TWO ROOT TEST EXAMPLES AND TWO RATIO TEST EXAMPLES

The TESTS FOR ABSOLUTE CONVERGENCE

Let $\sum_{n=1}^{\infty} a_n$ be a given series.

Consider the following limit L.
(Both tests interpret L in the SAME WAY.)

THE RATIO TEST	THE ROOT TEST
$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = L$

If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is Absolutely Convergent.

If $L > 1$, Then $\sum_{n=1}^{\infty} a_n$ is Divergent.

If $L = 1$, the test fails and some other test must be used.

In what follows,
in EXAMPLE 1, the RATIO Test is applied
In EXAMPLE 2, the ROOT Test is applied.
In EXAMPLE 3, the ROOT Test is applied.
In EXAMPLE 4, the RATIO Test is applied.

Example 1:

Is the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ Convergent OR Divergent?

Solution (using the Ratio Test):

$$|a_n| = \frac{n^3}{3^n} \quad \text{and} \quad |a_{n+1}| = \frac{(n+1)^3}{3^{(n+1)}}$$

$$L = \lim_{n\to\infty} \left(\frac{\frac{(n+1)^3}{3^{(n+1)}}}{\frac{n^3}{3^n}} \right) = \lim_{n\to\infty} \left(\frac{(n+1)^3}{3^{(n+1)}} \right)\left(\frac{3^n}{n^3} \right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{3}\right)\left(\frac{n+1}{n}\right)^3 = \left(\frac{1}{3}\right)\left(1^3\right) = \frac{1}{3} = L$$

since $\lim_{n\to\infty} \left(\frac{n+1}{n}\right) = 1$.

JUSTIFICATION IS REQUIRED

Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3}$ and $\frac{1}{3} < 1$, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is Absolutely Convergent by the Ratio Test.

[Of course, by an earlier theorem, since $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is absolutely convergent, it is convergent.]

EXAMPLE 2: Is $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3^n}{n^5}$ Convergent OR Divergent?

Solution (using the ROOT test):

$$\lim_{n\to\infty} \sqrt[n]{3^n/n^5} = \lim_{n\to\infty} \frac{\sqrt[n]{3^n}}{\sqrt[n]{n^5}} = \lim_{n\to\infty} \frac{3}{(\sqrt[n]{n})^5} = \frac{3}{1} = 3$$

since $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

JUSTIFICATION REQUIRED

Since $\lim_{n\to\infty} \sqrt[n]{3^n/n^5} = 3$ and $3>1$, the series $\sum_{n=1}^{\infty} \frac{3^n}{n^5}$ is Divergent by the ROOT TEST.

EXAMPLE 3: Is $\sum_{n=1}^{\infty} \frac{3}{[\ln(n)]^n}$ Convergent OR Divergent?

Solution (using the ROOT TEST):

Here, $|a_n| = \frac{3}{[\ln(n)]^n}$, so $\sqrt[n]{|a_n|} = \frac{\sqrt[n]{3}}{\ln(n)}$.

Now, $\lim_{n\to\infty} \sqrt[n]{3} = 1$ and $\lim_{n\to\infty} \ln(n) = \infty$, so

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{\sqrt[n]{3}}{\ln(n)} = 0 = L < 1.$$

JUSTIFICATION IS REQUIRED

Since $\lim_{n\to\infty} \sqrt[n]{\frac{3}{[\ln(n)]^n}} = 0$ and $0<1$, the series $\sum_{n=1}^{\infty} \frac{3}{[\ln(n)]^n}$ is Absolutely Convergent by the ROOT Test.

EXAMPLE 4:

Is the series $\sum_{n=1}^{\infty} \frac{5^n n^3}{(n!)}$ Convergent or Divergent?

Solution: In considering whether to use the ROOT TEST OR the RATIO TEST, seeing the presence of FACTORALS, like "$n!$" here, or perhaps "$(n+1)!$" or "$(n+2)!$", etc. tells you not to not to use the ROOT TEST (NOT WITH FACTORALS!). Use the RATIO Test with FACTORALS.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{\frac{5^{(n+1)}(n+1)^3}{(n+1)!}}{\frac{5^n n^3}{n!}}\right) = \left(\frac{5^{(n+1)}(n+1)^3}{(n+1)!}\right)\left(\frac{n!}{5^n n^3}\right).$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{5^{(n+1)}(n+1)^3}{(n+1)!}\right)\left(\frac{n!}{5^n n^3}\right)$$

$$= \lim_{n\to\infty}\frac{5}{(n+1)}\left(\frac{n+1}{n}\right)^3 = 0 \text{ since } \lim_{n\to\infty}\left(\frac{n+1}{n}\right) = 1 \text{ and }$$

$$\lim_{n\to\infty} n+1 = \infty.$$

JUSTIFICATION is REQUIRED: Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ and $0 < 1$, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{5^n n^3}{(n!)}$ is Absolutely Convergent by the RATIO TEST