LECTURE NOTES FOR THE FIRST LECTURE ON SERIES

Consider the sequence $\{r^n\}_{n=1}^{\infty}$

$= $ " $r, r^2, r^3, r^4, \ldots$ "

<u>Fact</u>: $\{r^n\}$ is a convergent sequence if $-1 < r \leq 1$

$\{r^n\}$ is a divergent sequence if $r > 1$ or if $r \leq -1$.

$\lim_{n\to\infty} r^n = 0$ if $-1 < r < 1$

$\lim_{n\to\infty} r^n = 1$ if $r = 1$

Thus, $\lim_{n\to\infty} (0.72)^n = 0$ since $-1 < 0.72 < 1$.

Series

$\{S_n\}_{n=1}^{\infty} = SOPS$

A series is a sequence which is produced from a different sequence $\{a_n\}$ called the sequence of terms of the series.

Recall, for a sequence $\{a_n\}$,

It is Divergent, if $\lim_{n\to\infty} a_n$ DNE.

It is Convergent if $\lim_{n\to\infty} a_n$ exists.

The Series $\sum_{n=1}^{\infty} a_n$

The Sequence of Terms of the Series.

Given a first sequence $a_1, a_2, a_3, \cdots$,

the series $\sum_{n=1}^{\infty} a_n$ is the sequence $S_1, S_2, S_3, S_4, \cdots$

(Called The Sequence of Partial Sums or SOPS) where

$S_k = a_1 + a_2 + \cdots + a_{k-1} + a_k$, for $k = 1, 2, 3, 4 \cdots$

$$\left.\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \end{aligned}\right\}$$

This $\{S_n\}$ sequence IS the series $\sum_{n=1}^{\infty} a_n$.

<u>Ex:</u> Let $\{a_n\}_{n=1}^{\infty} = \{(\frac{1}{10})^n\}_{n=1}^{\infty}$, which is

$$a_1, \quad a_2, \quad a_3, \quad a_4, \ldots$$
$$0.1, \quad 0.01, \quad 0.001, \quad 0.0001, \ldots \left(\xrightarrow{\text{Incidentally}} 0\right)$$

The <u>SERIES</u> $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (\frac{1}{10})^n$ is $s_1, s_2, s_3, s_4, \ldots$

and $s_1 = a_1 = 0.1$

$s_2 = a_1 + a_2 = 0.11$

$s_3 = a_1 + a_2 + a_3 = 0.111$

$\vdots \qquad \vdots$

(SOPS)

So, $\sum_{n=1}^{\infty} (0.1)^n$ = The Sequence "0.1, 0.11, 0.111, 0.1111, ..." $\to \frac{1}{9}$

" $s_1, s_2, s_3, s_4, \ldots$ "

If the series $\{s_n\}$, the SOPS, is a convergent (Divergent) sequence, we say the series $\{s_n\}$ is convergent (Divergent), that is

<u>The series</u> $\sum_{n=1}^{\infty} a_n$ is convergent (Divergent). "$\{s_n\}_{n=1}^{\infty}$" and "$\sum_{n=1}^{\infty} a_n$" mean the SAME THING (USUALLY)

For $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (\frac{1}{10})^n$, $\lim_{n\to\infty} s_n = \frac{1}{9}$, so

The Series $\sum_{n=1}^{\infty} (\frac{1}{10})^n$ is convergent.

Sequence $\searrow$ $=$ "$s_1, s_2, s_3, \ldots$"

FACT: When the series $\sum_{n=1}^{\infty} a_n$ is a convergent sequence with $\lim_{n\to\infty} s_n = s$, we write "$\sum_{n=1}^{\infty} a_n = s$"

NUMBER

C or D? Recall $\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

when $a_n = \left(\frac{1}{10}\right)^n$, $s = \lim_{n\to\infty} s_n = \frac{1}{9}$, so, $\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = \frac{1}{9}$.

C! So, the series is Convergent.

$\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ denotes the series $s_1, s_2, s_3, \ldots$ AND

$\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ denotes its limit $s = \frac{1}{9}$

You will know by the context whether the symbol $\sum_{n=1}^{\infty} a_n$ is denoting the sequence of P.S's or its limit.

Another SOPS Notation:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots \qquad = \text{"} s_1, s_2, s_3, s_4, \ldots \text{"}$$

s_2, s_3, s_4, $\vdots$

Ex: The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad a_n = \frac{1}{n}$

C or D? $\sum_{n=1}^{\infty} \frac{1}{n} = \underbrace{\frac{1}{1} + \frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{2}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{4}{8} = \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{17} + \cdots + \frac{1}{32}}_{\geq \frac{1}{2}} + \cdots$

$s_1 \geq \frac{1}{2},$

$s_2 \geq \frac{1}{2} + \frac{1}{2} = 1$

$s_4 \geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1.5$

$s_8 \geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \geq 2$

$s_{16} \geq 2.5$

$s_{32} \geq 3.0$

$s_{64} \geq 3.5$

$\vdots$

$\lim_{n \to \infty} s_n = \infty$

D! The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty.$$

Given a Series $\sum_{n=1}^{\infty} a_n$, we ask C or D?

See the Series Summation Laws on p. 714

Ex: (i) $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$.

C or D?

Ex: $\sum_{n=1}^{\infty} \frac{5}{n} = \sum_{n=1}^{\infty} 5 \cdot \left(\frac{1}{n}\right) = 5\left(\sum_{n=1}^{\infty} \frac{1}{n}\right)$, which is divergent because it is a non-zero constant multiple of the Harm. Series.

Def'n: Let a and r be real numbers.

The series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$

is called the Geometric Series with First Term $= a$ and

with Common Ratio $= r$, and

it is Divergent if $|r| \geq 1$, AND

it is Convergent if $-1 < r < 1$, and in this case,

See Ex. 2 on p. 703

$$\sum_{n=1}^{\infty} ar^{n-1} = S = \frac{a}{1-r} = \frac{\text{FIRST TERM}}{1 - (\text{Common Ratio})}$$

Ex: $\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots$

is the Geometric Sequence with $r = \frac{1}{10}$ and $a = \frac{1}{10}$

So, Because $r = \frac{1}{10}$ and $-1 < r < 1$, the Geometric Series

$\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ is Convergent & $S = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = \dfrac{\frac{1}{10}}{\underbrace{\left(1 - \frac{1}{10}\right)}_{= \frac{9}{10}}} = \frac{1}{9} = S$

Problem:

Is the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5^n}{8^{n-1}}$ C or D?

Here, $a_n = \frac{5^n}{8^{n-1}} = \frac{5 \cdot 5^{n-1}}{8^{n-1}} = 5\left(\frac{5}{8}\right)^{n-1}$, so

$\sum_{n=1}^{\infty} \frac{5^n}{8^{n-1}} = \sum_{n=1}^{\infty} 5\left(\frac{5}{8}\right)^{n-1}$ and this is a geometric series with Common Ratio $r = \frac{5}{8}$.

A JUSTIFICATION SENTENCE IS REQUIRED

The Series $\sum_{n=1}^{\infty} \frac{5^n}{8^{n-1}}$ is Convergent because it is a Geometric Series with $r = \frac{5}{8}$ and $-1 < \frac{5}{8} < 1$.

The Sum $s = \sum_{n=1}^{\infty} 5\left(\frac{5}{8}\right)^{n-1} = \frac{5}{1 - \frac{5}{8}} = 13\frac{1}{3}$

$$\frac{5}{\frac{8}{8} - \frac{5}{8}} = \frac{\left(\frac{5}{1}\right)}{\left(\frac{3}{8}\right)} = \frac{5}{1} \cdot \frac{8}{3} = \frac{40}{3} = \frac{39}{3} + \frac{1}{3} = 13\frac{1}{3}$$

Whenever you assert that a particular series is Convergent or Divergent, you must write a <u>Required</u> <u>Justification</u> <u>Sentence</u> which explains why we are justified in making that assertion. It must be as clear and complete as those suggested in the Handouts.

THE TEST FOR DIVERGENCE

Considering the Series $\sum_{n=1}^{\infty} a_n$, there are two (2) sequences involved:

(1) The Sequence of Terms: $a_1, a_2, a_3, a_4, \ldots = \{a_n\}_{n=1}^{\infty}$

and

(2) The Sequence of Partial Sums: $s_1, s_2, s_3, s_4, \ldots = \sum_{n=1}^{\infty} a_n$,
(SOPS)
which is the series itself.

FACT: If the Series $s_1, s_2, s_3, \ldots$ is Convergent,

That is, If the series $\sum_{n=1}^{\infty} a_n$ is Convergent, then

The Sequence of Terms $a_1, a_2, a_3, \ldots$ is Convergent AND

$\lim_{n\to\infty} a_n = 0$.

$$\boxed{\text{So, If } \lim_{n\to\infty} a_n \neq 0 \text{, Then the Series } \sum_{n=1}^{\infty} a_n \text{ is Divergent.}}$$

Applying this fact to conclude divergence is The <u>DIVERGENCE TEST</u>

Problem:

Is $\sum_{n=1}^{\infty} \frac{(n+3)^2}{n(2n+5)}$ Convergent or Divergent?

Is $\sum_{n=1}^{\infty} a_n$ C or D? First, Look at $\lim_{n\to\infty} a_n$.

Here, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+3)^2}{n(2n+5)} = \lim_{n\to\infty} \frac{n^2+6n+9}{2n^2+5n}$

$= \lim_{n\to\infty} \frac{n^2(1+\frac{6}{n}+9/n^2)}{n^2(2+5/n)} = \lim_{n\to\infty} \frac{(1+\frac{6}{n}+\frac{9}{n^2})}{(2+5/n)} = \frac{1}{2}$

A JUSTIFICATION SENTENCE is REQUIRED

Since $\lim_{n\to\infty} \frac{(n+3)^2}{n(2n+5)} \neq 0$,

the Series $\sum_{n=1}^{\infty} \frac{(n+3)^2}{n(2n+5)}$ is Divergent

by the "TEST for Divergence".