THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part I

Friday, Aug 23, 2019

Part I

Problem 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which an independent sequence of random variables $\{Y_n\}_{n\in\mathbb{N}_0}$ such that $\mathbb{P}[Y_n = \pm 1] = 1/2$ is defined. We set

$$X_n = \prod_{k=0}^{n} Y_k, \text{ and } \mathcal{F}_n = \sigma(X_k, k > n) \text{ for } n \in \mathbb{N}_0, \text{ as well as } \mathcal{C} = \sigma(Y_1, Y_2, \dots).$$

(1) Show that $\mathcal{D} = \cap_n \mathcal{F}_n$ is independent of each Y_k, $k \in \mathbb{N}_0$.
(2) Prove that $\bigcap_n (\mathcal{C} \vee \mathcal{F}_n) \neq \mathcal{C} \vee \Big(\cap_n \mathcal{F}_n \Big)$, where $\mathcal{A} \vee \mathcal{B} = \sigma(\mathcal{A} \cup \mathcal{B})$.

Problem 2.

(1) Let X and Y be two random variables taking values in $[0, 2\pi)$. Show that they must be equally distributed as soon as their characteristic functions agree on the set of all integers. (*Hint:* Use the Stone-Weierstrass theorem to approximate (uniformly) a continuous function by a trigonometric polynomial.)
(2) Let U and V be two independent random variables, uniformly distributed on $[0, 1)$. Given $m, n \in \mathbb{N}$, use (1) above to determine the distribution of $X = \{mU + nV\}$? (*Note:* $\{x\}$ denotes the fractional part of x, i.e., $\{x\} = x - \lfloor x \rfloor$ and $\lfloor x \rfloor$ is the largest integer $\leq x$.)

Problem 3. Construct two random variables $X \in \mathcal{L}^2$ and Y, defined on the same probability space, such that

(a) $\mathbb{E}[X|\sigma(Y)] = \mathbb{E}[X]$, a.s.,
(b) $\mathbb{E}[X^2|\sigma(Y)] = \mathbb{E}[X^2]$, a.s., but
(c) X and Y are not independent.

For partial credit, construct X and Y for which only (a) and (c) are required hold.