THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part I

Jan 10, 2014

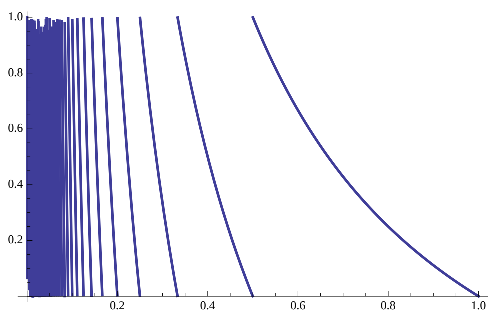
Problem 1 (35pts)**.** Let $T : [0,1] \to [0,1]$ be the **fractional-reciprocal map** given by

$$T(0) = 0 \text{ and } T(x) = \{1/x\} \text{ for } x \in (0,1],$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the **fractional part** of a. Show that the measure μ, given by $\mu(B) = \int_B \frac{1}{1+x}\, dx$ for $B \in \mathcal{B}([0,1])$, is preserved by T, i.e., that

$$\mu = T_* \mu,$$

where $T_*\mu$ denotes the pushforward of μ via T.



Problem 2 (35pts)**.** Let X_1 and X_2 be two independent unit normals and let $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ denote the rotation by the angle $\alpha \in [0, 2\pi)$.

(1) Show that the distribution of the random vector (X_1, X_2) is invariant under independent random rotations, i.e., that $T_\delta(X_1, X_2) \overset{(d)}{=} (X_1, X_2)$, where δ is a $[0, 2\pi)$-valued random variable, independent of (X_1, X_2)

(2) (*) Let γ be random variable, independent of (X_1, X_2) and uniformly distributed on $[0, 2\pi)$, and let ξ be a random variable measurable in $\sigma(X_1, X_2)$. Show that $T_{\gamma'}(X_1, X_2) \overset{(d)}{=} T_\gamma(X_1, X_2)$, where $\gamma' = \gamma + \xi \pmod{2\pi}$.

(3) (*) For a Borel function $f : [0, \infty) \to [0, 2\pi)$, determine the distribution of the random vector $(Y_1, Y_2) = T_{f(X_1^2 + X_2^2)}(X_1, X_2)$. (*Note:* In words, (Y_1, Y_2) is a rotated copy of (X_1, X_2), by the angle $\alpha = f(X_1^2 + X_2^2)$ which depends on (X_1, X_2) only through $X_1^2 + X_2^2$.)

(*Note:* Parts (2) and (3) - marked by (*) - are extra credit.)

Problem 3 (30pts)**.** For $X, Y \in \mathcal{L}^2(\mathcal{F})$ and a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, give detailed proofs of the following statements: (1) $X\mathbb{E}[Y|\mathcal{G}], Y\mathbb{E}[X|\mathcal{G}] \in \mathbb{L}^1$. (2) $\mathbb{E}[X\mathbb{E}[Y|\mathcal{G}]] = \mathbb{E}[Y\mathbb{E}[X|\mathcal{G}]]$.