ALTERNATING SERIES EXAMPLES

EXAMPLE 1: Determine if $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2^{n-1}}$ is C or D.

Solution: Let $b_n = |a_n| = \frac{1}{2^{n-1}}$ for each $n \geq 1$.

(i) Show that $b_{n+1} \leq b_n$ for each $n \geq 1$.

Since $2^n \geq 2^{n-1}$, $\frac{1}{2^n} \leq \frac{1}{2^{n-1}}$.

Since $b_{n+1} = \frac{1}{2^{(n+1)-1}} = \frac{1}{2^n}$ and $b_n = \frac{1}{2^{n-1}}$, $b_{n+1} \leq b_n$.

(ii) Show that $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\overset{\to 1}{1}}{\underset{\to\infty}{2^{n-1}}} = 0.$$

Justification is REQUIRED: Because ① $b_{n+1} \leq b_n$ for each $n \geq 1$ and ② $\lim_{n\to\infty} \frac{1}{2^{n-1}} = 0$, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2^{n-1}}$ is Convergent by the Alternating Series Test.

EXAMPLE 2: $\sum_{n=1}^{\infty} (-1)^n \cdot 1 = \sum_{n=1}^{\infty} (-1)^n b_n$ where $b_n = |(-1)^n \cdot 1|$.

Thus, $\sum_{n=1}^{\infty} (-1)^n \cdot 1 = -1+1-1+1-1+1-1+\cdots$ and the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ is $-1, 0, -1, 0, -1, 0, -1, \cdots$.

Clearly, $\lim_{n\to\infty} S_n$ Does not exist. Since the limit, as $n \to \infty$, of the sequence of Partial Sums Does NOT Exist, the series $\sum_{n=1}^{\infty} (-1)^n \cdot 1$ is **Divergent**.

EXAMPLE 3: The ALTERNATING HARMONIC SERIES.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ where } b_n = \frac{1}{n} \text{ for all } n.$$

We apply the Alternating Series Test to show that this series is Convergent.

STEP (i) Show that $b_{n+1} \le b_n$ for all $n \geq 1$.

Since $n+1 > n$, $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$, so

$b_{n+1} \le b_n$ for all $n \geq 1$.

STEP (ii) Show that $\lim_{n \to \infty} b_n = 0$.

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0.$$

JUSTIFICATION REQUIRED

"Because (1) $b_{n+1} \le b_n$ for all $n \geq 1$

and (2) $\lim_{n \to \infty} \frac{1}{n} = 0$,

the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ is convergent by the Alternating Series Test."

SAMPLE SOLUTIONS:

Problem 1: Test the series for Convergence or Divergence:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1} = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

where $b_n = \dfrac{n^2}{n^3+1}$.

SOL'N:

(i) Show $b_{n+1} \leq b_n$ [b_n decreases as n increases]

Let $f(x) = \dfrac{x^2}{x^3+1}$ and note: $f(n) = b_n$ for all n.

Show that $f(x)$ decreases as x increases!

Show $f'(x) < 0$ on $[1, \infty)$

$$f'(x) = \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

$f'(x) < 0$ when $x^3 > 2$; when $x > \sqrt[3]{2}$

so $f(x)$ decreases on $[\sqrt[3]{2}, \infty)$.

$\therefore f(n+1) < f(n)$ for $n \geq 2$

$\therefore b_{n+1} \leq b_n$ for $n \geq 2$.

Problem 1 (Continued):

(ii) Show $\lim_{n\to\infty} b_n = 0$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{n^2 \cdot 1}{n^3(1+1/n^3)}$$

$$= \lim_{n\to\infty} \frac{1}{n(1+1/n^3)} = 0 \text{ since } n \to \infty.$$

Justification is REQUIRED

"Therefore,

Because ① $b_{n+1} \le b_n$ for all $n \ge 2$

and ② $\lim_{n\to\infty} \dfrac{n^2}{n^3+1} = 0$

the Alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{n^2}{n^3+1}$

is Convergent by the ALTERNATING SERIES TEST."

PROBLEM 2: Let $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{12^n}$

1) Show that the series converges,

2) Approximate s with a partial sum s_n correct to four decimal places:
That is, $s_n \approx s$ so that
$|\text{ERROR}| \leq 0.00005$

SOL'N: 1) Show $b_{n+1} \leq b_n$ for all $n \geq 1$

$$b_n = \frac{n}{12^n} \quad ; \quad b_{n+1} = \frac{n+1}{12^{(n+1)}}$$

$$b_{n+1} = \frac{n+1}{12 \times (12^n)} \leq \frac{n+11n}{12 \times (12^n)} = \frac{12n}{12(12^n)} = \frac{n}{12^n} = b_n$$

$\therefore b_{n+1} \leq b_n$ for all $n \geq 1$

2) Show $\lim_{n \to \infty} b_n = 0$

Let $f(x) = \frac{x}{12^x}$. Then $f(n) = \frac{n}{12^n}$

PROBLEM 2 (continued):

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n/12^n = \lim_{x\to\infty} \frac{x}{12^x}$$

$$= \lim_{x\to\infty} x/12^x \stackrel{L'H}{=} \lim_{x\to\infty} \frac{1}{(12^x(\ln 12))} = 0$$

Form: $\frac{\infty}{\infty}$ Since $12^x \to \infty$ as $x \to \infty$.

RECALL $\frac{d}{dx}(a^x) = a^x(\ln a)$

So, Because ① $b_{n+1} \le b_n$ for all $n \ge 1$ and ② $\lim_{n\to\infty} n/12^n = 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{12^n}$ is Convergent by The ALTERNATING SERIES TEST.

2) For Approximation $S_n \approx S$ for an alternating series, $|\text{ERROR}| \le b_{n+1}$.

We need $b_{n+1} < 0.00005$. $|S - S_3| \le b_4$

We compute: $b_4 = \frac{4}{12^4} = 0.0001929$ X

$|S - S_4| \le b_5$; $b_5 = 5/12^5 = 0.00002$ ✓

$\therefore b_{n+1} = b_5 \Rightarrow n+1 = 5 \Rightarrow n = 4 \Rightarrow S_n = S_4$

$$S_4 = \frac{1}{12} - \frac{2}{12^2} + \frac{3}{12^3} - \frac{4}{12^4} = 0.07098$$

ROUND TO THE FOUR PLACES: $S \approx S_4 = 0.0710$