INTEGRAL TEST — A MODEL SOLUTION

<u>Sec 11.3, #17</u> Is $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ Convergent?

Here $a_n = \frac{1}{n^2+4}$.

Let $f(x) = \frac{1}{x^2+4}$. Then $\frac{1}{n^2+4} = a_n = f(n)$.

Function f is positive and continuous on $[1, \infty)$

Show that f is decreasing on $[2, \infty)$:

$$f'(x) = \frac{0 \cdot (x^2+4) - 1(2x)}{(x^2+4)} = \frac{-2x}{(x^2+4)}$$

$\therefore f'(x) < 0$ on $[2, \infty)$. $\therefore$ f is decreasing on $[2, \infty)$.

So function f and sequence $\{a_n\}$ satisfy the conditions of the Integral test on p. 716

$$\int_1^{\infty} \frac{1}{x^2+4}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^2+4}\,dx = \lim_{t \to \infty} \left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right)\Big]_1^t$$

$$= \lim_{t \to \infty} \left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right]$$

$\therefore \int_1^{\infty} \frac{1}{x^2+4}\,dx$ is convergent.

Since $\int_1^{\infty} \frac{1}{x^2+4}\,dx$ is convergent, $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ is convergent by the Integral Test.

Integral Test - Model Solution

PROBLEM: Is $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ Convergent?

Here $a_n = \frac{n}{n^2+1}$.

Let $f(x) = \frac{x}{x^2+1}$. Then $\frac{n}{n^2+1} = a_n = f(n)$

Function f is positive and continuous on $[1,\infty)$.

Show that f is decreasing on $[1,\infty)$:

$$f'(x) = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$ and so $f'(x) < 0$ on $[2,\infty)$.

$\therefore f$ is decreasing on $[2,\infty)$.

So function f and sequence $\{a_n\}$ satisfy the conditions of the Integral Test on p. 716.

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \frac{1}{2}\int_2^{t^2+1} \frac{1}{u}\,du$$

Let $u = x^2+1$, $du = 2x\,dx$, $x\,dx = \frac{1}{2}du$, at $x=1$, $u=2$, ...

$$= \lim_{t\to\infty} \frac{1}{2}(\ln u)\Big]_2^{t^2+1}$$

$$= \lim_{t\to\infty} \left[\frac{1}{2}\ln(t^2+1) - \frac{1}{2}\ln 2\right] = \infty.$$

$\therefore \int_1^{\infty} \frac{x}{x^2+1}\,dx$ diverges.

Since $\int_1^{\infty} \frac{x}{x^2+1}\,dx$ is divergent, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent by the Integral Test.

P-SERIES TEST - Model Solution

Is $\sum_{n=1}^{\infty} \frac{7\sqrt{n}+6}{n^4}$ Convergent?

Here $a_n = \frac{7\sqrt{n}+6}{n^4} = 7\frac{\sqrt{n}}{n^4} + 6\frac{1}{n^4} = 7\left(\frac{1}{n^{7/2}}\right) + 6\left(\frac{1}{n^4}\right)$

Now, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{7/2}}$ is convergent by the p-test with $p = 7/2$.

and $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n^4}$. is convergent by the p-test with $p = 4$.

Since $a_n = 7b_n + 6c_n$ and since $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ are convergent,

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{7\sqrt{n}+6}{n^4}$ is convergent,

By the theorem in Box 8 on p. 709