Algebraic Topology Prelim, August 2022

Do all three questions. The questions are weighted equally.

1. Let M_1 and M_2 be copies of the Möbius band, and let n be a positive integer. Parametrizing ∂M_1 and ∂M_2 as S^1, the unit circle in the complex plane, let $f : \partial M_1 \to \partial M_2$ be the map $f(z) = z^n$. Let X be the space obtained from the disjoint union of M_1 and M_2 by gluing ∂M_1 to ∂M_2 by the map f. Compute the homology of X.

2. (a) Let X be the wedge of two circles. Describe two connected 2-fold covering spaces of X that are not homeomorphic.

(b) Let S be a closed orientable surface of positive genus. Show that any two connected 2-fold covering spaces of S are homeomorphic.

3. Let X and Y be topological spaces, and let $x_0 \in X$, $y_0 \in Y$. Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.