverse Secant Function : $y = \sec^{-1}(x)$

$$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{x\sqrt{x^2 - 1}} \quad \textbf{and} \quad \frac{d}{dx}\left(\sec^{-1}(u)\right) = \left(\frac{1}{u\sqrt{u^2 - 1}}\right)\left(\frac{du}{dx}\right)$$

For any positive constant $a > 0$:

$$\int \frac{1}{x\sqrt{x^2 - 1}}\,dx = \sec^{-1}(x) + C \quad \textbf{and} \quad \int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

Using the formulas relating the other three inverse trig functions with these three, the derivatives of the other three functions are easily calculated:

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1 - x^2}} = -\frac{d}{dx}\left(\sin^{-1}(x)\right) \;;$$

$$\frac{d}{dx}\left(\cot^{-1}(x)\right) = \frac{-1}{1 + x^2} = -\frac{d}{dx}\left(\tan^{-1}(x)\right) \;;$$

$$\frac{d}{dx}\left(\csc^{-1}(x)\right) = \frac{-1}{x\sqrt{x^2 - 1}} = -\frac{d}{dx}\left(\sec\ (x)\right)$$

Derivation of derivatives for Inverse Trig functions

Derivation #1 : Proof that $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1 - x^2}}$ **:**

Write $u = \sin^{-1}(x)$ **.** **We seek** $\dfrac{du}{dx} = \dfrac{d}{dx}\left(\sin^{-1}(x)\right)$ **.**

Then, $\sin(u) = \sin(\sin^{-1}(x)) = x$ **;** **so,** $\dfrac{d}{dx}\left(\sin(u)\right) = \dfrac{d}{dx}\left(x\right) = 1$ **;**

so, $\cos(u)\dfrac{du}{dx} = 1$ **, and thus** $\dfrac{du}{dx} = \dfrac{1}{\cos(u)} = \dfrac{1}{\cos(\sin^{-1}(x))}$.

Claim: $\cos(\sin^{-1}(x)) = \sqrt{1 - x^2}$ *for all* $x \in [-1, 1]$.

Proof of Claim:

The method of drawing $u = \sin^{-1}(x)$ **presented above shows that**

$$\cos(\sin^{-1}(x)) = \sqrt{1 - x^2} \quad \textit{for all } x \in [-1, 1] \textit{ such that } x > 0 .$$

If $x < 0$, *then* $x = -|x|$, *and so* $\sin^{-1}(x) = \sin^{-1}(-|x|) = -\sin^{-1}(|x|)$ **. This last equality is a direct consequence of the fact that** $\sin(-z) = -\sin(z)$ *for all* z **.**

Thus, $\cos(\sin^{-1}(x)) = \cos\left(\sin^{-1}(-|x|)\right) = \cos\left(-\sin^{-1}(|x|)\right) = \cos\left(\sin^{-1}(|x|)\right)$ **,**

and so, $\cos(\sin^{-1}(x)) = \cos\left(\sin^{-1}(|x|)\right) = \sqrt{1 - |x|^2} = \sqrt{1 - x^2}$.

When x = 0, the claim is true because $\sin^{-1}(0) = 0$ and $\cos(0) = 1$, and the claim is proven.

Finally, $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{du}{dx} = \dfrac{1}{\cos(u)} = \dfrac{1}{\cos(\sin^{-1}(x))} = \dfrac{1}{\sqrt{1 - x^2}}$ **, and the proof is complete.**

Derivation #2 : Proof that $\frac{d}{dx}\left(\tan^{-1}(x) \right) = \frac{1}{1 + x^2}$ **:**

Write $u = \tan^{-1}(x)$ **.** **We seek** $\frac{du}{dx} = \frac{d}{dx}\left(\tan^{-1}(x) \right)$ **.**

Then, $\tan(u) = \tan(\tan^{-1}(x)) = x$ **;** **so,** $\frac{d}{dx}\left(\tan(u) \right) = \frac{d}{dx}\left(x \right) = 1$ **;**

so, $\sec^2(u)\frac{du}{dx} = 1$ **, and thus** $\frac{du}{dx} = \frac{1}{\sec^2(u)} = \frac{1}{\sec^2(\tan^{-1}(x))}$.

Claim: $\sec(\tan^{-1}(x)) = \sqrt{1 + x^2}$ *for all* $x \in \left(-\infty , \infty \right)$.

Proof of Claim:

The method of drawing $u = \tan^{-1}(x)$ **presented above shows that**

$$\sec(\tan^{-1}(x)) = \sqrt{1 + x^2} \quad \textit{for all } x \textit{ such that } x > 0 .$$

If $x < 0$, *then* $x = -|x|$, *and so* $\tan^{-1}(x) = \tan^{-1}(-|x|) = -\tan^{-1}(|x|)$ **. This last equality is a direct consequence of the fact that** $\tan(-z) = -\tan(z)$ *for all* z **.**

Thus, $\sec(\tan^{-1}(x)) = \sec\left(\tan^{-1}(-|x|) \right) = \sec\left(-\tan^{-1}(|x|) \right) = \sec\left(\tan^{-1}(|x|) \right)$ **,**

and so, $\sec(\tan^{-1}(x)) = \sec\left(\tan^{-1}(|x|) \right) = \sqrt{1 + |x|^2} = \sqrt{1 + x^2}$.

When x = 0, the claim is true because $\tan^{-1}(0) = 0$ **and sec(0) = 1 , and the claim is proven.**

Finally,

$$\frac{d}{dx}\left(\tan^{-1}(x) \right) = \frac{du}{dx} = \frac{1}{\sec^2(u)} = \frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{\left(\sqrt{1 + x^2}\right)^2} = \frac{1}{1 + x^2} ,$$

and the proof is complete.

Derivation #3 :

Proof that $\frac{d}{dx}\left(\sec^{-1}(x) \right) = \frac{1}{x\sqrt{x^2 - 1}}$ *for all* $x \in \left(-\infty, -1 \right) \cup \left(1, \infty \right)$ **:**

Write $u = \sec^{-1}(x)$ **.** **We seek** $\frac{du}{dx} = \frac{d}{dx}\left(\sec^{-1}(x) \right)$ **.**

Then, $\sec(u) = \sec(\sec^{-1}(x)) = x$; **so,** $\frac{d}{dx}\left(\sec(u) \right) = \frac{d}{dx}\left(x \right) = 1$;

so, $\sec(u)\tan(u)\,\frac{du}{dx} = 1$ **, and thus**

$$\frac{du}{dx} = \frac{1}{\sec(u)\tan(u)} = \frac{1}{\sec(\sec^{-1}(x))\ \tan(\sec^{-1}(x))} = \frac{1}{x\ \tan(\sec^{-1}(x))}$$

Claim:

$$\tan(\sec^{-1}(x)) = \left\{ \begin{array}{l} \sqrt{x^2 - 1} \ \ \textit{for all } x \in \left[1, \infty \right) \\ \\ \sqrt{x^2 - 1} \ \ \textit{for all } x \in \left(-\infty, -1 \right] \end{array} \right\}$$

Proof of Claim:

If x > 1 , then method of drawing $u = \sec^{-1}(x)$ **presented above shows that**

$$\tan(\sec^{-1}(x)) = \sqrt{x^2 - 1} .$$

If $x < -1$, *then* $x = -\left| x \right|$, *and so* $\sec^{-1}(x) = \pi + \sec^{-1}\left(|x|\right)$ **.**

This last equality is evident from the definition of u = sec^{-1}(x) presented above.

Thus, $\tan(\sec^{-1}(x)) = \tan\left(\pi + \sec^{-1}\left(|x|\right) \right) = \tan\left(\sin^{-1}\left(|x|\right) \right)$ **and so,**

$$\tan(\sec^{-1}(x)) = \tan\left(\sec^{-1}\left(|x|\right) \right) = \sqrt{|x|^2 - 1} = \sqrt{x^2 - 1} .$$

When x = 1, the claim is true because sec^{-1}(1) = 0 and tan(0) = 0 ;
when x = -1, the claim is true because sec^{-1}(-1) = π and tan(π) = 0 , and the claim is proven.

Thus, when x > +1 or x < -1 ,

$$\frac{du}{dx} = \frac{1}{x\tan(\sec^{-1}(x))} = \frac{1}{x\sqrt{x^2 - 1}} ,$$

and the proof is complete.