LYAPUNOV CENTER THEOREM FOR SOME NONLINEAR PDE's: A SIMPLE PROOF

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Abstract. We give a simple proof of existence of small oscillations in some nonlinear partial differential equations. The proof is based on the Lyapunov–Schmidt decomposition and the contraction mapping principle; the linear frequencies ω_j are assumed to satisfy a Diophantine type nonresonance condition (of the kind of the first Melnikov condition) slightly stronger than the usual one. If $\omega_j \sim j^d$ with $d > 1$, such Diophantine condition will be proved to have full measure in a sense specified below; if $d = 1$, we will prove that the condition is satisfied in a set of zero measure. Applications to nonlinear beam equations and to nonlinear wave equations with Dirichlet boundary condition are given. The result also applies to more general systems and boundary conditions (e.g. periodic).

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1. Introduction

In this paper we study the extension of the Lyapunov center theorem on the existence of nonlinear small oscillations to some partial differential equations. The result we are going to prove is not really new: even stronger results are known (see [1-6]), but our proof is new and so simple that we think it is of some interest.

Consider a finite dimensional Hamiltonian system having an elliptic equilibrium at 0; let ω_l be the frequencies of the linear oscillations about such an equilibrium, assume that $H(0) = 0$ and pick up one frequency, say ω_1; if

$$\omega_1 j - \omega_l \neq 0 \ , \quad j \geq 1, l \neq 1 \ , \tag{1.1}$$

then the Lyapunov center theorem ensures that, on each surface $H = \epsilon^2$ with small ϵ, there is a periodic orbit u_ϵ with frequency close to ω_1, smoothly dependent on ϵ, and $O(\epsilon^2)$ close

The paper is also available at http://www.mat.unimi.it/~bambusi to the linear mode with frequency ω_1. The proof is based on the implicit function theorem; condition (1.1) ensures that the eigenvalues of the linear operator to be inverted in order to apply the implicit function theorem, do not vanish.

In order to generalize this result to nonlinear PDE's (like nonlinear beam and wave equation, see eqs. (1.4), (1.5)) one is confronted with the problem that, since typically $\omega_l \sim l^d$ with some $d > 0$, the sequence formed by the quantities at l.h.s. of (1.1) has zero as an accumulation point. It follows that the linear operator whose eigenvalues are given by the l.h.s. of (1.1) is not surjective, and therefore the standard implicit function theorem cannot be applied. To overcome such problem KAM theory[1−3] or the Nash–Moser implicit function theorem[4−6] have been used. The results of [1-6] apply to systems whose linear frequencies fulfill a suitable nonresonance condition of Diophantine type; to discuss its generality, one can consider one parameter families of frequencies, it turns out that the nonresonance condition is fulfilled for values of the parameter forming a Cantor set of large measure. Existence of periodic orbits is then ensured for values of ϵ constituting a Cantor set of large measure.

In the present paper we give a proof of existence of small oscillations in some equations of the form

$$u_{tt} + Au = f(u) ,$$

where u belongs to a suitable Hilbert space, A is a selfadjoint, strictly positive, operator with pure point spectrum, and f is a nonlinear map vanishing at the origin. Our proof is based on the standard contraction mapping principle and holds when the linear frequencies fulfill a nonresonance condition slightly stronger than the usual one; its generality will be discussed shortly in this introduction and in detail in sect. 3.

We now present the main idea of our proof. We fix a frequency ω close to ω_1 and look for a periodic solution of frequency ω. We make a Lyapunov-Schmidt decomposition (see e.g. [7]), obtaining a 1-dimensional bifurcation equation on the Kernel K of the operator $L_{\omega_1} := \omega_1^2 \frac{d^2}{dt^2} + A$ (considered in an Hilbert space of 2π-periodic functions) and an infinite dimensional equation on $K^\perp$. This latter equation (which will be called the Q–equation) is the one where small denominators usually appear.

Our main point is that the linear operator obtained by linearizing at zero the Q–equation (i.e. $L_\omega|_{K^\perp}$) has eigenvalues given by

$$-\omega^2 j^2 + \omega_l^2 = (-\omega j + \omega_l)(\omega j + \omega_l) , \quad j \geq 0 , \quad (j, l) \neq (1, 1) \tag{1.2}$$

so that, if ω fulfills

$$|\omega j - \omega_l| \geq \frac{\gamma}{j} , \quad j \geq 1 , \quad l \geq 2 \tag{1.3}$$

(with some positive γ) then the quantities (1.2) lie outside a neighborhood of zero. Therefore one can hope to use the standard implicit function theorem to solve the Q equation. This remark is inspired by the trick used by R. De La LLave[8] to obtain a variational proof of existence of periodic orbits in some nonlinear wave equations. Then one proceeds more or less as in the finite dimensional case obtaining that, if ω is close enouth to ω_1, there exists a periodic orbit with frequency ω. To obtain small oscillations, i.e. a sequence of periodic orbits with frequencies tending to ω_1 and accumulating at the origin, we actually need a sequence of frequencies ω fulfilling (1.3) with a given γ and accumulating at ω_1. Its existence is the main assumption of our abstract theorem (see theorem 2.2).

A frequency vector $\Omega = (\omega_1, \omega_2, \omega_3, ...)$ such that there exists a sequence of ω's satisfying (1.3) with the same γ and converging to ω_1 will be said to have property γ-NR; this is fundamental for the applicability of our method. Then the problem is that of proving that such condition is non empty, and of clarifyinfg how general it is. To this end we assume $\omega_l \sim l^d$ and distinguish the cases $d > 1$ and $d = 1$. In the case $d > 1$ we assume that ω_l depends on a real parameter τ, and we will prove (under suitable conditions) that the property γ-NR is satisfied when the parameter τ belongs to a set of full measure. If $d = 1$ then the property γ-NR can be satisfied, but for a set of frequencies which, in some sense, is of zero measure.

In sect. 4 we will give some applications of our general result. Consider the nonlinear beam equation

$$u_{tt} + u_{xxxx} - \alpha u_{xx} + \beta u = \psi(u) \tag{1.4}$$

with Dirichlet boundary conditions on $[0, \pi]$ and α, β positive parameters; fix a linear mode, and one of the two parameters, e.g. $\alpha \geq 0$; then, if β belongs to a subset of $[0, \infty]$ of full measure the linear frequencies have the property γ–NR. If moreover the nonlinearity fulfills a nondegeneracy condition, automatic if ψ is an odd polynomial, then there exists a sequence of periodic orbits close to the considered linear mode and accumulating at zero.

An application to the nonlinear wave equation

$$u_{tt} - u_{xx} + mu = \psi(u) \ , \tag{1.5}$$

with Dirichlet boundary conditions on $[0, \pi]$ is also made. If m belongs to an uncountable subset of $\mathbb{R}$, then we will obtain the same result as for the nonlinear beam equation.

Our method can be easily generalized to other boundary conditions (e.g. periodic).

Finally we point out that the smoothness property needed for the nonlinear part ψ are much weaker than those required in [1-6] (see sect. 4), indeed it is enough to assume that it is C^4.

2. The abstract result

Fix a sequence $\Omega \equiv (\omega_1, \omega_2, \omega_3, ...)$ of positive numbers and consider the system of differential equations

$$\frac{d^2}{dt^2} u + Au = f^{(0)}(u) + f^{(1)}(u) \ , \tag{2.1}$$

where $u = (u_1, u_2, ...) \in \ell^2_s$ is a vector, A is a diagonal positive operator defined by $(Au)_i := \omega_i^2 u_i$, and $f^{(i)}$ are nonlinear operators. Here ℓ^2_s is the Hilbert space of the sequences such that

$$|u|^2_s := \sum_{k \geq 1} k^{2s} |u_k|^2 < \infty \ .$$

We fix the parameter s.

Concerning the nonlinearity we assume that its main part $f^{(0)}$ is a bounded homogeneous polynomial of degree r (see e.g. [9]), with some $r \geq 2$. Concerning the higher order part $f^{(1)}$ of the nonlinearity we assume that it vanishes at the origin, has a Lipschitz first derivative and satisfies the estimate

$$\sup_{\|u^i\| \leq 2\epsilon} \left\| Df^{(1)}(u^1) - Df^{(1)}(u^2) \right\|_{\ell^2_s \to \ell^2_s} \leq C_1 \epsilon^{r-1} \left| u^1 - u^2 \right|_s ,$$

where $Df^{(1)}$ denotes the derivative the map $f^{(1)}$. Remark that it follows that $f^{(1)}$ has a zero of order $r+1$ at the origin.

We will look for periodic solutions of (2.1) which are close to the first linear mode

$$\xi_1(\omega_1 t) := \cos(\omega_1 t) e_1$$

where $e_1 \in \ell^2_s$ is the vector (1,0,0,....). Analogously we will use the notation e_l for the vector having each component equal to 0 but the l-th which is equal to 1. We will also denote by Ω_c the sequence $(\omega_2, \omega_3, \omega_4, ...)$.

Remark 2.1. *We did not assume $\omega_l > \omega_1$ for $l > 1$, so the first mode is actually an arbitrary one.*

Definition. *The frequency ω will be said to be γ strongly nonresonant with Ω_c if the following inequality*

$$|\omega j - \omega_l| \geq \frac{\gamma}{j} \;, j \geq 1 \;, \quad l \neq 1 \tag{2.2}$$

holds.

Definition. PROPERTY γ-NR. *We will say that Ω has the property γ-NR if there exists a closed set $W_\gamma \subset \mathbb{R}$ such that any $\omega \in W_\gamma$ is γ strongly nonresonant with Ω_c and moreover ω_1 is an accumulation point of both $W_\gamma \cap (-\infty, \omega_1]$ and $W_\gamma \cap [\omega_1, +\infty)$.*

In sect. 3 we will discuss the generality of property γ–NR.

Theorem 2.2. *Consider system (2.1); assume that there exists a positive γ such that Ω has property γ–NR, and that $\omega_l \geq \gamma$, $\forall l \geq 1$; assume also that*

$$\beta_0 := \int_0^{2\pi} \langle f(\xi_1(t)); \xi_1(t) \rangle_{\ell^2} \, dt \neq 0 \;; \tag{2.3}$$

then there exist a strictly positive ω_*, a set $\mathcal{E} \subset \mathbb{R}$ having zero as an accumulation point, a 1-1 map $\mathcal{E} \ni \epsilon \mapsto \omega^\epsilon \in W_\gamma$, onto $W_\gamma \cap [\omega_1, \omega_1 + \omega_*)$, if $\beta_0 < 0$ or onto $W_\gamma \cap (\omega_1 - \omega_*, \omega_1]$, if $\beta_0 > 0$, and a family $\{u_\epsilon(t)\}_{\epsilon \in \mathcal{E}}$ of periodic solutions of (2.1), with the following properties

$$u_\epsilon(t + \frac{2\pi}{\omega_\epsilon}) = u_\epsilon(t) \;, \quad \left| u_\epsilon\left(\frac{t}{\omega^\epsilon}\right) - \epsilon \xi_1(t) \right|_s \leq C_2 \epsilon^r \;, \quad |\omega_1 - \omega^\epsilon| \leq C_3 \epsilon^{r-1} \;. \tag{2.4}$$

Moreover $u_\epsilon \in H^1([0, 2\pi/\omega^\epsilon], \ell^2_s)$.

Remark 2.3. *By the second of (2.4) u_ϵ has norm of order ϵ.*

Proof. First we introduce a suitable Hilbert space $\mathcal{H}$ of periodic functions. Let q have the representation

$$q(t) := \sum_{j\geq 0, l\geq 1} q_{jl}\cos(jt)e_l \tag{2.5}$$

Define $\mathcal{H} \subset H^1([0,2\pi],\ell^2_s)$ to be the space of all the function of the form (2.5) with

$$\|q\|^2 := \sum_{jl} |q_{jl}|^2(1+j^2)l^{2s} = \frac{1}{\pi}\int_0^{2\pi}\left(|q(t)|_s^2 + |\dot q(t)|_s^2\right)dt < \infty$$

Remark that $f^{(1)}$ induces a Lipschtiz map (Nemitski operator) $\mathcal{H} \ni q \mapsto f^{(1)}\circ q \in \mathcal{H}$ whose Lipschitz constant is of order ϵ^r in a neighborhood of order ϵ of the origin. The map defined analogously by $f^{(0)}$ is also regular since it is a bounded polynomial.

Fix $\omega \in W_\gamma$ close to ω_1, and look for periodic solutions of (2.1) with frequency ω of the form

$$u(t) = q(\omega t) \text{ with } q \in \mathcal{H}\ .$$

Define the operator L_ω by

$$L_\omega : \mathcal{H} \supset D(L_\omega) \to \mathcal{H}\ ; \qquad L_\omega q := \omega^2\frac{d^2}{dt^2}q + Aq\ ,$$

with $D(L_\omega)$ the maximal domain. Then any (weak) solution of the equation

$$L_\omega q = f^{(0)}(q) + f^{(1)}(q)\ , \qquad q \in \mathcal{H} \tag{2.6}$$

is a periodic solution with frequency ω of (2.1).

We will denote $K := Ker(L_{\omega_1}) = span\,\{\xi_1\}$, and $R := K^\perp$; correspondingly we will use the projectors $P : \mathcal{H} \to K$ and $Q = 1\!\!1 - P$. We look for solutions of (2.6) of the form

$$q = \epsilon\xi_1 + \epsilon^r q_\perp\ , \quad q_\perp \in R\ , \tag{2.7}$$

where ϵ is a (small) parameter that will be eventually determined as a function of ω. To this end we write

$$\omega^2 = \omega_1^2 + \beta\epsilon^{r-1} \tag{ω}$$

with a still undetermined β. Project (2.6) on K and R, using (ω) and (2.7) one obtains the system

$$-\beta\xi_1 = Pf^{(0)}(\xi_1) + P\left[f^{(0)}(\xi_1+\epsilon^{r-1}q_\perp) - f^{(0)}(\xi_1) + \frac{1}{\epsilon^r}f^{(1)}(\epsilon\xi_1+\epsilon^r q_\perp)\right] \tag{P}$$

$$L_\omega q_\perp = Qf^{(0)}(\xi_1) + Q\left[f^{(0)}(\xi_1+\epsilon^{r-1}q_\perp) - f^{(0)}(\xi_1) + \frac{1}{\epsilon^r}f^{(1)}(\epsilon\xi_1+\epsilon^r q_\perp)\right] \tag{Q}$$

It is useful to consider the system formed by the equations (P, Q, ω) for the unknowns $(\beta, q_\perp, \epsilon)$. It is clear that any solution of this system give rise to a periodic solution of (2.1), close to $\epsilon\xi_1$, with frequency ω.

We begin by studying the (Q)–equation. First we invert the operator $L_\omega|_R$. To this end remark that its eigenvalues λ_{jl} are given by

$$\lambda_{jl} := -j^2\omega^2 + \omega_l^2 = (-j\omega + \omega_l)(j\omega + \omega_l) \ , \quad j \geq 0 \ , \quad (j,l) \neq (1,1)$$

so that, if ω is γ strongly non resonant with Ω_c one has, for $(j,l) \neq (1,1)$ and $|\omega - \omega_1| < \omega_1/2$

$$|\lambda_{jl}| \geq \gamma \min\left\{ \frac{\omega_1}{2}, \min_l \omega_l \right\}$$

and therefore $L_\omega|_R$ has a bounded inverse L_ω^{-1}. Moreover one has $\left\| L_\omega^{-1} \right\| \leq \gamma C$. So we multiply (Q) by L_ω^{-1} and apply to the so obtained equation the implicit function theorem. We thus obtain that there exists a positive $\epsilon_\sharp$ such that, provided $0 \leq \epsilon < \epsilon_\sharp$, then there exists a Lipschtiz function $q_\perp^\omega(\epsilon)$ which solves (Q). Moreover it is easy to see that $\epsilon_\sharp$ depends on ω only through γ and that

$$q_\perp^\omega(\epsilon) = L_\omega^{-1} Q f^{(0)}(\xi_1) + O(\epsilon) \ , \qquad \sup_{\epsilon < \epsilon_\sharp} |q_\perp^\omega(\epsilon)|_s \leq C_4 \gamma^{-1}$$

with a C_4 independent of ω.

The (P) equation then defines

$$\beta := -\left[\beta_0 + \beta_1(\epsilon, q_\perp^\omega(\epsilon))\right] \tag{2.8}$$

where β_0 is given by (2.3) and

$$\beta_1(\epsilon, q_\perp^\omega(\epsilon)) := \left\langle \xi_1; \left[f^{(0)}(\xi_1 + \epsilon^{r-1} q_\perp^\omega(\epsilon)) - f^{(0)}(\xi_1) + \frac{1}{\epsilon^r} f^{(1)}(\epsilon\xi_1 + \epsilon^r q_\perp^\omega(\epsilon)) \right] \right\rangle_{\mathcal{H}}$$

and we denoted by $\langle .;. \rangle_{\mathcal{H}}$ the scalar product in $\mathcal{H}$. Remark that, for fixed $q_\perp$, the quantity β_1 tends to zero as $\epsilon \to 0$.

Finally, we need to solve eq. (ω). Recall that ω is fixed, so eq.(ω) is an equation for ϵ. Upon insertion of (2.8), it becomes

$$\omega^2 = \omega_1^2 - \epsilon^{r-1}(\beta_0 + \beta_1(\epsilon, q_\perp^\omega(\epsilon))) \ . \tag{2.9}$$

If the term β_1 were absent then (2.9) would have been a 1-1 relation between ϵ and ω, and therefore choosing ϵ in the set corresponding to strongly nonresonant ω's one would obtain the statement of theorem 2.2. To deal with the true case $\beta_1 \neq 0$ we use the contraction mapping principle, so, we rewrite (2.9) as

$$\mu = \frac{\omega_1^2 - \omega^2}{\beta_0} - \mu \frac{\beta_1(\mu^{1/(r-1)}, q_\perp^\omega(\mu^{1/(r-1)}))}{\beta_0} \ ,$$

with $\mu = \epsilon^{r-1}$. Fix a positive δ, then provided ω is close enough to ω_1 and μ and δ are small enough, the r.h.s. is a contraction of a ball centered at $(\omega_1^2 - \omega^2)/\beta_0$ having radius of order $((\omega_1^2 - \omega^2)/\beta_0)\delta$, and therefore, for any fixed ω there exists e *unique* $\mu = \epsilon^{r-1}$ fulfilling such an equation and the estimate

$$\left| \epsilon^{r-1} - \frac{\omega_1^2 - \omega^2}{\beta_0} \right| \leq C_5 \left(\frac{\omega_1^2 - \omega^2}{\beta_0} \right) \delta$$

with a suitable C_5. $\square$

3. On property $\gamma-$NR

To study the generality of our Diophantine type nonresonance condition we assume that $\omega_j \sim j^d$ with some $d \geq 1$. We will distinguish the cases $d > 1$ and $d = 1$.

3.1 The case $d > 1$

Assume that the frequency vector $\Omega = \Omega(\tau)$ depends in a differentiable way on a real parameter $\tau \in [\tau_m, \tau_M]$. We will assume that the following properties are fulfilled with some $d > 1$ and for all τ in the considered interval

$$\begin{aligned} \omega_l(\tau) \geq a l^d \ , \ |\omega_l'(\tau)| \leq \frac{c_2}{l^d} \ , \qquad \forall l \geq 2 \ , \\ \omega_1(\tau) \in [\omega_m, \omega_M] \ , \quad \omega_1'(\tau) \geq c_1 \end{aligned} \tag{3.1}$$

with some strictly positive $c_1, c_2, \omega_m, \omega_M, a$. Here $' \equiv \frac{d}{d\tau}$.

Define the sets

$$\mathcal{N}_{jl} := \{\tau \ : \ j\omega_1(\tau) - \omega_l(\tau) = 0\} \ ;$$

we assume

$$|\mathcal{N}_{jl}| = 0 \ , \quad \forall (j,l) \in N_+^2 \ , \tag{3.2}$$

where $|\mathcal{A}|$ denotes the Lebesgue measure of the set $\mathcal{A}$.

Finally we denote

$$\mathcal{A}_\gamma := \{\tau \in [\tau_m, \tau_M] \ : \ \Omega(\tau) \text{ has property } \gamma - \text{NR}\} \ .$$

Remark 3.1. *Property (3.2) must be verified only for a finite number of values of j, l, indeed, for large j, l it is a consequence of (3.1).*

Theorem 3.2. *Assume (3.1) and (3.2) then*

$$|\mathcal{A}_\gamma| \stackrel{\gamma \to 0}{\longrightarrow} [\tau_M - \tau_m] \ .$$

The proof of this theorem will be split into a few lemmas and will occupy the rest of this subsection.

We will use the notation

$$\mathcal{R}_{jl}(\gamma) := \left\{ \tau \ : \ |\omega_1(\tau) j - \omega_l(\tau)| < \frac{\gamma}{j} \right\}$$

Our first task will be to evaluate

$$\left| \bigcup_{\substack{j \geq 1 \\ l \geq 2}} \mathcal{R}_{jl}(\gamma) \right|$$

Remark 3.3. *For γ small enough $\mathcal{R}_{j,1}(\gamma)$ and $\mathcal{R}_{0,l}(\gamma)$ are empty.*

Lemma 3.4. *Assume that (3.1) holds and define*

$$l_*(j) := \left(\frac{2c_2}{c_1}\frac{1}{j}\right)^{1/d}$$

then, there exists a positive C^* such that, provided γ is small enough one has

$$\left|\bigcup_{\substack{l\geq l_*(j)\\ j\geq 1}} \mathcal{R}_{jl}\right| < C^*\gamma \ .$$

Proof. First remark that $\mathcal{R}_{jl}(\gamma)\neq\emptyset$ implies

$$\omega_l < \frac{\gamma}{j} + \omega_1 j \ ,$$

which, assuming $\omega_M/2 > \gamma$ implies

$$l < \left(\frac{\omega_M}{2a} j\right)^{1/d} \tag{3.3}$$

Secondly, for fixed j, l we have

$$\frac{d}{d\tau}(\omega_1 j - \omega_l) > c_1 j - \frac{c_2}{l^d} \ .$$

Assume $l \geq l_*(j)$ then the r.h.s. is larger than $c_1 j/2$. It follows that for such values of l and j the quantity $\omega_1(\tau)j - \omega_l(\tau)$ has at most one zero τ_{jl}. For $\tau > \tau_{jl}$ one has

$$\omega_1(\tau)j - \omega_l(\tau) = \int_{\tau_{jl}}^{\tau} \frac{d}{d\tau}\left[\omega_1(\tau)j - \omega_l(\tau)\right] d\tau \geq (\tau - \tau_{jl})\frac{c_1}{2}j$$

working out an analogous inequality for $\tau < \tau_{jl}$ we easily get that

$$|\mathcal{R}_{jl}(\gamma)| < \frac{\gamma}{j^2}\frac{4}{c_1} \ .$$

From this and from the limitation (3.3) one immediately gets

$$\left|\bigcup_{l\geq l_*(j)} \mathcal{R}_{jl}(\gamma)\right| \leq \frac{4}{c_1}\frac{\gamma}{j^2}\left(\frac{\omega_M j}{2a}\right)^{1/d} \ ,$$

from which, summing over j the thesis follows. $\square$

Lemma 3.5. *Define*

$$\tilde{\mathcal{A}}_\gamma := [\tau_M, \tau_m] \setminus \left[\bigcup_{\substack{j\geq 1\\ l\geq 2}} \mathcal{R}_{jl} \right] \tag{3.4}$$

then one has

$$|\tilde{\mathcal{A}}_\gamma| \stackrel{\gamma\to 0}{\longrightarrow} [\tau_M - \tau_m] \ . \tag{3.5}$$

Proof. We have to show that the measure of the union of the remaining $\mathcal{R}_{jl}(\gamma)$ (i.e. $l < l_*(j)$) tends to zero as $\gamma \to 0$. To this end remark that such sets are a finitely many. Indeed if j is larger than some J_* then $l_*(j) < 2$ and therefore the preceding lemma gives the estimates of all the considered sets. We prove that

$$\bigcap_{\gamma>0} \left[\bigcup_{\substack{l\leq l_*(j)\\ j\leq J_*}} \mathcal{R}_{jl}(\gamma) \right] = \bigcup_{\substack{l\leq l_*(j)\\ j\leq J_*}} \mathcal{N}_{jl} \ , \tag{3.6}$$

which is a finite union of sets with zero measure and therefore has zero measure. To prove (3.6) just remark that given any τ outside the set at r.h.s. of (3.6) one has

$$\min_{\substack{l\leq l_*(j)\\ j\leq J_*}} j \left|\omega_1(\tau) j - \omega_l(\tau)\right| = \gamma > 0$$

and therefore such a τ is outside of $\bigcup_{\substack{l\leq l_*(j)\\ j\leq J_*}} \mathcal{R}_{jl}(\gamma)$, and therefore outside of the l.h.s. of (3.6). So, the thesis follows. □

Lemma 3.6. *Fix $\Omega_c \equiv (\omega_2, \omega_3, \omega_4, ...)$, and let $\omega_1 \in [\omega_m, \omega_M]$ be γ strongly nonresonant with Ω_c. Let $\tilde{\omega}_l$ be a sequence fulfilling $\tilde{\omega}_l \geq a l^d$, $l \geq 2$ and*

$$|\omega_l - \tilde{\omega}_l| < \frac{c_\nu}{l^d} \ , \qquad l \geq 2$$

with $c_\nu < \omega_m \gamma / 2a$; then ω_1 is $\tilde{\gamma}$ nonresonant with $\tilde{\Omega}_c \equiv (\tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4, ...)$. Here

$$\tilde{\gamma} := \min\left\{ \frac{\omega_m}{2}, \gamma - \frac{2ac_\nu}{\omega_m} \right\} \ .$$

Proof. First remark that provided $\tilde{\gamma} \leq \omega_m/2$ the inequality $|j\omega_1 - \tilde{\omega}_l| < \tilde{\gamma}/j$ implies $l^d \geq j\omega_m/2a$. So, that $l^d \leq j\omega_m/2a \Longrightarrow \omega_1 \notin \mathcal{R}_{jl}(\tilde{\gamma})$. So we study only the case $l^d \geq j\omega_m/2a$. One has

$$|j\omega_1 - \tilde{\omega}_l| \geq |j\omega_1 - \omega_l| \geq \frac{\gamma}{j} - \frac{c_\nu}{l^d} \geq \frac{\gamma}{j} - \frac{2ac_\nu}{\omega_m j} \ .$$

□

Proof of theorem 3.2. We extract now the subset $\mathcal{A}_\gamma$ of $\tilde{\mathcal{A}}_\gamma$ formed by the points which are accumulation points both from the right and from the left.

Define

$$\mathcal{B}_1 := \left\{\tau \in \tilde{\mathcal{A}}_\gamma \ : \ \exists \epsilon_\tau > 0 \ s.t. \ (\tau, \tau+\epsilon_\tau) \cap \tilde{\mathcal{A}}_\gamma = \emptyset\right\}$$
$$\mathcal{B}_2 := \left\{\tau \in \tilde{\mathcal{A}}_\gamma \ : \ \exists \epsilon_\tau > 0 \ s.t. \ (\tau-\epsilon_\tau, \tau) \cap \tilde{\mathcal{A}}_\gamma = \emptyset\right\}$$

and $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2$. Then $\tilde{\mathcal{A}}_\gamma$ is the disjoint union of $\mathcal{B}$ and a set $\mathcal{A}_\gamma$ composed by points which are accumulation points both from the right and from the left. We choose the set $\mathcal{A}_\gamma$ of the statement of the theorem to be the one just defined. It is easy to see that $\mathcal{B}_i$ is at most numerable, so that $|\mathcal{A}_\gamma| = \left|\tilde{\mathcal{A}}_\gamma\right|$. Indeed, consider $\mathcal{B}_1$ and fix $\tau \in \mathcal{B}_1$, choose a rational number in $(\tau, \tau+\epsilon_\tau)$; this gives a biunivocal correspondence between $\mathcal{B}_1$ and a subset of the rational numbers.

It remains to prove that for $\tau \in \mathcal{A}_\gamma$ the frequency $\omega_1(\tau)$ is the accumulation point of a sequence of frequencies which are $\tilde{\gamma}$ nonresonant with $\Omega_c(\tau)$. But this is a trivial consequence of lemma 3.6. Indeed, let $\tau_k \in \mathcal{A}_\gamma$ be a sequence converging to τ. It follows that $\omega_1(\tau_k)$ is γ strongly nonresonant with $\Omega_c(\tau_k)$ and that $|\omega_l(\tau_k) - \omega_l(\tau)| \leq c_2 \, |\tau_k - \tau| / l^d$, so, for k large enough $\omega_1(\tau_k)$ is also $\tilde{\gamma}$ strongly nonresonant with $\Omega_c(\tau)$ (with any $\tilde{\gamma} < \gamma$). So, $\omega_1(\tau_k)$ is the wonted sequence of frequencies accumulating at $\omega_1(\tau)$. Remark also that, since $c_1 \neq 0$, if $\tau_k > \tau$ then also $\omega_1(\tau_k) > \omega_1(\tau)$ and vice-versa. $\square$

Corollary 3.7. *Let $\omega_l(\alpha,\beta) := \sqrt{l^4 + \alpha l^2 + \beta}$ be the square roots of the Dirichlet eigenvalues of $\partial_{xxxx} - \alpha\partial_{xx} + \beta$; fix $\alpha \geq 0$ then $\Omega(\alpha,\beta) \equiv (\omega_1(\alpha,\beta), \omega_2(\alpha,\beta), \omega_3(\alpha,\beta), ...)$ has the property γ–NR with some γ, for β in a subset of $[0,+\infty]$ having full measure.*

3.2 The case $d = 1$

We will use the continued fraction expansion of ω. Correspondingly we will use the standard notation $\omega = [a_0, a_1, a_2, ...]$ with a_i non negative integers to mean

$$\omega = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + ...}} .$$

Proposition 3.8. *Let $\omega_1 = a_0 + \alpha = [a_0, a_1, a_2, ...]$ with $0 < \alpha < 1$ and $\omega_1 \geq 1/2$; let $\{\omega_l\}_{l\geq 2}$ be a sequence of positive numbers satisfying*

$$|\omega_l - l| \leq \frac{c_\nu}{l} \quad l \geq 2$$

assume that the sequence a_i has infinitely many nonvanishing elements and is bounded; define $\gamma_1 := \inf_{i\geq 2} (a_i + 2)^{-1}$ assume also $c_\nu < \gamma_1/4$ and $3\alpha > c_\nu$, then there exists a positive γ such that Ω has the property γ-NR. Moreover, there is family of frequencies γ strongly nonresonant with Ω_c, accumulating at ω_1 and having the power of the continuum.

Proof. We begin by proving that one has

$$|j\omega_1 - l| \geq \frac{\gamma_1}{j} \qquad \text{for all } l \neq a_0 j \ . \tag{3.7}$$

This is a consequence of the following two facts (i) if $|j\omega_1 - l| < \frac{1}{2j}$ then l/j is a convergent of ω_1 (see [10, Th.5C]). (ii) if l_n/j_n is the n-th convergent of ω_1 then one has

$$|j_n\omega_1 - l_n| \geq \frac{1}{j_n} \inf_{i\geq n+1} \frac{1}{a_i + 2}$$

(see [10, p.23]). Then an argument equal to that of lemma 3.6 shows that

$$|j\omega_1 - \omega_l| \geq \left(\gamma_1 - \frac{c_\nu}{4}\right)\frac{1}{j} \ , \qquad l \neq a_0 j \ .$$

Then we study the case $l = a_0 j$. If $a_0 = 0$ then $l = 0$ is not allowed. Assume $a_0 \geq 1$, one has

$$|\omega_1 j - \omega_{a_0 j}| \geq \alpha j - \frac{c_\nu}{a_0 j} = \left(\alpha j^2 - \frac{c_\nu}{a_0}\right)\frac{1}{j} \ ;$$

if $a_0 \neq 1$ then it follows that ω_1 is γ strongly nonresonant with Ω_c. If $a_0 = 1$ then $j = 1$ is not allowed, so the same conclusion holds. Then remark that the frequencies ω of the form $1/2 < \omega = \omega_1 + \alpha_1 = [a_0, b_1, b_2, ...]$ with $|\alpha_1| < \min\{\alpha/4, 1-\alpha\}$ and $4c_\nu < \gamma_2 := \inf_{i\geq 2}(b_i+2)^{-1} < \infty$ accumulate at ω_1 and are $\tilde{\gamma}$ nonresonant with Ω_c with a suitable $\tilde{\gamma}$. □

Corollary 3.9. *Define $\omega_l := \sqrt{l^2 + m}$ to be the square roots of the Dirichlet eigenvalues of $-\partial_{xx} + m$ on $[0,\pi]$ then property γ-NR holds for m belonging to a non–numerable subset of $\mathbb{R}$. Moreover for any m in such a set, the family of ω's which are γ–strongly nonresonant with Ω_c and accumulate at ω_1, is not countable.*

A simple generalization of the above corollary is

Corollary 3.10. *Let $V_0 \in L^2[0,\pi]$ be a function with zero average. Denote by $\lambda_i = \lambda_i(\mu, m, V_0)$ the Dirichlet eigenvalues of $-\partial_{xx} + m^2 + \mu V_0$ on $[0,\pi]$. Then there exists an uncountable set $\mathcal{A} \subset \mathbb{R}$ and a constant $0 \leq C(V_0) < \infty$, such that, if $m \in \mathcal{A}$, and $C(V_0)|\mu| < m^2$, the frequencies $\omega_i := \sqrt{\lambda_i}$ fulfill property γ-NR with some γ.*

More general examples can be obtained taking into account known results of Sturm–Liouville theory [11].

4. Applications to nonlinear beam and wave equations

We are now ready to prove existence of small oscillations in nonlinear beam and wave equations. We assume that the nonlinearity has the form

$$\psi(u) = \psi_0 u^r + \psi^{(1)}(u) \ ,$$

with some $r \geq 2$ and $\psi^{(1)}$ which admits two Lipscitz derivatives, vanish together with its first derivative at zero and fulfills the inequality

$$\sup_{|x|<2\epsilon, |y|<2\epsilon} \left| \left(\psi^{(1)}\right)''(x) - \left(\psi^{(1)}\right)''(y) \right| \leq C_6 \epsilon^{r-2} |x-y| \ ,$$

from which in particular it follows that it has a zero of order $r+1$ at zero.

To apply theorem 2.2 we fix the parameters α, β for the nonlinear beam equation or m for the nonlinear wave equation in such a way that the linear frequencies have the property γ-NR with some γ. This is possible in view of corollaries 3.7 and 3.9.

It easy to verify that the nondegeneracy condition (2.3) holds *provided* r is odd. The smoothness assumptions are fulfilled fixing the index s of the space to be 1.

Remark 4.1. *If ψ is C^4 and fulfills*

$$\psi(0) = \psi'(0) = \psi''(0) = 0 \ , \quad \psi'''(0) \neq 0 \ .$$

all the assumptions on the nonlinearity are fulfilled.

We thus have the following theorem which holds identically for the beam and for the wave equations.

Theorem 4.2. *Under the above assumptions there exists a family $\{u_\epsilon\}_{\epsilon\in\mathcal{E}}$ of periodic solutions of the considered equation; moreover $\mathcal{E}$ has an accumulation point at zero, u_ϵ has frequency ω^ϵ which is γ strongly nonresonant with Ω_c, and the following inequalities hold*

$$|\omega^\epsilon - \omega_1| \leq C_7 \epsilon^{r-1} \ , \qquad \left\| u_\epsilon\left(\frac{t}{\omega^\epsilon}\right) - \epsilon\xi_1(t) \right\|_{H^1} \leq C_8 \epsilon^r$$

where $\xi_1(t) := \sin x \cos(t)$. Moreover one has $u_\epsilon \in H^1([0, 2\pi/\omega^\epsilon]; H^1[0,\pi])$

Remark 4.3. *In the case of the nonlinear wave equation we can ensure that there are uncountably many periodic orbits (see corollary 3.9).*

Remark 4.4. *In the case of the nonlinear beam equation the same result holds for all the other modes of oscillations, namely when ξ_1 is substituted by*

$$\xi_k(t) = \sin(kx)\cos(t) \ .$$

For the case of the nonlinear wave equation, corollary 3.9 applyes only when the frequency vector is orderd (i.e. $\omega_i < \omega_j$ for $i < j$). To prove existence of higher normal modes one should show that there exist some values of m corresponding to which the frequency vector with ω_k interchanged with ω_1 has property $\gamma-NR$. We did not try to obtain such result.

Remark 4.5. *The above theorem can be easily generalized to periodic boundary conditions and to nonlinearities which depend explicitly on the space variable.*

Remark 4.6. *The function $\psi(u)$ in (1.4) and in (1.5) does not need to be odd. However, if it is odd and has $k+1$ Lipschitz derivatives, then one can easily prove that $u_\epsilon(t) \in H^k([0,\pi])$.*

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