Statistics and error terms of occurrence times in mixing processes

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Abstract

We study the statistics of visits to a set with small measure in a stochastic process with different kinds of mixing properties. We prove an upper bound for the error when approximating the first return time law to a small set, with a mixture of an exponential law and a degenerated law at the origin. We also prove that once the process enter to the small set the law of the sojourn time in it can be approximated by a geometric law with parameter that depends only on the recurrence properties of the event. We also provide an expression for the error on this approximation. Finally, we prove an error term in the approximation of the number of visits in whole process to a small set that does not recur, to the Poisson law. We obtain also an approximation for the moments of this number of visits.

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1 Introduction.

This paper deals with the statistics of occurrence times of a long string of symbols in a mixing stochastic process. We prove an upper bound for the difference between the law of the first time the process returns to an n-string for large n and a law that is a mixture of an exponential law with a Dirac measure at the origin. We also prove an upper bound for the difference between the law of the number of adjacent repetitions of the n-string and a geometric law. These results hold for *all* finite strings. Finally, when the string does not recur very fast, we prove an upper bound for the difference between the law of the number of visits to the string and a Poisson law. Our results stand for ψ-mixing, ϕ-mixing and strong or α-mixing processes (see definitions below), each one with its corresponding error.

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We recall that those are a wide family of processes. For example an irreducible and aperiodic finite state Markov chain is ψ-mixing with exponential decay. Moreover, Gibbs states with a potential with exponential variations are exponentially ψ-mixing. See Bowen (1975) for definitions and properties. Also, chains of infinite order as defined in Fernández, Ferrari and Galves (2001) are exponentially ψ-mixing. We refer to Doukhan (1995) for examples and references of α-mixing processes that are not ϕ-mixing, ϕ-mixing processes that are not ψ-mixing and for ψ-mixing decaying at any rate.

Our interest is to describe the statistic of occurrence times of an untypical situation. The study of the time elapsed until the first occurrence of a such situation in non independent stochastic processes is a classical subject. The typical result is that when the process loses memory fast enough the law of the first visit to a rare event, suitably re-scaled, converges to an exponential law as the set becomes rare. The list of results of this kind for different classes of processes and events is large. The pioneer paper in this area is Döeblin (1940), who studied the Poisson approximation for the Gauss transformation. In the context of Markov chains, the convergence of the occurrence time of a rare event to the exponential law was first studied by Bellmann and Harris (1951) and Harris (1953). Since there, the list of results of this kind for different types of processes and events increases considerably. So, the typical result is

$$\lim_{n\to\infty} I\!P\left\{\tau_{A_n} > t\ b_n\right\} = e^{-t} \ , \tag{1}$$

where τ_{A_n} is the first time the process *hits* a measurable set A_n such that its measure $I\!P\left\{A_n\right\} \to 0$ as $n \to \infty$ and $\{b_n\}_{n\in I\!N}$ is a suitable rescaling sequence of positive numbers. We refer the reader to Galves and Schmitt (1997) for results of this kind before 1997. For a detailed review on the latest results that provide upper bounds for the rate of convergence of the limit (1) in the context of mixing processes we refer to Abadi and Galves (2001).

We want to describe the point process of occurrences of a rare event. Before to aboard this question we must differentiate three situations that appear in this point process. The first *hitting* time to an event that was just described. The first *return* time to it and then the *sojourn* time into the event.

For the problem not just of hitting time but rather *return* time, that is, the time the process takes to come back to the event, it arises that for a general small set it can no more be well approximated by an exponential law but rather by a law that is a mixture of an exponential law and a Dirac law concentrated at the point of recurrence of the string. The non-convergence to a pure exponential law of a return time law was already proved by Hirata (1993). He proves the almost sure convergence of the visits to a small cylinder around a point to the Poisson law for axiom A diffeomorphisms. That is

$$\lim_{I\!P\{A\}\to 0} I\!P\left\{N_{t/I\!P\{A\}} = k\right\} = \frac{e^t\ t^k}{k!} \ , \tag{2}$$

where N_s is the number of visits of the process to the cylinder up to time s. The result holds for almost every point. The one-level set of this convergence corresponds to the law of the first return time. Then, he proves that for periodic points, the asymptotic limit law of the first return time is a mixture of an exponential law and a Dirac measure law (which differs of the one-level Poisson law).

Hirata, Saussol and Vaienti (1999) show that the return time law and the hitting time law converge to the same law if and only if the return time converges to the exponential law. Then, they prove an upper bound for the rate of convergence of the limit (2) that depends on the distance between the hitting time and the return time laws. This bound is not so transparent and it diverges exponentially with the number of visits to the cylinder. Haydn and Vaienti (2001) proves also a rate of convergence of the limit (2) using a method due to Sevast'yanov (see Sevast'yanov (1972)) for $(\psi - f)$-mixing processes. This processes are another generalization of the ψ-mixing processes.

Our first result refers to the convergence of the return time law to a mixture of an exponential law and Dirac law. In Abadi (2001b) a sharp bound on the rate of convergence of the hitting time law to the exponential law is proven, (see Theorem 1 below). We emphasize that this result is not an almost sure one but it applies to *all* points. Trough this result on hitting times we prove here an upper bound for the rate of convergence of the return time. Our result still holds for *all* points. The result holds for the different types of mixing processes. It also extend that of Hirata, Saussol and Vaienti in the sense that it states that the laws of hitting and return time coincide only when the Dirac measure of the return time law is absent. We prove also that this is equivalent to consider a non-recurrent point.

The return time law lead us to consider the question of *sojourn* time, that is the time the process stays in the event. Our second result states that, under certain condition of regularity on the measure, this law converges to a geometric law. The parameter of this law does not depend on the length nor the measure of the string but just upon its recurrence properties. As far as we know, this is the first result proving this behavior.

Our third result refers to the Poisson approximation of the number of visits to the string A. We prove an upper bound for the difference between the number of occurrences of A and the Poisson law. We prove our result for the above mentioned ψ, ϕ and strong mixing processes, each one with its own error term, and for strings that do not return very fast to themselves. The different types of bounds obtained for the different types of mixture appears as a characterization for the different types of mixing.

The error bound that we obtain decreases as a factorial of N_t, the number of visits to the string up to time t. This control on the tail distribution of N_t allow us to obtain also and approximation for the moments of N_t.

We also want to remark that the convergence of the return time not to a purely exponential law but rather to a mixture of an exponential with a degenerated law proven in Theorem 4 and the convergence of the sojourn time to a geometric law for recurrent strings proven in Theorem 10 also arise as a counter-example of the convergence to the Poisson law of recurrent events. As an application of the error terms we are allow to estimate the moments of the number of visits to the small sets.

This paper is organized as follows. In section 2 we establish our framework. In section 3 we state and prove the approximation theorem for the return time law. In section 4 we prove the geometric approximation for the sojourn time law. An example of this approximation is also given. In section 5 we prove the convergence of the number of occurrences to a Poisson law for non-recurrent strings.

2 The framework

Let $\mathcal{E}$ be a finite set. Put $\Omega = \mathcal{E}^{\mathbb{Z}}$. For each $\omega = \{\omega_m\}_{m\in\mathbb{Z}} \in \Omega$ and $m \in \mathbb{Z}$, let $X_m : \Omega \to \mathcal{E}$ be the m-th coordinate projection, that is $X_m(\omega) = \omega_m$. We denote by $T : \Omega \to \Omega$ the one-step-left shift operator.

We denote by $\mathcal{F}$ the σ-algebra over Ω generated by cylinders. Moreover we denote by $\mathcal{F}_I$ the σ-algebra generated by cylinders with coordinates in I, $I \subseteq \mathbb{Z}$.

For a subset $A \subseteq \Omega$ we say that $A \in \mathcal{C}_n$ if and only if

$$A = \{X_0 = a_0, \ldots, X_{n-1} = a_{n-1}\} ,$$

with $a_i \in \mathcal{E},\ i = 0, \ldots, n-1$.

We consider a stationary probability measure $\mathbb{P}$ over $\mathcal{F}$. We shall assume without loss of generality that there are no singletons of probability 0 or 1.

Let $\alpha = (\alpha(l))_{l\geq 0}$, $\phi = (\phi(l))_{l\geq 0}$ and $\psi = (\psi(l))_{l\geq 0}$ be three decreasing sequences converging to zero of real numbers. We shall say that $\{X_m\}_{m\in\mathbb{Z}}$ is *α-mixing* or uniform mixing if for all integers $n \geq 1$ and $l \geq 0$, the following holds,

$$\sup_{B\in\mathcal{F}_{\{0,.,n\}}, C\in\mathcal{F}_{\{n\geq 0\}}} \left|\mathbb{P}\left\{B \cap T^{-(n+l+1)}C\right\} - \mathbb{P}\{B\}\mathbb{P}\{C\}\right| = \alpha(l) ,$$

ϕ-mixing if

$$\sup_{B\in\mathcal{F}_{\{0,.,n\}}, C\in\mathcal{F}_{\{n\geq 0\}}} \frac{\left|\mathbb{P}\left\{B \cap T^{-(n+l+1)}C\right\} - \mathbb{P}\{B\}\mathbb{P}\{C\}\right|}{\mathbb{P}\{B\}} = \phi(l) ,$$

and *ψ-mixing* if

$$\sup_{B\in\mathcal{F}_{\{0,.,n\}}, C\in\mathcal{F}_{\{n\geq 0\}}} \frac{\left|\mathbb{P}\left\{B \cap T^{-(n+l+1)}C\right\} - \mathbb{P}\{B\}\mathbb{P}\{C\}\right|}{\mathbb{P}\{B\}\mathbb{P}\{C\}} = \psi(l) ,$$

where in the above expressions the supremum is taken over the sets B and C, such that $I\!P(B) > 0$ in the second case and such that $I\!P(B)I\!P(C) > 0$ in the third one.

Given $A \in \mathcal{C}_n$, we define the *hitting time* $\tau_A : \Omega \to I\!N \cup \{\infty\}$ as the following random variable (r.v.) defined on the probability space $(\Omega, \mathcal{F}, I\!P)$. For any $\omega \in \Omega$

$$\tau_A(\omega) = \inf\{k \geq 1 : T^k(\omega) \in A\} \ .$$

Clearly, ψ-mixing implies ϕ-mixing that implies α-mixing. We recall that mixing (at any rate) implies ergodicity, and ergodicity ensures that τ_A is $I\!P$-almost surely finite (see e.g. Cornfeld, Fomin and Sinai, 1982).

We shall use the classic probabilistic shorthand notation for events defined through r.v.'s. We shall write $\{\tau_A = m\}$ instead of $\{\omega \in \Omega : \tau_A(\omega) = m\}$, $T^{-k}(A) = \{\omega \in \Omega : T^k(\omega) \in A\}$ and $\{X_r^s = x_r^s\} = \{X_r = x_r, \ldots, X_s = x_s\}$. For two measurables A and B, we shall write also $I\!P\{A \ ; \ B\} = I\!P\{A \ \cap \ B\}$. The conditional measure of B given A is denoted as usual $I\!P\{B \mid A\} = I\!P_A\{B\} = I\!P\{B \ ; \ A\} / I\!P\{A\}$. For $A = \{X_0^{n-1} = x_0^{n-1}\}$ and $1 \leq w \leq n$, we write $A^{(w)} = \{X_{n-w}^{n-1} = x_{n-w}^{n-1}\}$ for the w-cylinder belonging to the σ-algebra $\mathcal{F}_{\{n-w,\ldots,n-1\}}$ and consisting of the last w symbols of A. We also denote with $A^c = \Omega \backslash A$, the complement of a measurable A. As usual, the mean of a r.v. X will be denoted by $I\!E(X)$. Wherever it is not ambiguous we will write C and c for different positive constants even in the same sequence of equalities/inequalities.

3 Return times

In this section we prove our result concerning *return* times. We prove that the law of the first return time into an n-cylinder can be well approximated by a law that is a mixture of a degenerated law and an exponential law. The basic result is a result in Abadi (2001b) about *hitting* times. For easy reference to the reader we quote it here.

Theorem 1 (Abadi, 2001b) *Let $\{X_m\}_{m \in \mathbb{Z}}$ be ψ-mixing or ϕ-mixing with ϕ summable. Then, there exist constants $C > 0$, $0 < \Xi_1 < 1 \leq \Xi_2 < \infty$, such that for all $n \in I\!N$ and $A \in \mathcal{C}_n$ there exists $\xi_A \in [\Xi_1, \Xi_2]$, for which the following inequality holds for all $t > 0$*

$$\left| I\!P\{\tau_A > t\} - e^{-\xi_A \boldsymbol{P}\{A\} t} \right| \leq e_1(A) \ (t I\!P\{A\} \vee 1) \ e^{-\xi_A \boldsymbol{P}\{A\} t} \ ,$$

where

$$e_1(A) := C \inf_{n \leq \ell \leq \frac{1}{P\{A\}}} [\ell I\!P\{A\} + *(\ell)] \ , \tag{3}$$

and where $*$ stands for ψ or ϕ according to the case.

We remark that the theorem consider the law of τ_A starting from the equilibrium measure $I\!P$.

Definition 2 *For any $A \in \mathcal{C}_n$ we define the* periodicity *of A (with respect to the shift T) as the number r_A defined as follows:*

$$r_A = \inf\left\{x \in Z\!\!Z_+ \mid A \cap \; T^{-x}(A) \neq \emptyset\right\} \; .$$

We denote with $\mathcal{R}_r = \mathcal{R}_r(n)$, the family of n-cylinders of periodicity r, namely

$$\mathcal{R}_r = \{A \in \mathcal{C}_n | r_A = r\} \; .$$

We remark that $\mathcal{R}_r$ with $1 \leq r \leq n$, form a partition for $\mathcal{C}_n$. Let us take $A \in \mathcal{R}_r$ and write $n = q\,r + s$, with $q = [n/r]$ and $0 \leq s < r$. We have

$$A \;\; = \;\; \left\{X_0^{r-1} = X_r^{2r-1} = \ldots = X_{(q-1)r}^{qr-1} = a_0^{r-1} \; ; \; X_{qr}^{n-1} = a_0^{s-1}\right\} \; .$$

So, we say that A has *period* r and *rest* s. We remark that in this case, periods can be "readen backward" (and for future purpose it will be more useful to do it in this way), that is

$$\begin{aligned} A \;\; &= \;\; \left\{X_0^{s-1} = a_{n-s}^{n-1} \; ; \; X_{n-qr}^{n-(q-1)r-1} = \ldots = X_{n-2r}^{n-r-1} = X_{n-r}^{n-1} = a_{n-r}^{n-1}\right\} \\ &= \;\; \left\{T^{qr}A^{(s)}; T^{(q-1)r}A^{(r)}; \ldots ; T^{2r}A^{(r)}; T^{r}A^{(r)}; A^{(r)}\right\} \; . \end{aligned}$$

We recall the definition of $A^{(w)}, 1 \leq w \leq n$ at the end of section 2 Let $1 \leq j \leq n$. We note that $A \cap T^{-j}(A) \neq \emptyset$, if and only if, $j = kr$, with $1 \leq k \leq q$. Let us write $A_1 = \left\{X_n^{n+r-1} = a_{n-r}^{n-1}\right\}$. Since $A \in \mathcal{R}_r$ it is easy to see that

$$I\!P_A\{A_1^c\} = I\!P_A\left\{\tau_A \neq r\right\} = I\!P_A\left\{\tau_A > r\right\} = I\!P_A\left\{\tau_A > n-1\right\} \; .$$

Let us define $\mathcal{L}(t)$ as follows

$$\mathcal{L}(t) = \begin{cases} 1 & t < r \\ I\!P_A\left\{\tau_A > n-1\right\} & r \le t \le n-1 \\ I\!P_A\left\{\tau_A > n-1\right\} e^{-\xi_A I\!P\{A\} t} & t \ge n \end{cases} ,$$

where ξ_A is the same of Theorem 1. Before to state the theorem for return time we need to prove the following lemma

Lemma 3 *Let $\{X_m\}_{m\in\mathbb{Z}}$ be ϕ-mixing (at any rate). Let $A \in \mathcal{C}_n$ and $M \in I\!N$. Then, the following inequality holds*

$$I\!P_A\left\{\tau_A \circ T^{n-1} \le M\right\} \le \inf_{1\le w\le n} n\left[I\!P\{A^{(w)}\} + \phi(n-w)\right] + M I\!P\{A\} + \sum_{j=n}^{M}\phi(j) \ .$$

Proof. Consider $M > n$. Let $1 \le w \le n$. The mixing property implies that

$$\begin{aligned}
& I\!P\left\{A \ ; \ \tau_A \circ T^{n-1} \le M\right\} \\
= & \sum_{j=1}^{n-1} I\!P\left\{A \ ; \ \tau_A \circ T^{n-1} = j\right\} + \sum_{j=n}^{M} I\!P\left\{A \ ; \ \tau_A \circ T^{n-1} = j\right\} \\
\le & \sum_{j=1}^{n-1} I\!P\left\{A \ ; \ T^{-(n-1+j)}A^{(w)}\right\} + \sum_{j=n}^{M} I\!P\left\{A \ ; \ T^{-(n-1+j)}A\right\} \\
\le & \sum_{j=1}^{n-1} I\!P\{A\}\left(I\!P\left\{T^{-(n-1+j)}A^{(w)}\right\} + \phi(n-w+j)\right) \\
+ & \sum_{j=n}^{M} I\!P\{A\}\left(I\!P\left\{T^{-(n-1+j)}A\right\} + \phi(j)\right) \ .
\end{aligned}$$

The first inequality follows since $\{\tau_A = j\} \subset \{T^{-j}A\} \subset \{T^{-j}A^{(w)}\}$. Using the stationarity of the measure we have

$$I\!P_A\left\{\tau_A \circ T^{n-1} \le M\right\} \le n\left[I\!P\{A^{(w)}\} + \phi(n-w)\right] + M I\!P\{A\} + \sum_{j=n}^{M}\phi(j) \ .$$

If $M \le n$ the second term disappears. The bound is the same in both cases. This ends the proof since w is arbitrary.

Theorem 4 *Let the process $\{X_m\}_{m\in\mathbb{Z}}$ be ϕ-mixing with ϕ summable. Then, there exist strictly positive constants C_1 and C_2 such that for any n and any $A \in \mathcal{R}_r(n)$, the following inequality holds*

$$|I\!P_A\{\tau_A > t\} - \mathcal{L}(t)| \le e_2(A) \ ,$$

where

$$\begin{aligned} e_2(A) &:= C_1 \inf_{n\le \ell \le \frac{1}{\boldsymbol{P}\{A\}}} \left[\ell I\!P\{A\} + \sum_{j=n}^{\ell} \phi(j)\right] \\ &+ C_2\, n \inf_{1\le w\le n} \left[I\!P\{A^{(w)}\} + \phi(n-w)\right] \ . \end{aligned} \tag{4}$$

We remark that the theorem considers the law of τ_A starting from the measure concentrated in A, that is $I\!P_A$. We call this the first *return* time (to A).

Proof. For $t \le n-1$, since r is the periodicity of A, we have immediately

$$I\!P_A\{\tau_A > t\} = \mathcal{L}(t) \ .$$

Now let ℓ be the one that defines $e_1(A)$ in (4). For $t \ge \ell + 2n$ we have

$$\begin{aligned} & |I\!P_A\{\tau_A > t\} - \mathcal{L}(t)| \\ \le\ & \left|I\!P_A\{\tau_A > t\} - I\!P_A\left\{A_1^c\ ;\ \tau_A \circ T^{\ell+2n} > t-(\ell+2n)\right\}\right| \\ +\ & \left|I\!P_A\left\{A_1^c\ ;\ \tau_A \circ T^{\ell+2n} > t-(\ell+2n)\right\} - I\!P_A\{A_1^c\} I\!P\left\{\tau_A \circ T^{\ell+2n} > t-(\ell+2n)\right\}\right| \\ +\ & I\!P_A\{A_1^c\}\left|I\!P\left\{\tau_A \circ T^{\ell+2n} > t-(\ell+2n)\right\} - I\!P\{\tau_A > t\}\right| \\ +\ & I\!P_A\{A_1^c\}\left|I\!P\{\tau_A > t\} - e^{-\xi_A \boldsymbol{P}\{A\} t}\right| \ . \end{aligned} \tag{5}$$

The first term in the right hand side of the above inequality is bounded as follows

$$\left|I\!P_A\{\tau_A > t\} - I\!P_A\left\{A_1^c\ ;\ \tau_A \circ T^{\ell+2n} > t-(\ell+2n)\right\}\right|$$

$$
\begin{aligned}
&= \left| I\!P_A \left\{ A_1^c \ ; \ \tau_A \circ T^{n-1} > t - (n-1) \right\} - I\!P_A \left\{ A_1^c \ ; \ \tau_A \circ T^{\ell+2n} > t - (\ell+2n) \right\} \right| \\
&\le I\!P_A \left\{ A_1^c \ ; \ \tau_A \circ T^{n-1} \le (\ell+2n) - (n-1) \right\} \\
&\le I\!P_A \left\{ \tau_A \circ T^{n-1} \le \ell + n + 1 \right\} \ .
\end{aligned}
$$

The last bound is still upper-bounded using Lemma 3. The second term in the right hand side of (5) is bounded using the mixing property

$$
\begin{aligned}
& \left| I\!P_A \left\{ A_1^c \ ; \ \tau_A \circ T^{\ell+2n} > t - (\ell+2n) \right\} - I\!P_A \left\{ A_1^c \right\} I\!P \left\{ \tau_A \circ T^{\ell+2n} > t - (\ell+2n) \right\} \right| \\
\le \ & \phi \left(\ell + 2n - (n+r) \right) \\
\le \ & \phi(\ell) \ .
\end{aligned}
$$

For the third term in the right hand side of (5) we have

$$
\begin{aligned}
& \left| I\!P \left\{ \tau_A \circ T^{\ell+2n} > t - (\ell+2n) \right\} - I\!P \left\{ \tau_A > t \right\} \right| \\
\le \ & I\!P \left\{ \tau_A \le \ell + 2n \right\} \\
\le \ & (\ell + 2n) I\!P\{A\} \\
\le \ & C \ \ell I\!P\{A\} \ .
\end{aligned}
$$

The last term in the right hand side of (5) is obviously bounded using Theorem 1 by $e_1(A)$.

For t such that, $n \le t < \ell + 2n$, we have

$$
\begin{aligned}
& \left| I\!P_A \left\{ \tau_A > t \right\} - \mathcal{L}(t) \right| \\
\le \ & \left| I\!P_A \left\{ A_1^c \ ; \ \tau_A \circ T^{n-1} > t - n + 1 \right\} - I\!P_A \left\{ A_1^c \right\} \right| + \left| I\!P_A \left\{ A_1^c \right\} - I\!P_A \left\{ A_1^c \right\} e^{-\xi_A \boldsymbol{P}\{A\} t} \right| \\
= \ & I\!P_A \left\{ A_1^c \ ; \ \tau_A \circ T^{n-1} \le t - n + 1 \right\} + I\!P_A \left\{ A_1^c \right\} \left(1 - e^{-\xi_A \boldsymbol{P}\{A\} t} \right) \\
\le \ & I\!P_A \left\{ \tau_A \circ T^{n-1} \le \ell + n + 1 \right\} + 1 - e^{-\xi_A \boldsymbol{P}\{A\} \ell} \\
\le \ & I\!P_A \left\{ \tau_A \circ T^{n-1} \le \ell + n + 1 \right\} + \Xi_2 I\!P\{A\} \ell \ .
\end{aligned}
$$

Finally, the most left term of the above bound is bounded using again Lemma 3. The most right term follows by Theorem 1. This ends the proof of the theorem.

Remark. We want to emphasize the relationship between ξ_A and $I\!P_A\{A_1^c\}$. The parameter ξ_A has an explicit form given in Abadi (2001b). It is usually difficult to compute it. It is proven in Theorem 4 of Abadi (2001a) that when the system is exponentially ϕ-mixing , $I\!P_A\{A_1^c\}$ is an (usually easy computable) approximation of ξ_A. Then we allow a larger error (due to this approximation), we have the following corollary: let us define $\widetilde{\mathcal{L}}(t)$ as follows

$$\widetilde{\mathcal{L}}(t) = \begin{cases} 1 & t < r \\ I\!P_A\{\tau_A > n-1\} & r \le t \le n-1 \\ I\!P_A\{\tau_A > n-1\} \; e^{-\boldsymbol{P}_A\{\tau_A > n-1\} \; \boldsymbol{P}\{A\} t} & t \ge n \end{cases} .$$

Corollary 5 *Let $\{X_m\}_{m\in\boldsymbol{Z}}$ be a ϕ-mixing process with exponential function ϕ. Then, there exist strictly positive constants c and C such that for any n and any $A \in \mathcal{R}_r(n)$, the following inequality holds*

$$\left| I\!P_A\{\tau_A > t\} - \widetilde{\mathcal{L}}(t) \right| \le Ce^{-cn} .$$

Proof. It follows directly from Theorem 4 and Theorem 4 in Abadi (2001a).

We can state now a corollary that extend Theorem 2.1 in Hirata, Saussol and Vaienti (1999).

Corollary 6 *Let $\{X_m\}_{m\in\boldsymbol{Z}}$ be a ϕ-mixing process with summable function ϕ. Suppose that $A \in \mathcal{C}_n$. Then, the following conditions are equivalent*

(a) $\left| I\!P_A\{\tau_A > t\} - e^{-\boldsymbol{P}\{A\}t} \right| \le C \; e^{-cn}$,

(b) $|I\!P_A\{\tau_A > t\} - I\!P\{\tau_A > t\}| \le C \; e^{-cn}$,

(c) $|I\!P_A\{\tau_A > n-1\} - 1| \le C \; e^{-cn}$.

In particular, the law of the return time to A, converges to an exponential law if and only if the law of the return time and the law of the hitting time converge to the same law, if and only if A is non-recurrent.

Proof. $(a) \Rightarrow (b)$ By hypothesis, Theorem 4 in Abadi (2001a) and Theorem 1

$$
\begin{aligned}
& |I\!P_A\{\tau_A > t\} - I\!P\{\tau_A > t\}| \\
\leq\ & \left| I\!P_A\{\tau_A > t\} - e^{-\boldsymbol{P}\{A\}t} \right| + \left| e^{-\boldsymbol{P}\{A\}t} - e^{-\xi_A \boldsymbol{P}\{A\}t} \right| + \left| e^{-\xi_A \boldsymbol{P}\{A\}t} - I\!P\{\tau_A > t\} \right| \\
\leq\ & C\ e^{-cn} + C\ e^{-cn} + e_1(A) \leq C\ e^{-cn}\ .
\end{aligned}
$$

$(b) \Rightarrow (c)$ Follows from Theorem 4 in Abadi (2001a), Corollary 5 and Theorem 4 applied at $t = 1/(I\!P_A\{\tau_A > n-1\}\ \ I\!P\{A\})$

$(c) \Rightarrow (a)$ Follows directly from Corollary 5. This ends the proof of the corollary.

4 Sojourn time

We proved in the previous section that the return time to a rare event A can be well approximated by a mixture of two laws, one being a δ_r, that is, a Dirac measure concentrated at the point of recurrence of the event (namely, r if $A \in \mathcal{R}_r$) and the other being an exponential law with mean $1/(I\!P_A\{A_1^c\} I\!P\{A\})$. Loosely, given that the process is in A, either it returns immediately to A or it takes an exponential time to re-enter A. In the first case, the process could keep re-entering A, forming a cluster of visits. The cardinal of this cluster is a r.v. We want to study its distribution law.

We prove in this section that, under certain regularity on a conditional measure, the distribution of this r.v. can be well approximated by a geometric distribution that does *not* depend on the measure $I\!P\{A\}$ nor the length n of A, but just upon its recursive properties.

Definition 7 *Let $A \in \mathcal{R}_r$. We define the* sojourn time *on the set A as the r.v. $S_A : A \to I\!N \cup \{\infty\}$*

$$
S_A(\omega) = \sup\left\{ s \in I\!N \mid \omega \in A\ ;\ T^{-sr}A\ ;\ \forall j = 1, \ldots, s \right\}\ .
$$

The followings lemma and definition allow us to define the idea of regularity of the measure above mentioned.

Lemma 8 *Let $\{l_i\}_{i \in \boldsymbol{N}}$ be a sequence of real numbers such that $0 < l_i < 1$, for all $i \in I\!N$. Let $0 < l < 1$ be such that $|l_i - l| \leq d_i$ for all $i \in I\!N$ with decreasing going to zero d_i's. Then, there is a constant c, with $0 < c < 1$, such that the following inequality holds for all $k \in I\!N$*

$$
\left| \prod_{i=1}^{k} l_i - l^k \right| \leq k\ c^{k-1} d_1\ .
$$

Proof.

$$\begin{aligned}\left|\prod_{i=1}^{k} l_i - l^k\right| &= \left|\prod_{i=1}^{k} l_i - \prod_{i=1}^{k-1} l_i l + \prod_{i=1}^{k-1} l_i l - \prod_{i=1}^{k-2} l_i l^2 + \prod_{i=1}^{k-2} l_i l^2 - \ldots - l^k\right| \\ &\leq \sum_{i=1}^{k} \left(\prod_{j=1}^{k-i} l_j\right) |l_{k-i+1} - l|\ l^{i-1} \leq c^{k-1} \sum_{i=1}^{k} d_i \\ &\leq k\ c^{k-1} d_1 \ ,\end{aligned}$$

where $c = \max\{l_0, l\}$.

We still recall the definition of $A^{(r)}$ at the end of section 2. Let $A \in \mathcal{R}_r$. For short-hand notation let us write $A_j = T^{-jr} A^{(r)}$ for all $j \in \mathbb{Z}$. We introduce the following definition

Definition 9 *Given $A \in \mathcal{R}_r$, we define the sequence $r_i(A)$; $i = 1, 2, \ldots$ as follows*

$$r_i(A) := \mathbb{P}\left\{A_1 \mid A_0; A_{-1}; \ldots; A_{-(i-1)}; T^{i-1}A^{(s)}\right\} \ .$$

That is, if A is an n-cylinder with period r and rest s, r_i is the probability that the process repeats the period of A, given that it already occurred i times and once the rest in the immediate past, all of them adjacently.

Since A is fixed we will avoid to write the dependence on A in $r_i(A)$ writing just r_i. In the following theorem, we assume that there exists the limit of the sequence $\{r_i\}_{i\in\mathbb{N}}$, with velocity d_i, namely, there is a real number r_A such that

$$|r_i - r_A| \leq d_i \ , \tag{6}$$

for all $i \geq 1$, where d_i is a decreasing sequence of real numbers going to zero and $0 < r_A < 1$.

Theorem 10 *Let $\{X_m\}_{m\in\mathbb{Z}}$ be a stationary process. Let $A \in \mathcal{R}_r$. Assume that the limit in (6) exists. Then, there is a constant c, with $0 < c < 1$, such that the following inequalities hold for all $k \in \mathbb{N}$*

$$\left|\mathbb{P}_A\{S_A = k\} - (1 - r_A)\ r_A^k\right| \leq\ c^k \sum_{i=q}^{k+q} d_i \leq\ (k+1)\ c^k d_q \ ,$$

where $q = [n/r]$.

Proof. For $k = 0$ we just note that $I\!\!P_A\{S_A = 0\} = r_q$ and $|r_q - r_A| = |1 - r_q - (1 - r_A)| \leq d_q$. Suppose $k \geq 1$, we have

$$\begin{aligned} I\!\!P_A\{S_A = k\} &= \frac{I\!\!P\left\{A; A_1; \ldots; A_k; A^c_{k+1}\right\}}{I\!\!P\{A\}} \\ &= \frac{I\!\!P\left\{A; A_1; \ldots; A_k; A^c_{k+1}\right\}}{I\!\!P\{A; A_1; \ldots; A_k\}} \frac{I\!\!P\{A; A_1; \ldots; A_k\}}{I\!\!P\{A; A_1; \ldots; A_{k-1}\}} \cdots \frac{I\!\!P\{A; A_1\}}{I\!\!P\{A\}} . \end{aligned}$$

Note that by stationarity

$$\begin{aligned} I\!\!P\{A; A_1; \ldots; A_k\} &= I\!\!P\left\{T^{qr} A^{(s)}; A_{-(q-1)}; \ldots; A_0; A_1; \ldots; A_{k-1}; A_k\right\} \\ &= I\!\!P\left\{T^{(k+q-1)r} A^{(s)}; A_{-(k+q-2)}; \ldots; A_{-k+1}; A_{-k+2}; \ldots; A_0; A_1\right\} . \end{aligned}$$

Therefore

$$\frac{I\!\!P\{A; A_1; \ldots; A_k\}}{I\!\!P\{A; A_1; \ldots; A_{k-1}\}} = r_{k+q-1} .$$

Hence

$$I\!\!P_A\{S_A = k\} = (1 - r_{k+q})\, r_{k+q-1} \ \ldots \ r_q .$$

By Lemma 8

$$\begin{aligned} \left| I\!\!P_A\{S_A = k\} - (1 - r_A) r_A^k \right| &= \left| (1 - r_{k+q}) \prod_{i=q}^{k+q-1} r_i - (1 - r_A) r_A^k \right| \\ &\leq c^k \sum_{i=q}^{k+q} d_i \leq (k+1)\, c^k\, d_q , \end{aligned}$$

where $c = \max\{r_0, r_A\}$. This ends the proof of the Theorem.

Example 11 *Let* $\{X_m\}_{m\in\mathbb{Z}}$ *be a irreducible and aperiodic finite state Markov chain. Let* $A \in \mathcal{R}_r$. *Then the sequence* r_i *is constant, more precisely, by the Markovian property and for all* $i \in I\!N$

$$r_i = I\!P\left\{A_1 \mid A_0; A_{-1}; \ldots; A_{-(i-1)}, T^{i-1}A^{(s)}\right\} = I\!P\left\{A_1 \mid A^{(1)}\right\} .$$

Thus, $r_A = r_i$ *for all* $i \in I\!N$. *Furthermore,* $I\!P_A\{A_1^c\} = 1 - r_A$, *and moreover*

$$I\!P_A\{S_A = k\} = (1 - r_A)\; r_A^k .$$

Example 12 *For a Gibbs measure, the condition (6) is warranted by the continuity of the potential. For a chain of infinite order (see v.g. Fernández, Ferrari and Galves, 2001) condition (6) is assured by the nullness hypothesis.*

5 Poisson limit law

In this section we prove that the number of occurrences of the string A in an interval of time of length t has approximately Poisson distribution with parameter $tI\!P\{A\}$. The result holds when the string does not recur very fast.

Definition 13 *Given* $A \in \mathcal{C}_n$, *we define for* $j \in I\!N$, *the* j-occurrence time *of* A *as the r.v.* ${\tau_A}^{(j)} : \Omega \to I\!N \cup \{\infty\}$, *defined on the probability space* $(\Omega, \mathcal{F}, I\!P)$ *as follows: for any* $\omega \in \Omega$, ${\tau_A}^{(1)}(\omega) = \tau_A(\omega)$ *and for* $j \geq 2$

$${\tau_A}^{(j)}(\omega) = \inf\{x > {\tau_A}^{(j-1)}(\omega) : T^x(\omega) \in A\} .$$

We introduce the following notation. For $0 \leq s < t$ integers let

$$N_s^t = \sum_{i=s}^{t} 1\!\!1_{T^{-i}A} .$$

So, N_s^t counts the number of occurrences of A between s and t.

Definition 14 *Let* $s \in I\!N$. *Define* $\mathcal{B}_n = \mathcal{B}_n(s)$ *as the set of those* $A \in \mathcal{C}_n$ *which recur before time* n/s, *namely,* $A \in \mathcal{B}_n$ *if and only if there is an integer* $1 \leq j \leq n/s$ *such that* $A \cap T^{-j}A \neq \emptyset$ *for some* $1 \leq j \leq n/s$.

The following Lemma is the base of the proof of the Poisson statistics of occurrence times of a measurable $A \notin \mathcal{B}_n(2)$. It establish an asymptotic weak dependence of the occurrences of A.

Proposition 15 *Let $\{X_m\}_{m\in\mathbb{Z}}$ be a ϕ-mixing process. Let $A \notin \mathcal{B}_n(2)$. Let $t_0 = 0 < t_1 < t_2 < \ldots < t_k \le t = t_{k+1} - 1$. Let ℓ_A be the ℓ that defines $\varepsilon_2(A)$ below. Suppose that*

$$\min\{t_j - t_{j-1} : j = 2, \ldots, k\} > 2(\ell_A + n) \ .$$

Then there exists a positive constant $C > 0$ independent of A, n, t and k such that the following inequality holds

$$\begin{aligned} &\left| I\!P\left\{ \bigcap_{j=1}^{k} {\tau_A}^{(j)} = t_j \ ; \ {\tau_A}^{(k+1)} > t \right\} - I\!P\{A\}^k \prod_{j=1}^{k+1} I\!P\left\{\tau_A > \Delta_j - 2(\ell_A + n)\right\} \right| \\ \le \ & C \ \left(I\!P\{A\} + \phi(\ell_A)\right)^k \ \varepsilon(A) \ , \end{aligned}$$

where $\varepsilon(A) = \varepsilon_1(A) + \varepsilon_2(A)$,

$$\varepsilon_1(A) := \inf_{1\le w\le n/2} n\left\{ I\!P\{A^{(w)}\} + \phi\left(\frac{n}{2} - w\right)\right\} \ ,$$

and

$$\varepsilon_2(A) := \inf_{n\le \ell\le 1/\boldsymbol{P}\{A\}} \left\{ \ell I\!P\{A\} + \sum_{j=n}^{3\ell} \phi(j) + \frac{\phi(\ell)}{I\!P\{A\}} \right\} \ . \tag{7}$$

We remark that $A \in \mathcal{C}_n$ and $\varepsilon_1(A) \to 0$ when $n \to \infty$. The same is true for $\varepsilon_2(A)$ if the sequence $\{\phi(l)\}_{l\in I\!N}$ is summable.

Proof. We prove the proposition by induction on k. First we note that by stationarity and for any positive integer t (see v.g. pag. 16 in Abadi, 2001a)

$$I\!P\{\tau_A = t\} = I\!P\{A \ ; \ \tau_A > t - 1\} \ . \tag{8}$$

Let us write for short-hand notation $\overline{\ell}_A = 2(\ell_A + n)$. Let us write also $\Delta_j = t_j - t_{j-1}$, for $j = 1, \ldots, k+1$. For each non negative integer i denote $\mathcal{P}_i = I\!P\left\{\tau_A > \Delta_i - \overline{\ell}_A\right\}$. For $k = 1$

$$
\begin{aligned}
&\left| \mathbb{P}\left\{ \tau_A = t_1 \ ; \ {\tau_A}^{(2)} > t \right\} - \mathbb{P}\{A\} \prod_{j=1}^{2} \mathcal{P}_j \right| && (9)\\
\leq \ &\left| \mathbb{P}\left\{ \tau_A = t_1 \ ; \ {\tau_A}^{(2)} > t \right\} - \mathbb{P}\left\{ \tau_A = t_1 \ ; \ N^t_{t_1+\overline{\ell}_A} = 0 \right\} \right| \\
+ \ &\left| \mathbb{P}\left\{ \tau_A = t_1 \ ; \ N^t_{t_1+\overline{\ell}_A} = 0 \right\} - \mathbb{P}\left\{ \tau_A = t_1 \right\} \mathcal{P}_2 \right| \\
+ \ &\left| \mathbb{P}\left\{ A \ ; \ \tau_A > t_1 - 1 \right\} - \mathbb{P}\left\{ A \ ; \ N^{t_1-1}_{\overline{\ell}_A} = 0 \right\} \right| \ \mathcal{P}_2 \\
+ \ &\left| \mathbb{P}\left\{ A \ ; \ N^{t_1-1}_{\overline{\ell}_A} = 0 \right\} \mathcal{P}_2 - \mathbb{P}\{A\} \prod_{j=1}^{2} \mathcal{P}_j \right| \ .
\end{aligned}
$$

In the third term in the right hand side of (9) we used (8). The first term in the above bound is equal to

$$
\begin{aligned}
\mathbb{P}\left\{ \tau_A = t_1 \ ; \ \bigcup_{i=t_1+1}^{t_1+\overline{\ell}_A-1} T^{-i}A \ ; \ N^t_{t_1+\overline{\ell}_A} = 0 \right\} \ &\leq \ \sum_{i=t_1+1}^{t_1+\overline{\ell}_A-1} \mathbb{P}\left\{ T^{-t_1}A \ ; \ T^{-i}A \right\} && (10)\\
&= \ \sum_{i=1}^{\overline{\ell}_A-1} \mathbb{P}\left\{ A \ ; \ T^{-i}A \right\} \ .
\end{aligned}
$$

We divide the above sum in those terms with $1 \leq i \leq n/2$, $n/2 < i \leq 2n$, and $i > 2n$. For $1 \leq i \leq n/2$, the leading term of the above sum is zero since $A \notin \mathcal{B}_n(2)$. Moreover

$$
\begin{aligned}
\sum_{i=(n/2)+1}^{2n} \mathbb{P}\left\{ A \ ; \ T^{-i}A \right\} \ &\leq \ \sum_{i=(n/2)+1}^{2n} \mathbb{P}\left\{ A \ ; \ T^{-i}A^{(w)} \right\} \\
&\leq \ \sum_{i=(n/2)+1}^{2n} \mathbb{P}\{A\} \left(\mathbb{P}\left\{ A^{(w)} \right\} + \phi\left(i-w\right) \right) \\
&\leq \ C \ \mathbb{P}\{A\} \ n \ \left(\mathbb{P}\left\{ A^{(w)} \right\} + \phi\left(\frac{n}{2} - w \right) \right) \ ,
\end{aligned}
$$

for any $1 \leq w \leq n/2$. We still recall the definition of $A^{(w)}$ at the end of section 2. Similarly

$$
\sum_{i=2n}^{\overline{\ell}_A-1} \mathbb{P}\left\{ A \ ; \ T^{-i}A \right\} \ \leq \ \sum_{i=n}^{\overline{\ell}_A-n-1} \mathbb{P}\{A\} \left(\mathbb{P}\{A\} + \phi(i) \right)
$$

$$\leq \; C\, I\!P\{A\} \left(\ell_A I\!P\{A\} + \sum_{i=n}^{3\ell_A} \phi(i) \right) \; .$$

The second term in the right hand side of inequality (9) is bounded using the mixing property by

$$\phi(\overline{\ell}_A - n)\; I\!P\left\{\tau_A = t_1\right\} \leq \phi(\ell_A)\; I\!P\{A\}\; .$$

Analogous computations are used to bound the third and fourth terms in the right hand side of inequality (9). This ends the proof for $k = 1$.

Now let us suppose that the theorem holds for $k-1$ and prove for k. We use a triangular inequality where the terms involved are defined below. We briefly comment the idea behind each term. For each non negative i denote $\mathcal{S}_i = \{\tau_A = t_i\}$.

$$\left| I\!P\left\{ \bigcap_{j=1}^{k} \mathcal{S}_j \;\; ; \; {\tau_A}^{(k+1)} > t \right\} - I\!P\{A\}^k \prod_{j=1}^{k+1} \mathcal{P}_j \; \right| \leq I + II + III + IV + V \; .$$

In I we introduce a gap of length $\overline{\ell}_A$ at the left of the k-th occurrence of A, namely, between coordinates $t_k - \overline{\ell}_A + 1$ and $t_k - 1$

$$\begin{aligned}
I \;\; &:= \;\; \left| I\!P\left\{ \bigcap_{j=1}^{k} \mathcal{S}_j \;\; ; \; {\tau_A}^{(k+1)} > t \right\} - I\!P\left\{ \bigcap_{j=1}^{k-1} \mathcal{S}_j \;\; ; \; N_{t_{k-1}+1}^{t_k - \overline{\ell}_A} = 0 \; ; \; T^{-t_k} A \; ; \; N_{t_k+1}^{t} = 0 \right\} \right| \\
&= \;\; I\!P\left\{ \bigcap_{j=1}^{k-1} \mathcal{S}_j \;\; ; \; N_{t_{k-1}+1}^{t_k - \overline{\ell}_A} = 0 \; ; \; \bigcup_{i=t_k - \overline{\ell}_A + 1}^{t_k - 1} T^{-i} A \; ; \; T^{-t_k} A \; ; \; N_{t_k+1}^{t} = 0 \right\} \\
&\leq \;\; \sum_{i=t_k - \overline{\ell}_A + 1}^{t_k - 1} I\!P\left\{ \bigcap_{j=1}^{k-1} T^{-t_j} A \; ; \; T^{-i} A \; ; \; T^{-t_k} A \right\} \; .
\end{aligned}$$

As in (10) we split the above sum in terms with $t_k - \overline{\ell}_A + 1 \leq i \leq t_k - 2n$, $t_k - 2n + 1 \leq i \leq t_k - n/2 - 1$ and $t_k - n/2 \leq i \leq t_k - 1$. The bound is obtained iterating the mixing property. We recall that by hypothesis $\Delta_j > \overline{\ell}_A$ for all $i = 1, \ldots, k$.

$$\sum_{i=t_k - \overline{\ell}_A + 1}^{t_k - 2n} I\!P\left\{ \bigcap_{j=1}^{k-1} T^{-t_j} A \; ; \; T^{-i} A \; ; \; T^{-t_k} A \right\}$$

$$
\begin{aligned}
&\leq \sum_{i=n}^{\overline{\ell}_A - n} I\!P\{A\} \ \left(I\!P\{A\} + \phi(\overline{\ell}_A - n)\right)^{k-1} \ (I\!P\{A\} + \phi(i)) \\
&\leq (I\!P\{A\} + \phi(\ell_A))^k \ \left(C \ \ell_A I\!P\{A\} + \sum_{i=n}^{3\ell_A} \phi(i)\right) \ .
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
& \sum_{i=t_k-2n+1}^{t_k-(n/2)-1} I\!P\left\{\bigcap_{j=1}^{k-1} T^{-t_j}A \ ; \ T^{-i}A \ ; \ T^{-t_k}A\right\} \\
\leq & \sum_{i=t_k-2n+1}^{t_k-(n/2)-1} I\!P\left\{\bigcap_{j=1}^{k-1} T^{-t_j}A \ ; \ T^{-i}A \ ; \ T^{-t_k}A^{(w)}\right\} \\
\leq & \sum_{i=(n/2)+1}^{2n-1} I\!P\{A\} \ \left(I\!P\{A\} + \phi(\overline{\ell}_A - n)\right)^{k-1} \ \left(I\!P\left\{A^{(w)}\right\} + \phi\left(i - w\right)\right) \\
\leq & \ (I\!P\{A\} + \phi(\ell_A))^k \ 2n \ \left(I\!P\left\{A^{(w)}\right\} + \phi\left(\frac{n}{2} - w\right)\right) \ ,
\end{aligned}
$$

for any $1 \leq w \leq n/2$. The remaining terms are zero since $A \notin \mathcal{B}_n(2)$.

In II we apply the mixing property. The last inequality follows as above

$$
\begin{aligned}
II \ := & \ \Bigg| \, I\!P\left\{\left(\bigcap_{j=1}^{k-1} \mathcal{S}_j \ ; \ N_{t_{k-1}+1}^{t_k-\overline{\ell}_A} = 0\right) \ ; \ \left(T^{-t_k}A \ ; \ N_{t_k+1}^{t} = 0\right)\right\} \\
& - I\!P\left\{\bigcap_{j=1}^{k-1} \mathcal{S}_j \ ; \ N_{t_{k-1}+1}^{t_k-\overline{\ell}_A} = 0\right\} \ I\!P\left\{A \ ; \ N_1^{t-t_k} = 0\right\} \, \Bigg| \\
\leq & \ I\!P\left\{\bigcap_{j=1}^{k-1} \mathcal{S}_j \ ; \ N_{t_{k-1}+1}^{t_k-\overline{\ell}_A} = 0\right\} \phi(\overline{\ell}_A - n) \\
\leq & \ I\!P\left\{\bigcap_{j=1}^{k-1} T^{-t_j}A\right\} \phi(\ell_A) \\
\leq & \ (I\!P\{A\} + \phi(\ell_A))^k \ \frac{\phi(\ell_A)}{I\!P\{A\}} \ .
\end{aligned}
$$

In III we "fill-up" the gap we created in I

$$
\begin{aligned}
III &:= \left| I\!P\left\{\bigcap_{j=1}^{k-1} \mathcal{S}_j \ ; \ N_{t_{k-1}+1}^{t_k-\overline{\ell}_A} = 0\right\} - I\!P\left\{\bigcap_{j=1}^{k-1} \mathcal{S}_j \ ; \ N_{t_{k-1}+1}^{t_k-1} = 0\right\}\right| I\!P\left\{A \ ; \ N_1^{t-t_k} = 0\right\} \\
&\leq I\!P\left\{\bigcap_{j=1}^{k-1} \mathcal{S}_j \ ; \ N_{t_{k-1}+1}^{t_k-\overline{\ell}_A} = 0 \ ; \ \bigcup_{t_k-\overline{\ell}_A+1}^{t_k-1} T^{-i}A\right\} \ I\!P\{A\} \\
&\leq \sum_{t_k-\overline{\ell}_A+1}^{t_k-1} I\!P\left\{\bigcap_{j=1}^{k-1} T^{-j}A \ ; \ T^{-i}A\right\} \ I\!P\{A\} \\
&\leq \left(I\!P\{A\} + \phi(\ell_A)\right)^k \ \ell_A I\!P\{A\} \ .
\end{aligned}
$$

In IV now we can use the inductive hypothesis

$$
\begin{aligned}
IV &:= \left| I\!P\left\{\bigcap_{j=1}^{k-1} \mathcal{S}_j \ ; \ N_{t_{k-1}+1}^{t_k-\overline{\ell}_A} = 0\right\} - I\!P\{A\}^{k-1}\prod_{j=1}^{k} \mathcal{P}_j \right| \ I\!P\left\{A \ ; \ N_1^{t-t_k} = 0\right\} \\
&\leq C\left(I\!P\{A\} + \phi(\ell_A)\right)^{k-1} \ \varepsilon(A) \ I\!P\{A\} \ .
\end{aligned}
$$

In V we use again the inductive hypothesis with $k = 1$

$$
\begin{aligned}
V &:= I\!P\{A\}^{k-1}\prod_{j=1}^{k} \mathcal{P}_j \ \left| I\!P\left\{A \ ; \ N_1^{t-t_k} = 0\right\} - I\!P\{A\}\mathcal{P}_{k+1}\right| \\
&\leq I\!P\{A\}^{k-1}\prod_{j=1}^{k} \mathcal{P}_j \ I\!P\{A\} \ \varepsilon(A) \ .
\end{aligned}
$$

Summing the bounds above we end the proof of the proposition.

Following the above notation and for the sake of simplicity we write $N_t = N_0^t$.

Theorem 16 *Let* $\{X_m\}_{m\in\boldsymbol{Z}}$ *be a* ϕ*-mixing process. Let* $A \notin \mathcal{B}_n(2)$*. Then, the following inequality holds*

$$
\left| I\!P\left\{N_{\frac{t}{\boldsymbol{P}\{A\}}} = k\right\} - \frac{e^{-t}t^k}{k!}\right| \leq
\begin{cases}
C \ \varepsilon(A) \ \frac{e^k}{(k-1)!} \ \lambda_M & 1 \ \leq k \ \leq \frac{1}{\varepsilon(A)} \\
C \ \varepsilon(A) \ \frac{e^k}{\left(\frac{1}{\varepsilon(A)}\right)! \ \left(\frac{1}{\varepsilon(A)}\right)^{k-\frac{1}{\varepsilon(A)}}} \ \lambda_M & \frac{1}{\varepsilon(A)} \ < k \ \leq \frac{2t}{n\boldsymbol{P}\{A\}} \\
C \ nI\!P\{A\} \ \frac{e^{-t}}{(k-1)! \ t} & \frac{2t}{n\boldsymbol{P}\{A\}} \ < k
\end{cases} \ .
$$

where $\varepsilon(A) = \varepsilon_1(A) + \varepsilon_2(A)$, $\varepsilon_1(A)$ *and* $\varepsilon_2(A)$ *were defined in Proposition 15;*

$$\lambda_M = \max\left\{1\ ,\ \left(t\ \left[1 + \frac{\phi(\ell_A)}{I\!P\{A\}}\right]\right)^k\right\}$$

and ℓ_A *is the* ℓ *that defines* $\varepsilon_2(A)$.

Moreover, suppose that ϕ *is summable. Let* $\{A_n\}_{n\in I\!N}$ *be such that* $A_n \notin \mathcal{B}_n(2)$ *for all* $n \in I\!N$. *Then the law of the number of occurrences of* A_n *up to the re-scaled time* $t/I\!P\{A_n\}$ *converges weakly to a Poisson law as* $n \to \infty$ *(with the above rate).*

Proof. Take $t \in I\!N$. Let us write for the sake of simplicity $N = N_t$. For $k = 0$, the result comes from Theorem 1. For $k = 1$, the result comes from Theorem 4 and an application of Proposition 15 with $k = 1$. For $k > 2t/n$, since $A \notin \mathcal{B}_n(2)$, we have that $I\!P\{N = k\} = 0$. Then the error is exactly

$$\frac{e^{-t\boldsymbol{P}\{A\}}t^k I\!P\{A\}^k}{k!} \leq \frac{e^{-t\boldsymbol{P}\{A\}}t^k I\!P\{A\}^k}{(k-1)!}\ \frac{n}{2t}\ .$$

Let us consider k with $2 \leq k \leq 2t/n$. The idea of the proof is the following: consider a realization $\omega = \{\omega_m\}_{m\in\mathbb{Z}}$ of the process $\{X_m\}_{m\in\mathbb{Z}}$ such that the sequence $\{\omega_1, \ldots, \omega_{t+n-1}\}$ contains exactly k occurrences of A. The occurrences can appear all isolated or they can appear in clusters. We prove that the set of all the configurations with isolated A's gives the approximation to the Poisson law and that the set of configurations with clusters has small measure. Given $t > 0$, and $1 \leq t_1 < \ldots < t_k \leq t$ let us define the following measurable set

$$\bar{t} = \bar{t}(t_1, \ldots, t_k) = \left\{\bigcap_{j=1}^{k} {\tau_A}^{(j)} = t_j\ ;\ {\tau_A}^{(k+1)} > t\right\}\ .$$

As in Proposition 15 we put $\Delta_j = t_j - t_{j-1}$, for $j = 2, \ldots, k$. Let

$$I(\bar{t}) = \min_{2\leq j\leq k}\{\Delta_j\}\ .$$

As before and for short-hand notation put $\overline{\ell}_A = 2(\ell_A + n)$. Whenever $I(\bar{t}) \geq \overline{\ell}_A$, we will say that the occurrences of A's are *isolated* while whenever $I(\bar{t}) < \overline{\ell}_A$, we will say that the there exists at *least one cluster*. Let us denote

$$B_k = \left\{ \bar{t} \mid I(\bar{t}) < \overline{\ell}_A \right\} \quad \text{and} \quad G_k = \left\{ \bar{t} \mid I(\bar{t}) \geq \overline{\ell}_A \right\} \ .$$

We look at the difference

$$\left| I\!P\left\{ N = k \right\} - \frac{e^{-tI\!P\{A\}} t^k I\!P\{A\}^k}{k!} \right| \leq I\!P\left\{ B_k \right\} \ + \ \left| I\!P\left\{ G_k \right\} - \frac{e^{-tI\!P\{A\}} t^k I\!P\{A\}^k}{k!} \right| \ . \qquad (11)$$

We will prove that both quantities in the right hand side of (11) are small.

Proof: configurations with clusters have small measure.

We will prove an upper bound for $I\!P\left\{ B_k \right\}$. The following quantity counts how many clusters there are in a given $\bar{t} \in B_k$

$$C(\bar{t}) := \sum_{j=2}^{k} 1\!\!1_{\left\{ \Delta_j > \overline{\ell}_A \right\}} + 1 \ .$$

Suppose that $\bar{t}$ is such that $C(\bar{t}) = 1$. The same iterative argument of the mixing property that we used in the proof of Proposition 15 together with the fact that $A \notin \mathcal{B}_n(2)$ lead to the bound

$$I\!P\left\{ \bar{t} \right\} = I\!P\left\{ \bigcap_{j=1}^{k} {\tau_A}^{(j)} = t_j \ ; \ {\tau_A}^{(k+1)} > t \right\} \leq I\!P\left\{ \bigcap_{j=1}^{k} T^{-t_j} A \right\} \leq I\!P\{A\} \ \prod_{j=2}^{k} \rho_j \ , \qquad (12)$$

where

$$\rho_j := \begin{cases} I\!P\{A\} + \phi\,(j) & 2n < \Delta_j \leq \overline{\ell}_A \\ I\!P\left\{ A^{(w)} \right\} + \phi\left(\frac{n}{2} - w \right) & n/2 < \Delta_j \leq 2n \end{cases} \ .$$

with $1 \leq w \leq n/2$. Moreover, by the same argument, if $\bar{t}$ is such that $C(\bar{t}) = j$, we have the following bound

$$\mathbb{P}\{\bar{t}\} \;\leq\; \mathbb{P}\{A\}\,(\mathbb{P}\{A\}+\phi(\ell_A))^{j-1} \prod_{i \;|\; \Delta_i \leq \overline{\ell}_A} \rho_i \;.$$

To obtain an upper bound for $\mathbb{P}\{B_k\}$ we must sum the above bound over all $\bar{t}$ such that $C(\bar{t}) = j$, and then sum over $j = 1, \ldots, k-1$.

To do so, first fix j and then fix $1 \leq t(1) < \ldots < t(j) \leq t$. For any $\bar{t}(t_1, \ldots, t_k)$ such that $C(\bar{t}) = j$ consider the subsequence $t_{j_1}, \ldots, t_{j_j}$ of $t_1, \ldots, t_k$ that are the most left occurrences of A for each one of the j's clusters. Now sum over the measures $\mathbb{P}\{\bar{t}\}$ of all those $\bar{t}$ such that $(t_{j_1}, \ldots, t_{j_j}) = (t(1), \ldots, t(j))$. Namely, sum over $\Delta_i = n/2+1, \ldots, \overline{\ell}_A$ with $i \neq j_1, \ldots, i \neq j_j$. The sum can be divided as in (10) in Δ_i with $i = n/2+1, \ldots, 2n-1$ and Δ_i with $i = 2n, \ldots, \overline{\ell}_A$. We have that

$$\sum_{(t_{j_1},\ldots,t_{j_j})=(t(1),\ldots,t(j))} \mathbb{P}\{\bar{t}(t_1,\ldots,t_k)\} \;\leq\; \frac{(k-j)^j}{j!} \;(\mathbb{P}\{A\}+\phi(\ell_A))^j \;(\varepsilon_1(A)+\varepsilon_2(A))^{k-j} \;.$$

Summing over j's we have that $\mathbb{P}\{B_k\}$ is bounded by

$$\begin{aligned}
\mathbb{P}\{B_k\} \;&\leq\; \sum_{j=1}^{k-1} \frac{t^j}{j!}\;\frac{k^j}{j!}\;(\mathbb{P}\{A\}+\phi(\ell_A))^j \;\varepsilon^{k-j} \\
&\leq\; \max\Big\{1, (\,t\,[\mathbb{P}\{A\}+\phi(\ell_A)])^k \;\Big\} \sum_{j=1}^{k-1} \frac{\varepsilon(A)^{k-j}\;k^j}{j!\;j!} \;.
\end{aligned}$$

Finally, we note that for any positive integer k and any $\varepsilon > 0$

$$\begin{aligned}
\sigma_k \;&:=\; \sum_{j=1}^{k} \frac{\varepsilon^{(k+1)-j}\;(k+1)^j}{j!\;j!} = \varepsilon \sum_{j=1}^{k-1} \frac{\varepsilon^{k-j}\;k^j}{j!\;j!}\;\frac{(k+1)^j}{k^j} + \varepsilon\;\frac{(k+1)^k}{k!\;k!} \\
&\leq\; \varepsilon\;\left(\frac{k+1}{k}\right)^k \sigma_{k-1} + \varepsilon\;\frac{e^k}{k!} \;\leq \varepsilon\;e\;\sigma_{k-1} + \varepsilon\;\frac{e^k}{k!} \;.
\end{aligned}$$

A telescopic sum can be used to prove that

$$\sum_{j=1}^{k-1} \frac{\varepsilon(A)^{k-j}\ k^j}{j!\ j!} \leq \varepsilon(A)\ . \begin{cases} \frac{e^k}{(k-1)!} & 1 \leq k \leq \frac{1}{\varepsilon(A)} \\ \\ \frac{e^k}{\left(\frac{1}{\varepsilon(A)}\right)!\ \left(\frac{1}{\varepsilon(A)}\right)^{k-\frac{1}{\varepsilon(A)}}} & \frac{1}{\varepsilon(A)} < k \end{cases} .$$

This ends the proof of the bound for $I\!P\{B_k\}$.

Proof: A's isolated provide the Poisson limit law.

We can bound the most right term on the right-hand side of (11) by

$$\sum_{\bar{t}\in G_k} \left| I\!P\left\{ \bigcap_{j=1}^{k} \tau_A{}^{(j)} = t_j\ ;\ \tau_A{}^{(k+1)} > t \right\} - I\!P\{A\}^k \prod_{j=1}^{k+1} \mathcal{P}_j \right| \tag{13}$$

$$+ \quad \sum_{\bar{t}\in G_k} I\!P\{A\}^k \left| \prod_{j=1}^{k+1} \mathcal{P}_j - \prod_{j=1}^{k+1} e^{-(\Delta_j - \overline{\ell}_A)\boldsymbol{P}\{A\}} \right| \tag{14}$$

$$+ \quad \sum_{\bar{t}\in G_k} I\!P\{A\}^k \left| e^{-(t-(k+1)\overline{\ell}_A)\boldsymbol{P}\{A\}} - e^{-t\boldsymbol{P}\{A\}} \right| \tag{15}$$

$$+ \quad \left| \frac{\#G_k\ k!}{t^k}\ \frac{e^{-t\boldsymbol{P}\{A\}} t^k I\!P\{A\}^k}{k!} - \frac{e^{-t\boldsymbol{P}\{A\}} t^k I\!P\{A\}^k}{k!} \right| . \tag{16}$$

A combinatorial argument allow us to get the bound

$$\begin{aligned} \#G_k &\leq \frac{t\ (t-1-2\overline{\ell}_A)\ (t-2-2\overline{\ell}_A)\ (t-3-2\ell)\ \ldots (t-k+1-2\overline{\ell}_A)}{k!} \\ &\leq \binom{t}{k} \leq \frac{t^k}{k!}\ . \end{aligned}$$

Moreover, the leading term in (13) is bounded using Proposition 15. Then we have that (13) is bounded by

$$C\ \frac{t^k}{k!}\ (I\!P\{A\} + \phi(\ell_A))^k\ \varepsilon(A)\ .$$

The leading term in (14) is bounded by Theorem 1 and Lemma 8. Then (14) is bounded by

$$
\begin{aligned}
& C\, \frac{t^k}{k!}\, I\!P\{A\}^k \sum_{j=1}^{k} \prod_{i=1}^{j-1} \mathcal{P}_i \, \left|\mathcal{P}_j - e^{-\Delta_j \boldsymbol{P}\{A\}}\right| \prod_{i=j+1}^{k} e^{-\Delta_i \boldsymbol{P}\{A\}} \\
\leq\ & C\, \frac{t^k}{k!}\, I\!P\{A\}^k \ k \ e_1(A) \\
\leq\ & C\, \frac{t^k\ (I\!P\{A\} + \phi(\ell_A))^k}{(k-1)!}\, \varepsilon(A)\ .
\end{aligned}
$$

where $e_1(A)$ was defined in (3).

Term (15) is clearly bounded using the Mean Value Theorem by

$$
\frac{t^k\ I\!P\{A\}^k}{k!}(k+1)\overline{\ell}_A I\!P\{A\} \leq C\ \ \frac{t^k\ I\!P\{A\}^k}{(k-1)!}\ \ \ell_A I\!P\{A\}\ .
$$

Finally by a direct computation we can bound the difference below

$$
\left| \frac{\#G_k\ k!}{t^k} \ -1 \right| \leq \left| \frac{(t-k-2\overline{\ell}_A)^k}{t^k} -1 \right| = \left| e^{k\log(1-(k+\overline{\ell}_A)/t)} -1 \right| \leq C \frac{k\ (k+\ell_A)}{t}\ .
$$

So, (16) is bounded by

$$
C \frac{k\ (k+2\ell_A)}{t}\ \frac{e^{-t\boldsymbol{P}\{A\}} t^k I\!P\{A\}^k}{k!}\ .
$$

The exchange of variables $\tilde{t} = tI\!P\{A\}$ ends the proof of the theorem.

Remark. We proved Theorem 16 for ϕ-mixing processes for the clarity of the exposition. The same proof of Theorem 16 as well as the proof of Proposition 15 can be carried on to prove the ψ-mixing and α-mixing cases. Only we need to adapt the error to each case. We indicate briefly the modifications needed for the ψ-mixing case. We bound the left hand side of inequality (10) (where $k=1$) as follows: we still divide it in terms with closer and longer occurrences of A. Then, it is bounded by,

$$
\begin{aligned}
&\left(\sum_{i=t_1+1}^{t_1+n-1}+\sum_{i=t_1+n}^{t_1+\overline{\ell}_A-1}\right) I\!P\left\{N_1^{t_1-1}=0\ ;\ T^{-t_1}A\ ;\ T^{-i}A\ ;\ N_{t_1+\overline{\ell}_A}^t=0\right\}\\
\leq\ & \sum_{i=t_1+(n/2)+1}^{t_1+n-1} I\!P\left\{N_1^{t_1-1}=0\right\}\ I\!P\{A\}\ (1+\psi(0))^3\ I\!P\left\{A^{(i-t_1)}\right\}\ I\!P\left\{N_{t_1+\overline{\ell}_A}^t=0\right\}\\
&+\sum_{i=t_1+n}^{t_1+\overline{\ell}_A-1} I\!P\left\{N_1^{t_1-1}=0\right\}\ I\!P\{A\}\ (1+\psi(0))^3\ I\!P\{A\}\ I\!P\left\{N_{t_1+\overline{\ell}_A}^t=0\right\}\\
\leq\ & (1+\psi(0))^3 I\!P\{A\}\left(n\ I\!P\left\{A^{(n/2)}\right\}+\overline{\ell}_A I\!P\{A\}\right)\prod_{j=1}^{2} I\!P\left\{\tau_A>\Delta_i-\overline{\ell}_A\right\}\\
\leq\ & C\ \left[(1+\psi(0))^2\ I\!P\{A\}\right]^k \varepsilon(A)\ \prod_{j=1}^{2} e^{-t_i\boldsymbol{P}\{A\}}\,(1+e_1(A))\\
\leq\ & C\ [C\ I\!P\{A\}]^k\,\varepsilon(A)\ e^{-t\boldsymbol{P}\{A\}}\ .
\end{aligned}
$$

For the first inequality we iterate the mixing property together with the fact that $A\notin\mathcal{B}_n(2)$. In the third inequality we applied Theorem 1. The same argument is carried on the rest of the proofs. Then we get the following theorem

Theorem 17 *Let $\{X_m\}_{m\in\mathbb{Z}}$ be a ψ-mixing or a α-mixing process. Let $A\notin\mathcal{B}_n(2)$. Then, the following inequality holds*

$$
\left|I\!P\left\{N_{\frac{t}{\boldsymbol{P}\{A\}}}=k\right\}-\frac{e^{-t}t^k}{k!}\right|\leq
\begin{cases}
\varepsilon(A)\ \frac{e^k}{(k-1)!}\ \lambda_M & 1\ \leq k\ \leq\frac{1}{\varepsilon(A)}\\[2ex]
\varepsilon(A)\ \dfrac{e^k}{\left(\frac{1}{\varepsilon(A)}\right)!\ \left(\frac{1}{\varepsilon(A)}\right)^{k-\frac{1}{\varepsilon(A)}}}\lambda_M & \frac{1}{\varepsilon(A)}\ <k\ \leq\frac{2t}{n\boldsymbol{P}\{A\}}\\[2ex]
nI\!P\{A\}\ \frac{e^{-t}}{(k-1)!\ t} & \frac{2t}{n\boldsymbol{P}\{A\}}\ <k
\end{cases}
.
$$

where $\varepsilon(A)=\varepsilon_1(A)+\varepsilon_2(A)$, with $\varepsilon_1(A)$ defined in Proposition 15 (replacing ϕ by ψ or α) and for ψ-mixing processes,

$$
\varepsilon_2(A)=(1+\psi(0))^{2k+1}\ \inf_{\ell}\left\{I\!P\{A\}\left(1+\sum_{j\geq n}\psi(j)\right)+\psi(\ell)\right\}\ e^{-t}\ .
$$

So, if $\{A_n\}_{n\in\boldsymbol{N}}$ is such that $A_n\notin\mathcal{B}_n(2)$ for all $n\in I\!N$, then the law of the number of occurrences of A_n up to the re-scaled time $t/I\!P\{A_n\}$ converges weakly to a Poisson law as $n\to\infty$ (with the above rate). For α-mixing processes,

$$\varepsilon_2(A) = \inf_{\ell} \left\{ I\!P\{A\} \left(1 + \sum_{j=n}^{2n} \alpha(j) \right) + \frac{\alpha(n)}{I\!P\{A\}^k} \right\} .$$

The law of the number of occurrences of A_n up to the re-scaled time $t/I\!P\{A_n\}$ converges to a Poisson process if $\alpha(n)/I\!P\{A_n\}^k \to 0$ as $n \to \infty$ for all k.

Remark. The condition $\alpha(n)/I\!P\{A_n\}^k \to 0$ as $n \to \infty$ for all k means basically that α must decay super-exponentially fast.

Remark. In Theorem 16 and Theorem 17 and according to Definition 14 we chose $s = 2$. This value of s is purely formal and can be replaced by any other fixed value. The corresponding modification must be done in the definition of $\varepsilon_1(A)$. We recall that in Collet, Galves and Schmitt (1999) it was proved that for exponentially ψ-mixing processes there exist positive constants C and c such that $I\!P\left\{\mathcal{B}_n(3)\right\} \le Ce^{-cn}$. In Abadi (2001) the same inequality was proven for ϕ-mixing processes and for a value of s that just depends on the cardinality of the alphabet. This shows that Theorem 16 holds for typical cylinders. The *almost everywhere* Poisson limit law is also an immediate corollary of this fact.

The error terms given by Theorem 16 and Theorem 17 decreasing factorially fast on k allow us to estimated the moments of $N_{t/\boldsymbol{P}\{A\}}$ by those known of a r.v. with Poisson distribution.

Corollary 18 *Let $\beta > 0$. Let X be a r.v. with Poisson distribution of parameter $t > 0$. Under the conditions of Theorem 16 (Theorem 17)*

$$\left| I\!E\left(N^{\beta}_{t/\boldsymbol{P}\{A\}} \right) - I\!E(Z^{\beta}) \right| \le C_{\beta}\, \varepsilon(A) .$$

where $\varepsilon(A)$ was defined in Theorem 16 (Theorem 17) and C_β is a constant that just depends on β.

Proof. By definition

$$\begin{aligned} \left| I\!E\left(N^{\beta}_{t/\boldsymbol{P}\{A\}} \right) - I\!E(Z^{\beta}) \right| &= \left| \sum_{k \ge 0} k^{\beta}\; I\!P\left\{ N_{\frac{t}{\boldsymbol{P}\{A\}}} = k \right\} - \sum_{k \ge 0} k^{\beta}\; \frac{e^{-t}\; t^k}{k!} \right| \\ &\le \sum_{k \ge 0} k^{\beta} \left| I\!P\left\{ N_{\frac{t}{\boldsymbol{P}\{A\}}} = k \right\} - \frac{e^{-t}\; t^k}{k!} \right| . \end{aligned}$$

Theorems 16 (Theorem 17) end the proof of the corollary.

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