1. Find all the complex numbers described by the following formulas and sketch their location in the complex plane.

a $(-i)^{2/3}$.

b $(\frac{1+i}{1-i})^{1/2}$.

c $(1)^{i}$.

d $log(-i/3)$.

2. Compute the following limits

a $\lim_{n\to\infty} \frac{n^2}{n^2+2n+3}$

b $\lim_{z\to -3i} \frac{z+3i}{z^2+9}$

c $\lim_{z\to 2} \frac{e^z}{z^2-4}$

d $\lim_{z\to 0} \frac{1-\cos z}{z^2}$.

3. Find the integral of the function $f(x, y) = 3x^2 + ixy$ along the contour formed by the straight line from $(-2, 1) = -2 + i$ to $(2, -1) = 2 - i$.

4. Let C be a simple closed contour in the plane. Use Green's theorem to show that $\int_C \bar{z}dz = 2iArea(Q)$. Here Q is the region bounded by C. Illustrate the theorem by letting C be the circle of radius R about the origin and computing both the contour integral and the area separately.

5 Compute the following integrals: explain in each case whether you use the properties of anti-derivatives, Cauchy's theorem, or one of the Cauchy integral formulas.

a $\displaystyle\int_C \frac{1}{z} dz$ where C is the unit semi-circle from $-i$ to i.

b $\displaystyle\int_C 2^z dz$ where C is the circle of radius 2 about 2 and $\arg 2 = 0$.

c $\displaystyle\int_C \frac{dz}{z^2 - 9}$ where C is the same circle as in part b).

d $\displaystyle\int_C \frac{\cos z}{(z-\pi)^3} dz$ where C is again the circle as in part b).

6.

T F Every function $f(x, y) = u(x, y) + iv(x, y)$, where u and v are defined and smooth in the plane, is analytic.

T F There is a function $f(z)$which is analytic in, for which $|f(e^{i\theta})| \leq 1$ and $f(0) = i + 1$.

T F If $f(z) = u(x, y) + iv(x, y)$ is analytic, then $v(x, y$ is harmonic.

T F If $f(z) = u(x, y) + iv(x, y)$ and u and v are harmonic, then f is analytic.

7. Prove the fundamental theorem of algebra: Every polynomial $P(z)$ of degree $n > 0$ can be factored as the product of n linear factors $z - z_i$ and a constant.

a Every polynomial $P(z)$ of degree $n > 0$ has at least one point z_0 where $P(z_0) = 0$. Assume this is false and find a contradiction by looking at the properties of $\frac{1}{P(z)}$.

b Prove that if $P(z_0) = 0$, then $\frac{P(z)}{z-z_0}$ has a removeable singularity at z_0.

c Now show that $\frac{P(z)}{z-z_0}$ is a polynomial of degree $n-1$. Either follow the suggestion in the book, or use the Cauchy integral formulas and the behavior of this function for large $|z|$.

d Finish the proof by using induction on the degree of the polynomial.

8. a Expand $\log \frac{1+z}{1-z}$ in a Taylor series around $z = 0$. Use the principle value of log for your computations.

b Expand $\frac{z+3}{z^2+z}$ in a Laurent series about $z = -1$.

9. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}$ for $a > 1$.

10. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}\,dx}{(1+x^2)(4+x^2)}$ for $a \geq 0$.

Extra Credit

11. Express the Cauchy Riemann equation in coordinates t, s, where $x = e^t \cosh s, y = e^t \sinh s$.Use an extra page if necessary.

12. Evaluate $\displaystyle\int_0^\infty \frac{x^a}{1+x^2}dx$, where $-1 < a < 1$.