Algebra Preliminary Exam: Part II

January 14, 2023

Problem 1

Let p be an odd prime and $\mathbb{F}_p$ denote the field with p elements.

a) Let F be a finite extension of $\mathbb{F}_p$. Prove that the product of the non-zero elements of F equals -1.

b) Let ℓ be a prime. Determine the number of irreducible monic equations of degree ℓ over $\mathbb{F}_p$.

Problem 2

Let $\alpha = 1 + \sqrt[3]{2} + \sqrt[3]{4}$ and F be the Galois closure of $\mathbb{Q}(\alpha)$.

a) Determine the minimal polynomial of α over $\mathbb{Q}$

b) Determine the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ as a permutation group.

c) Determine all the intermediate fields of $F/\mathbb{Q}$.

d) Determine a primitive generator of $F/\mathbb{Q}$.

Problem 3

Consider $f(x) = x^4 + 7x + 7$.

a) Determine the degree of the splitting field of $f(x)$ over $\mathbb{F}_3$

b) Let F be the splitting field of $f(x)$ over $\mathbb{Q}$. Prove that $\mathrm{Gal}(F/\mathbb{Q}) \simeq \mathbf{S}_4$.

Recall: The discriminant of $x^n + ax + b$ is

$$(-1)^{n(n-1)/2}[(1-n)^{n-1}a^n + n^n b^{n-1}].$$