Lecture 16, Oct 7, M 408C

Homework: 1, 3, 7, 9, 13, 15, 16

Exponential Growth and Decay

- There are equations where the value of a quantity, $y(t)$, is proportional to the change of y with respect to t.

$$\frac{dy}{dt} = ky$$

k is the proportionality constant.

- We have seen that if

$y(t) = y_0 e^{kt}$, y_0 is a constant

then $\frac{dy}{dt} = y_0 \cdot \underbrace{e^{kt} \cdot k}_{\text{chain-rule}} = k \cdot y$

- $y = y_0 e^{kt}$ is the only function that solves the equation

$$\frac{dy}{dt} = ky.$$

Today: Do word problems in this form.

Application: Population Growth

$$\frac{dP}{dt} = kP \quad , \quad P(t) = P_0 e^{kt}$$

P is population, t is time
k is called the relative growth rate and is a constant.

3.8 #2 'rephrased'

A cell divides into 2 cells every 20 minutes. The initial number of cells is 60.

a) Find the relative growth rate.
→ what is k? not immediatly given
→ what do we know?

$P(t=0) = P_0 e^{k \cdot 0} = P_0$

$P(t=0) = 60$

$P_0 = 60$

$P(t = 1/3) = P_0 e^{k \cdot \frac{1}{3}} = 60 e^{k/3}$

20 min = 1/3 hr

$P(t = 1/3) = 60 \cdot 2 = 120$

population doubles ... 20 min

$$60\, e^{k/3} = 120$$

$$e^{k/3} = 2$$

$$k/3 = \ln 2$$

$$k = 3\ln 2 = \ln 2^3 = \ln 8$$

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b) find an expression for the number of cells after t hours

$$P(t) = P_0 e^{kt} = 60\, e^{t \ln 8}$$

or $$= 60\, e^{\ln 8^t} = 60 \cdot 8^t$$

c) Find the number of cells after 8 hours

$$P(8) = 60 \cdot 8^8$$ use calculator

d) Find the rate of growth after 8 hours

$\frac{dP}{dt}$ at $t = 8$

$$\frac{dP}{dt} = kP \quad \text{(initial model)}$$

$$= P \ln 8$$

at $t = 8$

$$= P(8)\ln(8) = 60 \cdot 8^8 \cdot \ln 8$$

e) When will the population reach 20,000 cells?

$$P(t) = 60\, e^{t \ln 8} = 20{,}000$$

$$e^{t \ln 8} = \frac{20{,}000}{60}$$

$$t \ln 8 = \ln\left(\frac{20{,}000}{60}\right)$$

$$t = \frac{\ln\left(\frac{20{,}000}{60}\right)}{\ln(8)}.$$

Application: Radio active Decay

is modelled by

$$\frac{dm}{dt} = km \quad , \quad m(t) = m_0 e^{kt}$$

m is mass

t is time

The <u>half-life</u> is the time it takes for ½ of the substance to disapear.

3.8 # 8

Stronium - 90 has a half life of 28 days. We are given an initial concentration of 50 mg.

a) Find a formula for the mass after t days.

$$m(t) = m_0 e^{kt}$$

we know

$$m(0) = m_0 = 50$$

Also

$$m(28) = m_0 e^{k \cdot 28} = 50 e^{k \cdot 28}$$

$$m(28) = \frac{50}{2} = 25$$

↑ half-life

$$50 e^{k \cdot 28} = 25$$

$$e^{k \cdot 28} = \frac{25}{50} = \frac{1}{2}$$

$$k \cdot 28 = \ln\left(\tfrac{1}{2}\right) = \ln(2^{-1}) = -\ln(2)$$

$$k = -\ln(2)/28$$, k is always negative for radioactive decay problems!

$$\boxed{m(t) = 50 e^{-t \ln(2)/28}}$$

b) Find the mass after 40 days.

$$m(40) = 50\, e^{-40 \ln(2)/28} \quad \text{(use calculator)}$$

or

$$= 50\, e^{\ln 2^{-40/28}}$$

$$= 50 \cdot 2^{-40/28} = 50 \cdot 2^{-10/7} \quad \text{(use calculator)}$$

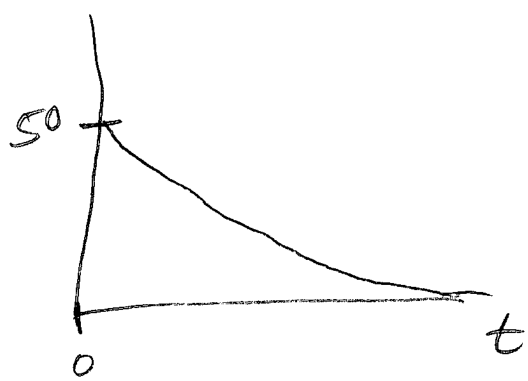
c) How long does it take to decay to 2mg?

$$m(t) = 2 = 50\, e^{-t \ln(2)/28}$$

$$\ln\left(\tfrac{2}{50}\right) = \ln\left(\tfrac{1}{25}\right) = -\ln(25) = -t \ln(2)/28$$

$$t = \frac{\ln(25)}{\ln(2)} \cdot 28$$

d) sketch graph of $m(t)$



Note: $m(0) = 50$

and $\lim_{t \to \infty} 50 e^{-t\ln(2)/28} = 0$

Temperature: Newtons Law of Cooling.

$$\frac{dT}{dt} = k(T - T_s)$$

T: temperature, t: time

T_s: temperature of surroundings. (assumed constant)

Not in for $\frac{dy}{dt} = ky$

Use a change in variable / substitution

Let $y = T - T_s$

then $\frac{dy}{dt} = \frac{dT}{dt} - \frac{dT_s}{dt} = \frac{dT}{dt} + 0 = \frac{dT}{dt}$

Plug into original equation:

$$\frac{dy}{dt} = ky.$$

So $y(t) = y_0 e^{kt}$

3.8 #14 'rephrased'

$T(a) = 32.5\,°C$

$T(a+1) = 30.3\,°C$

$T(0) = 37\,°C$

$T_s = 20\,°C$

a is unknown amount of time after the murder
This is at absolute time 1:30pm.

put in terms of y.

$y(a) = 32.5 - 20 = 12.5$

$y(a+1) = 30.3 - 20 = 10.3$

$y(0) = 37 - 20 = 17$

$y(t) = y_0 e^{kt}$ solve for k and y_0

$y(0) = y_0 = 17$

$y(a) = 17e^{ka} = 12.5$

$y(a+1) = 17e^{k(a+1)} = 10.3$

Gives Equations

① : $e^{ka} = \frac{12.5}{17}$ or $ka = \ln\left(\frac{12.5}{17}\right)$

② : $e^{k(a+1)} = \frac{10.3}{17}$ or $k(a+1) = \ln\left(\frac{10.3}{17}\right)$

or $ka + k = \ln\left(\frac{10.3}{17}\right)$

solve for k by:

Eqn 2 − Eqn 1

$= ka + k - ka = k$

$$k = \ln\left(\frac{10.3}{17}\right) - \ln\left(\frac{12.5}{17}\right)$$

solve for a by:

$\frac{\text{eqn } 1}{k}$: $\frac{k \cdot a}{k} = a$

$$a = \frac{\ln(12.5/17)}{\ln\left(\frac{10.3}{17}\right) - \ln\left(\frac{12.5}{17}\right)}$$

use calculator

$a \sim 1.59$ hours

~ 1 hour 35 min.

a is the time passed after death.
which is at 1:30 pm

death occured at 1:30pm − 1hr 35min

$\approx$ 11:55am