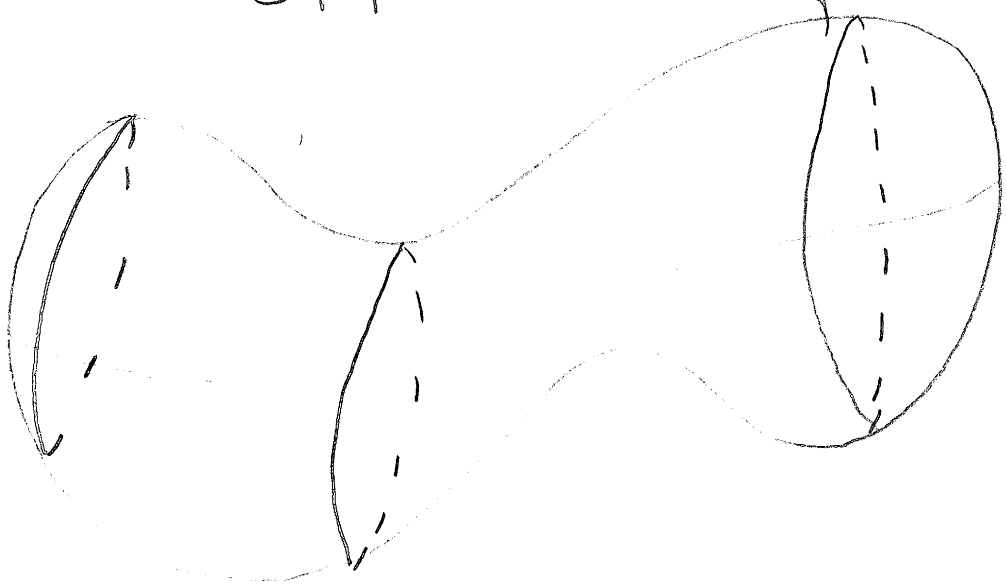
Lecture 33 M408C Nov 23^{rd} ①

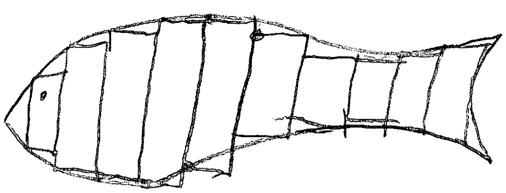
Homework: 6.2 # 1, 3, 7, 9, 11, 19, 23, 27, 47, 49

How can we use Calculus to compute volumes of 3D shapes?



An exact formula for an arbitrary shape is hard!

How Did we compute Areas of 2D shapes?

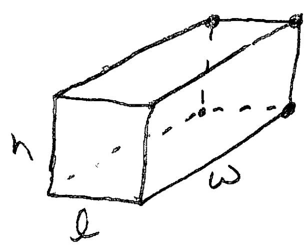


• We approximated by rectangles!

- We took a shape we knew the are of and fit it in there!

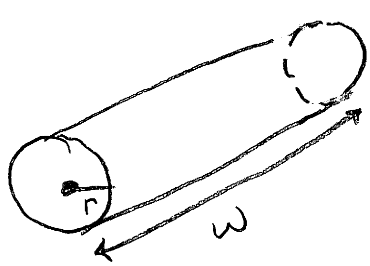
- What are some useful shapes in 3D?

Box:



$$V = \underbrace{l\,h}_{A} \cdot \underbrace{w}_{\Delta x}$$

Circular cylindar:



$$V = \underbrace{\pi r^2}_{A} \cdot \underbrace{w}_{\Delta x}$$

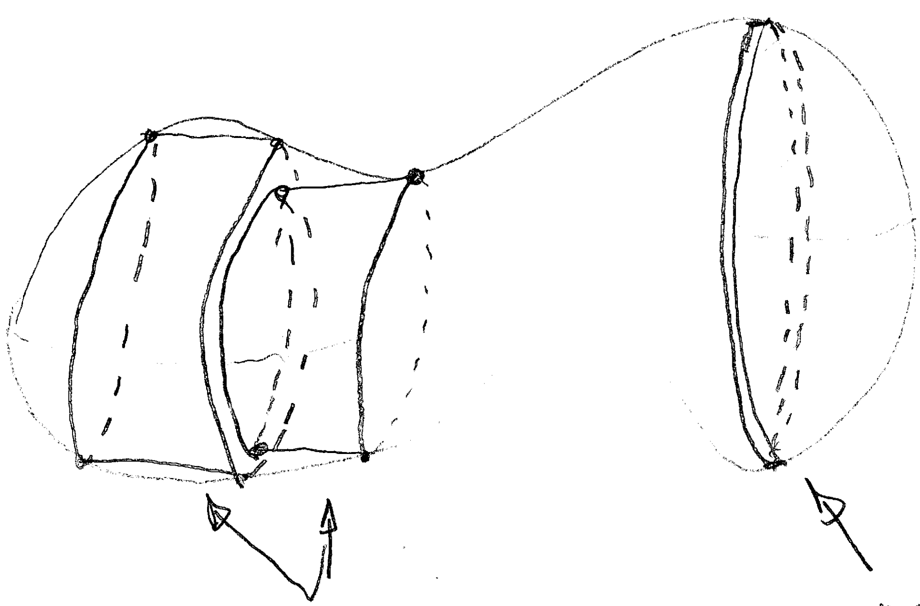
③

Idea:

Volume (Weird Shape)

= Sum (Volumes (Nice Shapes))

= Infinite sum (smaller & smaller Nice Shapes)



fit cylinders into 'weird' shape

make the cylinders super thin.

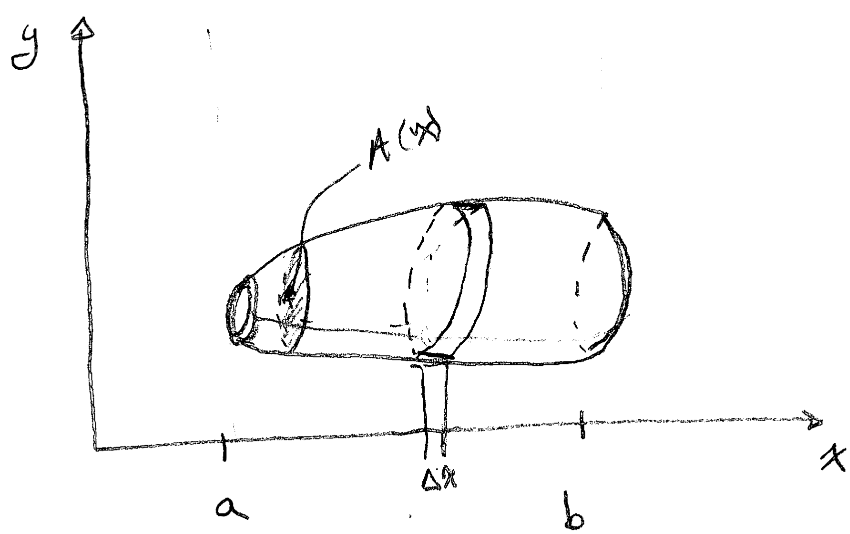
- Note: ~~these~~ super thin cylinders are like the cross-sectional area!

Volume:

$$V = \lim_{n \to \infty} \underbrace{\sum_{i=1}^{n} A(x_i)\,\Delta x}_{\text{the infinite sum of crossectional areas}} = \int_a^b A(x)\,dx$$

↑ the integral!

This Equation is for a solid between x=a and x=b.

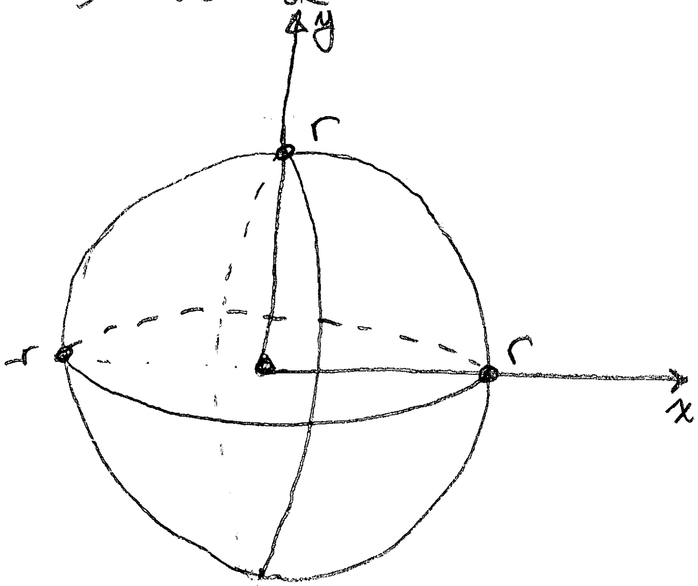


A(x) is the cross-sectional area and it varies as we change x!

Q: How can we use this to compute Volumes?

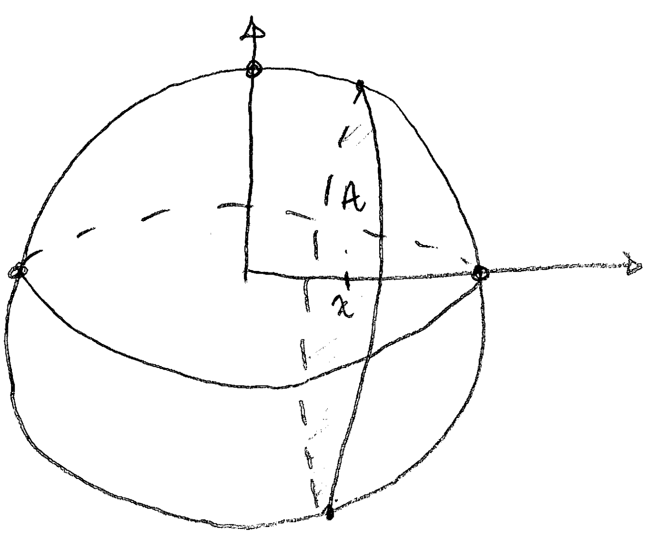
Ex: Show the Volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.

Draw a Picture!



Picture of sphere centered at the orgin!

Take a cross sectional slice in the x-direction



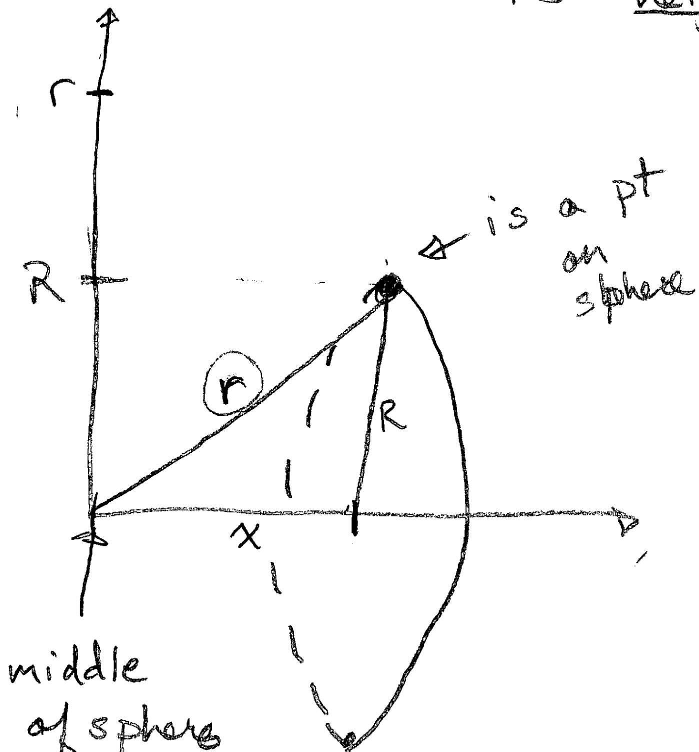
The cross-sectional slice is a circle!

The Eqn of Area

$$A = \pi R^2$$

what is R ? for the cross-section?

Note: it is height or y-value!



To Find R use the Pythagorean Thm!

$$R^2 + x^2 = r^2 \quad , \quad R^2 = r^2 - x^2$$

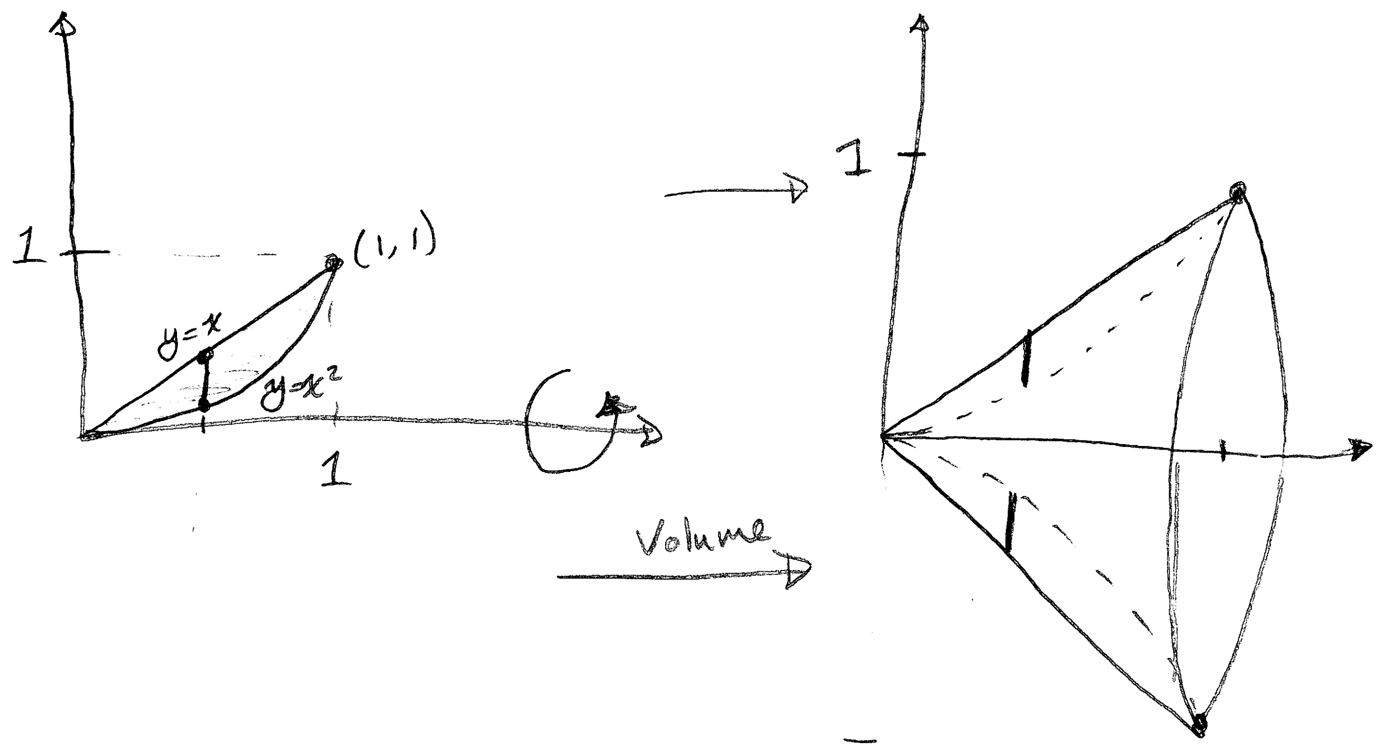
$$A = \pi r^2 - \pi x^2$$

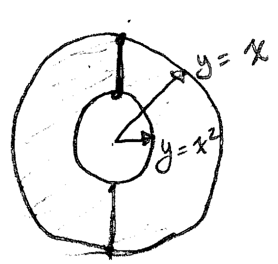
$$V = \int_{-r}^{r} \pi r^2 - \pi x^2 \, dx = \pi r^2 \cdot x - \frac{\pi}{3} x^3 \Big|_{-r}^{r} = \boxed{\frac{4}{3} \pi r^3}$$

Ex:

Find The Volume of the solid if you take the area enclosed by $y=x$, $y=x^2$ rotated about the x-axis.

Picture:





← Picture at cross section.

The Area of Washer:

$$A = \pi(r_o^2 - r_i^2)$$

The Difference in area of the outer and inner circles

$r_o = x \quad , \quad r_i = x^2$

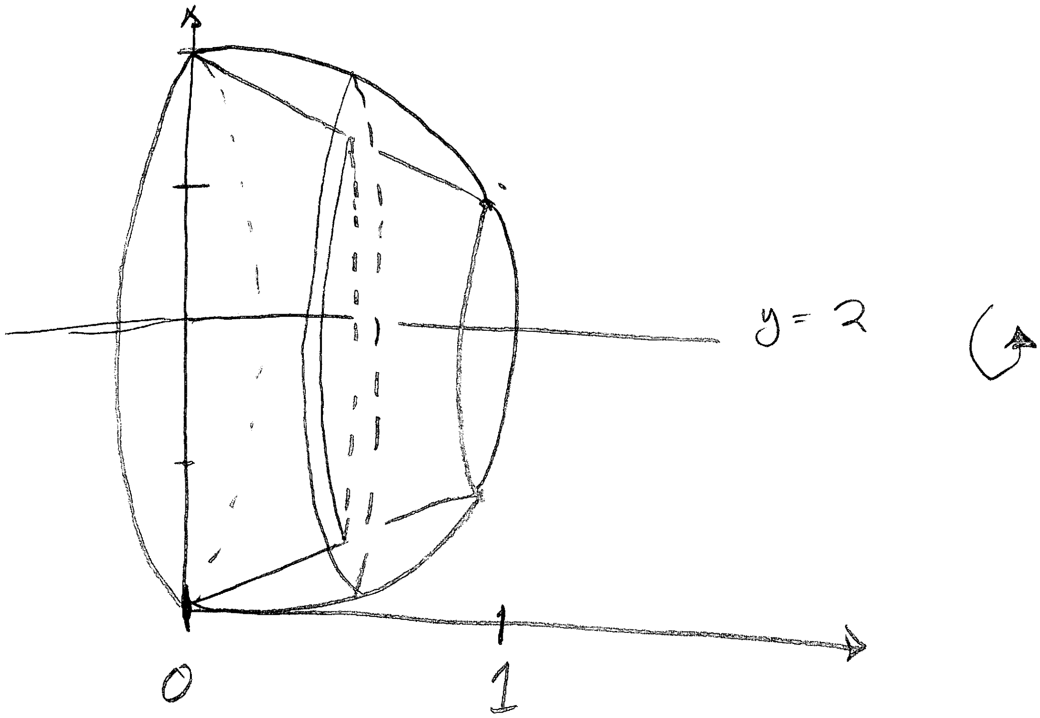
$A = \pi\left(x^2 - (x^2)^2\right)$

$$V = \int_0^1 \pi x^2 - \pi x^4 \, dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15}$$

x goes from 0 to 1

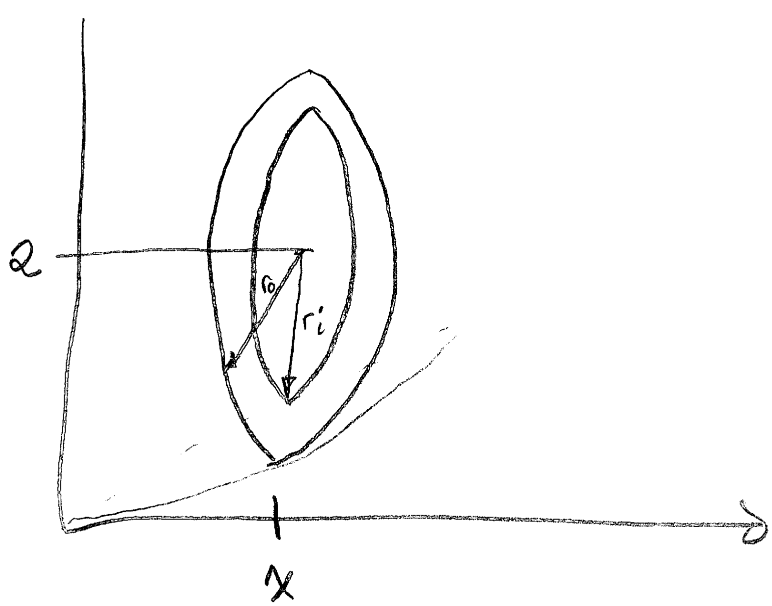
* The complicated part is setting up the integral!

What if instead we rotated about $y = 2$?

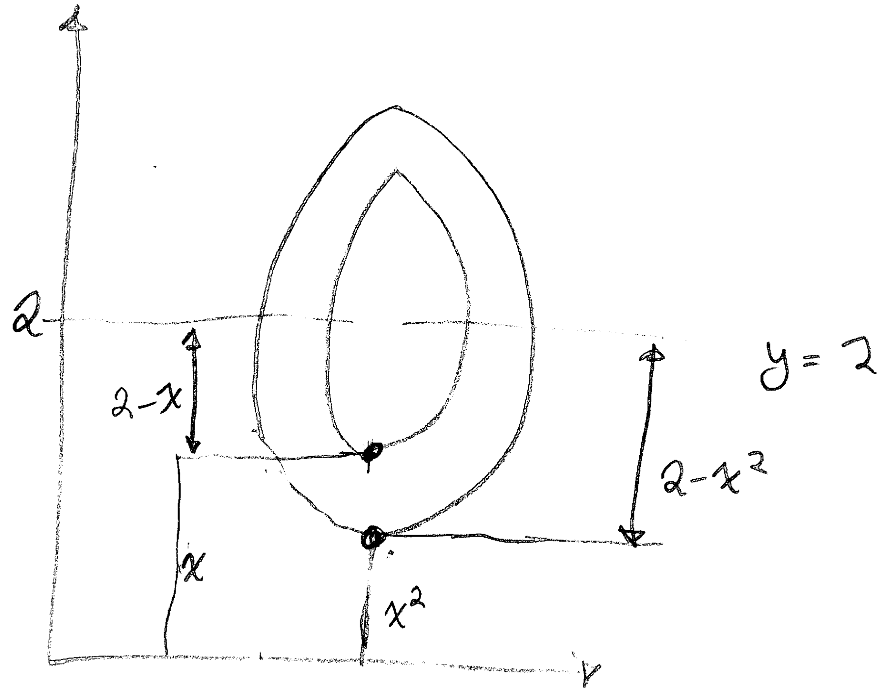


What does the cross-section look like?

A Washer!



What are r_i and r_o?



$r_i = (2-x)$ $\qquad$ $r_o = 2-x^2$

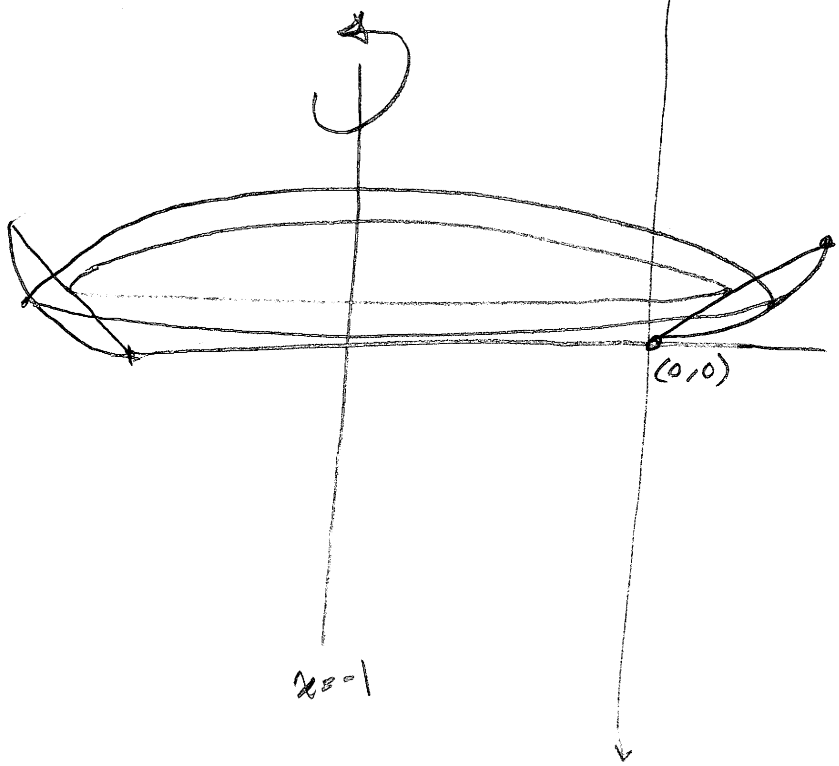
$A = \pi\left((2-x^2)^2 - (2-x)^2\right)$

$V = \int_0^1 A(x)\,dx$

$= \int_0^1 \pi\,(x^4 - 5x^2 + 4x)\,dx$

$= \pi\left(\frac{x^5}{5} - 5\,\frac{x^3}{3} + 4\,\frac{x^2}{2}\right)\Big|_0^1 = \frac{8\pi}{15}$

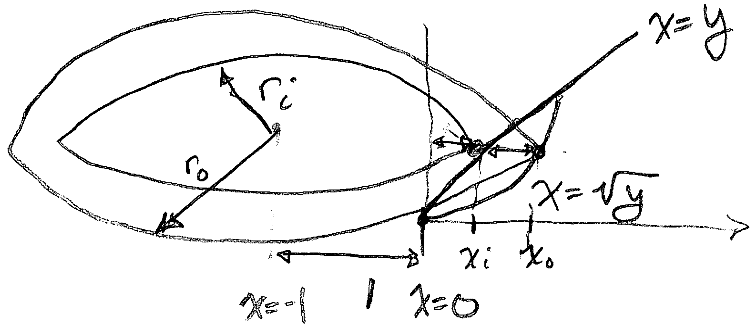
Rotated about $x=-1$?



Note: When we rotate about $x=-1$ our washers are stacked on top of each other.

$$V = \int_{y=0}^{1} A(y)\,dy,$$ Integrate in y-dir!

What is $A(y)$?



$$A = \pi (r_i^2 - r_o^2)$$

$$r_i = 1 + x_i = 1 + y$$

$$r_o = 1 + x_o = 1 + \sqrt{y}$$

$$A = \pi \left((1+y)^2 - (1+\sqrt{y})^2 \right)$$

$$V = \int_0^1 A(y)\,dy = \pi \int_0^1 2\sqrt{y} - y - y^2\,dy$$

$$= \pi \left[\frac{4}{3} y^{3/2} - \frac{y^2}{2} - \frac{y^3}{3} \right]_0^1$$

$$= \pi/2.$$