The University of Texas at Austin
Department of Mathematics

The Preliminary Examination in Probability
Part I

Thu, Jan 14, 2021

Problem 1. Let μ be a probability measure on $\mathcal{B}([0,\infty))$ (the Borel subsets of $[0,\infty)$) with the following property:

$$\mu([a,b]) = e^{-a} - e^{-b}, \text{ for all } 0 \le a < b.$$

Following the instructions below, show that μ is absolutely continuous with respect to the Lebesgue measure λ on $[0,\infty)$.

Instructions: Give a detailed proof, from first principles, with clear references to all theorems you are using. You are allowed to use the following without proof (but with a clear reference): basic facts and theorems from measure theory on general measurable spaces, as well as the fact that $\int \mathbf{1}_{[a,b]} e^{-x}\,\lambda(dx) = e^{-a} - e^{-b}$ for $a < b$ in the Lebesgue sense. In particular, you cannot use the notion of a derivative at all!

Problem 2. Let Y be a standard normal random variable, and let X be a random variable such that both pairs (X,Y) and $(X, X-Y)$ are independent. Show that X is constant with probability 1.

Problem 3. Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a simple symmetric random walk[1] and let $|X| = M + A$ be the Doob-Meyer decomposition of the submartingale $|X|$, with respect to filtration generated by X, into a martingale M with $M_0 = 0$ and a non-decreasing predictable process A. Show that M admits the representation[2]

$$M = H \cdot X, \tag{1}$$

for some predictable process H and find an explicit expression for H.

[1] $X_0 = 0$, $X_n = \sum_{k=1}^n \xi_k$, for $n \in \mathbb{N}$, where $\{\xi_n\}_{n\in\mathbb{N}}$ is an iid sequence with $\mathbb{P}[\xi_1 = -1] = \mathbb{P}[\xi_1 = 1] = \frac{1}{2}$.

[2] $H \cdot X$ denotes the martingale transform: $(H\cdot X)_0 = 0$ and $(H \cdot X)_n = \sum_{k=1}^n H_k(X_k - X_{k-1})$ for $n \ge 1$.