FIRST NOTES ON SERIES

DEFINITION OF "SERIES" (SEC 11.2)

GIVEN $\{a_n\}$: $a_1, a_2, a_3, a_4, \ldots$

1st P.S. $S_1 = \sum_{i=1}^{1} a_i = a_1 = s_1$

2nd P.S. $S_2 = \sum_{i=1}^{2} a_i = a_1 + a_2 = s_2$

THIRD PARTIAL SUM $S_3 = \sum_{i=1}^{3} a_i = a_1 + a_2 + a_3 = s_3$

$$\vdots$$

nth PARTIAL SUM $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n = s_n$

$$\vdots$$

The <u>SEQUENCE OF PARTIAL SUMS</u>

$s_1, s_2, s_3, s_4, \ldots$

is called the <u>series</u> $\sum_{i=1}^{\infty} a_i$.

If $\lim_{n\to\infty} s_n$ exists and $\lim_{n\to\infty} s_n = s$,

then we say the series

$\sum_{i=1}^{\infty} a_i$ is <u>convergent</u>.

The limit $s = \lim_{n\to\infty} s_n$ we call

THE <u>SUM</u> OF THE SERIES

and we write $\sum_{i=1}^{\infty} a_i = s$.

IF $\lim_{n\to\infty} s_n$ does not exist, then

we say the series $\sum_{i=1}^{\infty} a_i$ is <u>divergent</u>.

NOTE: THE NOTATION "$\sum_{i=1}^{\infty} a_i$" is used in <u>two</u> <u>ways</u>!

① As the <u>sequence</u> of PARTIAL sums $\{s_n\}_{n=1}^{\infty}$

AND

② As the <u>LIMIT</u> s of this sequence.

AN EXAMPLE SERIES:

$\{a_n\}_{n=1}^{\infty}$ is given by "a_n = the n^{th} ODD POSITIVE INTEGER"

a_n: $\underbrace{1, 3, 5}$, 7, 9, 11, ...

$s_1 = 1$, $s_2 = 4$, $s_3 = 9$, $s_4 = 16$, ...

It turns out that $s_n = n^2$, $n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} n^2 = \infty \quad \text{(D.N.E.)}$$

So $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}$ "n^{th} ODD #" is divergent.

GEOMETRIC SEQUENCES AND GEOMETRIC SERIES

GIVEN constants a and r, the sequence

$$a_n = a \cdot r^{n-1}, \quad n = 1, 2, 3, \ldots$$

is called a <u>GEOMETRIC SEQUENCE WITH COMMON RATIO r</u>.

$$a_n: \quad a \xrightarrow{\times r} ar \xrightarrow{\times r} ar^2 \xrightarrow{\times r} ar^3 \xrightarrow{\times r} \ldots \text{etc.}$$

$a \cdot r^0$, ar^1,

$$a_n = ar^{n-1}, \quad n = 1, 2, 3, \ldots$$

Given the geometric sequence $\{a_n = ar^{n-1}\}_{n=1}^{\infty}$, its sequence of partial sums is called a <u>Geometric Series</u> with commm ratio r

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$$

Such a GEOMETRIC SERIES is:

1. Convergent when $-1 < r < 1$; $|r| < 1$.
2. Divergent when $r \leq -1$ or $1 \leq r$. That is, when $|r| \geq 1$.

EXAMPLE OF A GEOMETRIC SERIES:

Here: $a = 1$, $r = \frac{1}{2}$

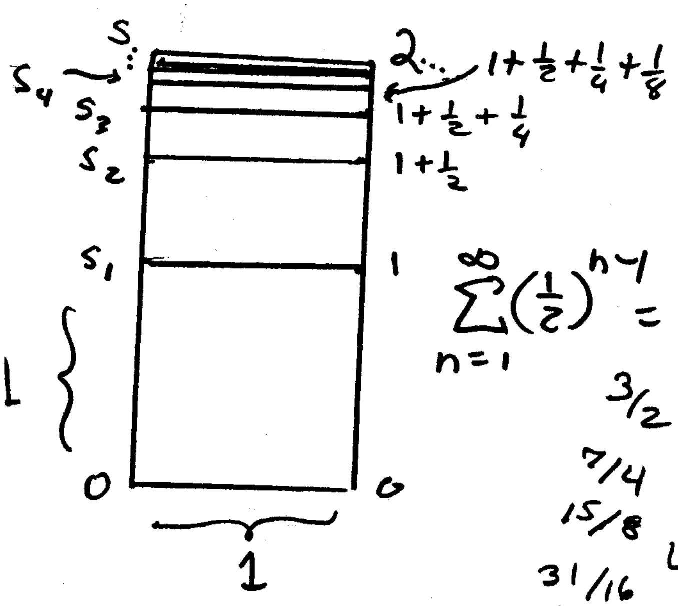
$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^{n-1} = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \dots$$

FORMULA FOR THE SUM s for a CONVERGENT GEOMETRIC SERIES.

For constants a and r with $|r| < 1$,

$$s = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r}$$

When $a = 1$ and $r = \frac{1}{2}$, $s = \dfrac{1}{1 - 1/2} = 2$



$$\sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^{n-1} = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \dots = 2$$

$3/2$

$7/4$

$15/8$

$31/16$

$\vdots$

$2 - \frac{1}{2^n}$ $\qquad \sum_{i=1}^{n} \left(\tfrac{1}{2}\right)^{i-1}$

$\swarrow$

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Two Equivalent Forms of the Geometric Series:

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

when $|r| < 1$

$$ar^0 + ar^1 + ar^2 + \cdots$$

$\boxed{\text{C OR D?}}$

Convergent or Divergent?

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{3}^{\,n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{3}} \cdot \left(\frac{1}{\sqrt{3}^{\,n}}\right) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}\right)^n = S$$

$$= \sum_{n=0}^{\infty} ar^n = S$$

Here $r = \frac{1}{\sqrt{3}} < 1$

Series is convergent!

$$S = \frac{a}{1-r} = \frac{1/\sqrt{3}}{1-(1/\sqrt{3})} = \frac{1/\sqrt{3}}{\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)}$$

$$= \frac{1}{\sqrt{3}} \cdot \left(\frac{\sqrt{3}}{\sqrt{3}-1}\right) = \frac{1}{\sqrt{3}-1}$$

$$= \frac{1}{\sqrt{3}-1} \cdot \left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right) = \frac{\sqrt{3}+1}{3-1} = \frac{\sqrt{3}+1}{2}$$

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{3}^{\,n+1}} = S = \frac{\sqrt{3}+1}{2}$$

$\boxed{\text{C or D ?}}$

$$\sum_{n=0}^{\infty} \frac{9^n}{17(5^n)} = \sum_{n=0}^{\infty} \frac{1}{17}\left(\frac{9}{5}\right)^n$$

Here $a = \frac{1}{17}$, $r = 9/5 > 1$. Series is Divergent!

THE HARMONIC SERIES DIVERGES

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \underbrace{1}_{>\frac{1}{2}} + \underbrace{\frac{1}{2}}_{\geq\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>2\cdot\frac{1}{4}\ >\frac{1}{2}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{>4\cdot\frac{1}{8}\ >\frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{>8\cdot\frac{1}{16}\ >\frac{1}{2}} + \underbrace{\frac{1}{17} + \cdots + \frac{1}{32}}_{>16\times\frac{1}{32}\ >\frac{1}{2}} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad (\text{D.N.E})$$

AN EXAMPLE APPLICATION

$$\sum_{n=1}^{\infty} \frac{7}{n} = \sum_{n=1}^{\infty} 7 \cdot \frac{1}{n} = 7\cdot\frac{1}{1} + 7\cdot\frac{1}{2} + 7\cdot\frac{1}{3} + \cdots$$

$$= 7\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots\right) = 7\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

THE SERIES is DIVERGENT

(Because the Harmonic Series Is Divergent.)

8 Theorem If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\,(a_n + b_n)$, and $\Sigma\,(a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ (ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

These properties of convergent series follow from the corresponding Limit Laws for Sequences in Section 11.1. For instance, here is how part (ii) of Theorem 8 is proved:

Let

$$s_n = \sum_{i=1}^{n} a_i \qquad s = \sum_{n=1}^{\infty} a_n \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

The nth partial sum for the series $\Sigma\,(a_n + b_n)$ is

$$u_n = \sum_{i=1}^{n} (a_i + b_i)$$

and, using Equation 5.2.9, we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sum_{i=1}^{n} (a_i + b_i) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right)$$

THE TEST FOR DIVERGENCE

In considering the series $\sum_{n=1}^{\infty} a_n$,

If $\lim_{n\to\infty} a_n$ does not exist

or $\lim_{n\to\infty} a_n \neq 0$,

Then the series is divergent.

$\boxed{\text{C or D ?}}$ $\sum_{n=1}^{\infty} \frac{(n+3)^2}{n(2n+5)} = \sum_{n=1}^{\infty} a_n$

Here $a_n = \frac{(n+3)^2}{n(2n+5)}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+3)^2}{n(2n+5)} = \lim_{n\to\infty} \frac{[n(1+3/n)]^2}{n[n(2+5/n)]}$$

$$= \lim_{n\to\infty} \frac{n^2(1+3/n)^2}{n^2(2+5/n)} = \lim_{n\to\infty} \frac{(1+3/n)^2}{(2+5/n)} = \frac{1^2}{2} = \frac{1}{2} \neq 0$$

$\therefore \lim_{n\to\infty} a_n \neq 0$ so the series diverges.

Since $\lim_{n\to\infty} \frac{(n+3)^2}{n(2n+5)} = \frac{1}{2} \neq 0$, $\sum_{n=1}^{\infty} \frac{(n+3)^2}{n(2n+5)}$ Diverges

by THE TEST FOR DIVERGENCE.

Problem: Find the values of x for which the given series converges and find the sum s for those values of x.

$$\sum_{n=1}^{\infty} (3x-5)^n$$

$$= \sum_{n=1}^{\infty} a r^{n-1} = \sum_{n=1}^{\infty} (3x-5)(3x-5)^{n-1}$$

Here $a = 3x-5$
$r = 3x-5$, also.

The geometric series converges when $-1 < r < 1$,

So $-1 < 3x-5 < 1$

(Add 5) $4 < 3x < 6$

(Divide by 3) $\frac{4}{3} < x < 2$

THE SERIES CONVERGES FOR ALL x such that $\frac{4}{3} < x < 2$

The sum s is

$$s = \frac{a}{1-r} = \frac{3x-5}{1-(3x-5)} = \frac{3x-5}{1-3x+5}$$

$$\boxed{s = \frac{3x-5}{6-3x}}$$

$\boxed{\text{C or D?}}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, 2^{n+3}}{3^{n+1}}$$

$$\lim_{n\to\infty} \frac{2^{n+3}}{3^{n+1}} =$$

$$\lim_{n\to\infty} \frac{8\cdot 2^n}{3\cdot 3^n} = \frac{8}{3}\lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$$

Series might Converge!

$\boxed{\text{Pr. 10}}$

$$= \sum_{n=1}^{\infty} \frac{(-1)(-1)^{n-1}(2^4)(2^{n-1})}{3^2(3^{n-1})}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)(2^4)\ (-1)^{n-1}(2^{n-1})}{3^2\ \cdot\ (3^{n-1})}$$

$$= \sum_{n=1}^{\infty} \left(\frac{-16}{9}\right)\left(\frac{(-2)^{n-1}}{3^{n-1}}\right)$$

$$= \sum_{n=1}^{\infty} \left(-\frac{16}{9}\right)\left(\frac{-2}{3}\right)^{n-1} = \sum_{n=1}^{\infty} a r^{n-1} = S$$

where $a = -\frac{16}{9}$, $r = -\frac{2}{3}$

The Series is Convergent since it is a GEOMETRIC Series with $-1 < r < 1$.

$$S = \frac{-16/9}{1-(-2/3)} = \frac{-16/9}{5/3} = -\frac{16}{9} \times \frac{3}{5} = -16/15 = S$$

$1 + 2/3$