On the Euler–Poincaré Characteristic of the Random Cluster Model

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Dedicated to Leonid A. Pastur on the occasion of his 65th birthday

Abstract: Recent results concerning the topological properties of random geometrical sets have been successfully applied to the study of the morphology of clusters in percolation theory.

We present here new results about the behaviour of the Euler characteristic of the clusters of the (Fortuin–Kasteleyn) random cluster measure.

Key words: Euler-Poincaré characteristic, Fortuin–Kasteleyn representation, Alexander duality, Phase transitions.

1 Introduction

Recently, new insights in the study of the critical properties of clusters in percolation theory have emerged based on ideas coming from mathematical

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morphology [Se] and integral geometry [Ha, S]. These mathematical theories provide a set of geometrical and topological measures allowing to quantify the morphological properties of random systems. In particular these tools have been applied to the study of random cluster configurations in percolation theory and statistical physics [MW, O, Wa1, M1].

One of these measures is the Euler-Poincaré characteristic which is a well known descriptor of the topological features of geometric patterns.

The application of these description tools for the study of random systems in statistical mechanics has already provided interesting results. In [MW, O], the computation of the Euler-Poincaré characteristic χ for a system of penetrable disks in several models of continuum percolation has led to conjectured new bounds for the critical value of the continuum percolation density. In [MW], an exact calculation shows that a close relation exists between the value where Euler-Poincaré characteristic vanishes and the critical threshold for continuum percolation in dimensions 2 and 3.

Of similar interest in the same domain, we recall that for the problem of bond percolation on regular lattices, Sykes and Essam [SyEs] were able to show, using standard planar duality arguments, that for the case of self-dual matching lattices (e.g. $\mathbb{Z}^2$), the mean value of the Euler-Poincaré characteristic changes sign at the critical point, see also [Gri].

More recently, H. Wagner [Wa1] has computed the Euler-Poincaré characteristic of random clusters on all plane regular mosaics (the 11 Archimedean lattices) as a function of the site occupancy probability $p \in [0, 1]$ and showed that a close connection exists between the threshold for site percolation on these lattices and the point where the Euler-Poincaré characteristic (expressed as a function of p) changes sign.

The aim of this work, motivated by discussions with H. Wagner about his recent results, is to present new ones about the behaviour of the Euler characteristic in statistical mechanics in the case of the Fortuin–Kasteleyn representation of the q–state Potts model on $\mathbb{Z}^d$. This representation includes the well known Ising model ($q = 2$) and the lattice bond percolation problem ($q = 1$), see [FK].

Our first result relies on existence properties of the mean Euler characteristic with respect to the Fortuin–Kasteleyn measure. We introduce a local Euler characteristic and show that its mean coincides with the mean Euler characteristic per site. Denote this limiting quantity by $\chi^{\mathrm{bc}}(\beta, q)$ where β

is the (inverse) temperature and the superscript bc refers to the boundary conditions. We then show that, in dimension 2, $\chi^{\mathrm{bc}}(\beta, q)$ satisfies the duality property $\chi^{\mathrm{f}}(\beta, q) = -\chi^{\mathrm{w}}(\beta^*, q)$, where β^* is the dual (inverse) temperature and f and w refer respectively to the free and wired boundary conditions. This implies that the mean local Euler characteristic is either zero or exhibits a jump at the self dual temperature. It vanishes when $q = 1$, and we prove that a jump actually occurs for q large enough. This jump turns out to be of order $q^{-1/2}$.

Similar results hold in higher dimensions d. Namely, for q large, the mean local Euler characteristic exhibits a jump, with change of sign, of order $q^{-\frac{1}{d}}$ when d is even, while when d is odd it exhibits a jump, without change of sign, of order $q^{-\frac{d-1}{d}}$.

Let us mention that this work relates on the large q analysis of the FK representation of the q–states Potts model given in [LMMRS].

The paper will be organized as follows. Section 2 is devoted to the 2–dimensional case. In Section 3, we present the results in higher dimensions. Concluding remarks are given in Section 4.

2 Euler Characteristic on $\mathbb{Z}^2$

2.1 Definitions

Let us consider the square lattice

$$\mathbb{Z}^2 = \{x = (x_1, x_2) : x_i \in \mathbb{Z},\, i = 1, 2\}$$

whose elements are called sites. Two sites x and y are nearest neighbors if $|x_1 - y_1| + |x_2 - y_2| = 1$. We call bonds $b = \langle xy \rangle$ the subsets of $\mathbb{R}^2$ which are the straight line segments with the nearest neighbors sites x and y as endpoints. We call plaquettes $p = [x, y, z, t]$ the subsets of $\mathbb{R}^2$ which are unit squares whose corners are the sites x, y, z, t. The boundary ∂b of the bond $b = \langle xy \rangle$ is the set $\{x, y\}$ and the boundary ∂p of the plaquette $p = [x, y, z, t]$ is the set of bonds $\{\langle xy \rangle, \langle yz \rangle, \langle zt \rangle, \langle tx \rangle\}$.

With this structure the lattice becomes a cell-complex $\mathbb{L} = \{\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2\}$, where $\mathbb{L}_0 = \mathbb{Z}^2$ is the set of sites, $\mathbb{L}_1$ is the set of bonds and $\mathbb{L}_2$ is the set of plaquettes (see e.g. [DW]). More generally, the elements of $\mathbb{L}_p$ will be called p–cells. We also introduce the co–boundary $\delta x = \{\langle xy \rangle \in \mathbb{L}_1 : y \in \mathbb{L}_0\}$.

To a set of bonds $X \subset \mathbb{L}_1$ we associate the subcomplex $\Lambda(X) \subset \mathbb{L}$ defined as the maximal closed subcomplex that contains X : this subcomplex is the union $\Lambda(X) = \Lambda_0(X) \cup \Lambda_1(X) \cup \Lambda_2(X)$ where

$$\Lambda_1(X) = X$$

is the set of bonds itself,

$$\Lambda_0(X) = \{x \in \mathbb{L}_0 : x \cap X \neq \emptyset\}$$

is the set of sites which are endpoints of bonds of X, and

$$\Lambda_2(X) = \{p = [x, y, z, t] \in \mathbb{L}_2 : \langle xy \rangle, \langle yz \rangle, \langle zt \rangle, \langle tx \rangle \in X\}$$

is the set of plaquettes whose all four bounds in their boundary belongs to X. Figure 1 shows a cell complex $\Lambda(X)$ associated to a given set of bonds X.

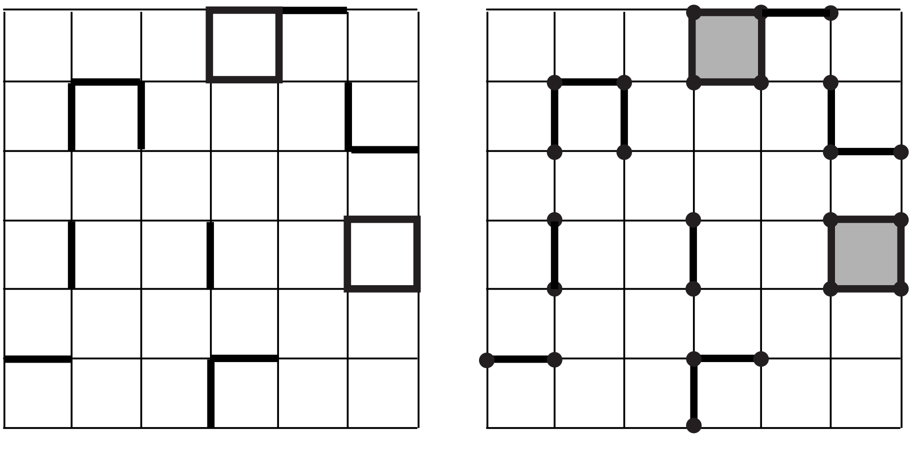

Figure 1: A set of bonds X and its associated cell complex $\Lambda(X)$.

For a set of bonds X, we define the boundary $B(X)$of X as the set of bonds of X that belong to a plaquette of $L_2 \setminus \Lambda_2(X)$. We also associate to the set of bonds X the occupation number

$$n_b = \begin{cases} 1 & \text{if} \quad b \in X \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and denote by $n \equiv \{n_b\}_{b\in\mathbb{L}_1}$ the associated configuration. We then use $N^0(n) \equiv |\Lambda_0(X)|$ to denote the number of sites of $\Lambda_0(X)$, $N^1(n) \equiv |\Lambda_1(X)|$ to denote the number of bonds of $\Lambda_1(X)$, and $N^2(n) \equiv |\Lambda_2(X)|$ to denote the number of plaquettes of $\Lambda_2(X)$. Hereafter we will use $|E|$ to denote the cardinality of the set E.

The Euler characteristic of $\Lambda(X)$ is defined by

$$\boldsymbol{\chi}(n) = N^0(n) - N^1(n) + N^2(n) \tag{2}$$

It satisfies the Euler–Poincaré formula

$$\boldsymbol{\chi}(n) = \pi^0(n) - \pi^1(n) + \pi^2(n) \tag{3}$$

where $\pi^0(n)$ and $\pi^1(n)$ are respectively the number of connected components and the maximal number of independent 1-cycles of $\Lambda(X)$. Here $\pi^2(n) = 0$ because $\Lambda(X)$ has no 2-cycles.

Our aim is to study the mean value of the Euler characteristic with respect to the Fortuin–Kasteleyn measure. To introduce this measure we define, for a "volume" $V \subset \mathbb{L}_1$, the partition functions with boundary conditions

$$Z^{\rm bc}(V,\beta,q) = \sum_{n\in\{0,1\}^V} (e^\beta - 1)^{N^1(n)} q^{|\Lambda_0(V)|-N^0(n)+\pi^0(n)}\, \kappa^{\rm bc}(n) \tag{4}$$

and the corresponding finite volume expectations of local functions f of the bonds variable $\{n_b\}$ by

$$\langle f\rangle^{\rm bc}(V,\beta,q) = \frac{1}{Z^{bc}(V,\beta,q)} \sum_{n\in\{0,1\}^V} f(e^\beta - 1)^{N^1(n)} q^{|\Lambda_0(V)|-N^0(n)+\pi^0(n)}\, \kappa^{\rm bc}(n) \tag{5}$$

where $\kappa^{\rm bc}(n)$ refers to the boundary condition bc. In particular, we will be interested to the free boundary condition $\kappa^{\rm f}(n) = 1$ and to the ordered or wired boundary condition $\kappa^{\rm w}(n) = \prod_{b\in B(V)} n_b$.

Notice that for a configuration $n \in \{0,1\}^{|V|}$, one has

$$N^0(n) = \Lambda_0(V) - \sum_{x\in\Lambda_0(V)} \prod_{b\in\delta x\cap V} (1-n_b)$$

$$N^1(n) = \sum_{x\in\Lambda_0(V)} \frac{1}{2} \sum_{b\in\delta x\cap V} n_b$$

$$N^2(n) = \sum_{x\in\Lambda_0(V)} \frac{1}{4} \sum_{p\in P(x)} \prod_{b\in\partial p\cap V} n_b$$

where $P(x)$ is the set of the four plaquettes that intersect x. This representation motivates the following definition of the local Euler characteristic.

$$\chi_x(n) \equiv 1 - \prod_{b\in\delta x}(1-n_b) - \frac{1}{2}\sum_{b\in\delta x} n_b + \frac{1}{4}\sum_{p\in P(x)}\prod_{b\in\partial p} n_b \tag{6}$$

2.2 Results

We first prove existence.

Theorem 1 *a) The following limits exist and coincide*

$$\lim_{V\to\mathbb{L}_1}\left\langle \frac{\boldsymbol{\chi}}{|\Lambda_0(V)|}\right\rangle^{\mathrm{f}}(V,\beta,q) = \lim_{V\to\mathbb{L}_1}\langle\chi_0\rangle^{\mathrm{f}}(V,\beta,q) \equiv \chi^{\mathrm{f}}(\beta,q) \tag{7}$$

and define the mean local Euler characteristic with free boundary condition $\chi^{\mathrm{f}}(\beta,q)$*.*

b) Analogously, the following limits exist and coincide

$$\lim_{V\to\mathbb{L}_1}\left\langle \frac{\boldsymbol{\chi}}{|\Lambda_0(V)|}\right\rangle^{\mathrm{w}}(V,\beta,q) = \lim_{V\to\mathbb{L}_1}\langle\chi_0\rangle^{\mathrm{w}}(V,\beta,q) \equiv \chi^{\mathrm{w}}(\beta,q) \tag{8}$$

and define the mean local Euler characteristic with wired boundary condition $\chi^{\mathrm{w}}(\beta,q)$*.*

Proof. The proof of these results are standard consequences of FKG inequalities [FKG]. Let us first recall basic definitions and properties.

A configuration n dominates n', $n \succ n'$, if $n_b \geq n'_b$ for all bonds b.

A function f is said to be increasing if $n \succ n' \Rightarrow f(n) \geq f(n')$. It is said decreasing if $-f$ is increasing.

A measure $\langle\ \rangle$ is said to have the FKG property if :

$$\langle fg\rangle \geq \langle f\rangle\langle g\rangle$$

for all increasing functions f and g.

The measures (5) with free and wired boundary conditions are FKG, see e.g. Theorem 2.2 in [ACCN]. This implies the following monotonicity properties:

$$\langle f\rangle^{\mathrm{f}}(V',\beta,q) \geq \langle f\rangle^{\mathrm{f}}(V,\beta,q), \quad \langle f\rangle^{\mathrm{w}}(V',\beta,q) \leq \langle f\rangle^{\mathrm{w}}(V,\beta,q)$$

for any increasing function and any pair of subsets $V' \subset V$.

These properties give, by standard arguments, the existence of the infinite volume limits with the corresponding boundary conditions, through sequence of increasing volumes, see e.g. Theorem 2.3 in [ACCN]. They imply also, following [G], that both limiting states are translation invariant. The translation invariance allows finally to show that the two limits in (7) (resp. (8)) coincide provided the first one is taken in such a way that $|B(V)/V| \to 0$ as $V \to \mathbb{L}_1$. ■

Theorem 2 *a) The mean local Euler characteristics satisfies the duality relation*

$$\chi^{\mathrm{f}}(\beta,q) = -\chi^{\mathrm{w}}(\beta^*,q) \tag{9}$$

where the dual inverse temperature β^* is given by

$$(e^{\beta}-1)(e^{\beta^*}-1) = q \tag{10}$$

b) At the self-dual point $\beta_t = \log(1+\sqrt{q})$ solution of (10) with $\beta = \beta^*$ we have

$$\chi^{\mathrm{f}}(\beta_t,q) = -\chi^{\mathrm{w}}(\beta_t,q) \tag{11}$$

Proof. Duality properties between FK measures with free and wired b.c. have been already derived in more general situation, (see [PV]), and we will use some notations inspired from Section 3.3 of that reference. First, we recall some basic geometrical results about Poincaré duality and Alexander duality. Namely, we introduce the dual lattice

$$(\mathbb{Z}^2)^* = \{x = (x_1,x_2) : x_i + 1/2 \in \mathbb{Z},\, i = 1,2\}$$

and use $\mathbb{L}^*$ to denote the corresponding cell complex. $\mathbb{L}^*$ is the dual complex of $\mathbb{L}$ and the following geometrical relations hold:

1. each site $x \in \mathbb{L}$ is the center of a unique plaquette $p^* \in \mathbb{L}^*$,

2. each bond $b \in \mathbb{L}$ is crossed by a unique bond $b^* \in \mathbb{L}^*$,

3. each plaquette $p \in \mathbb{L}$ has a unique site $x^* \in \mathbb{L}^*$ at its center.

This correspondence between the p–cells of $\mathbb{L}$ and the $(2-p)$–cells of $\mathbb{L}^*$ is one–to–one

$$s_p \longleftrightarrow s^*_{2-p} \tag{12}$$

For a subcomplex $\Lambda \subset \mathbb{L}$, the dual Λ^* of Λ is the set of elements of $\mathbb{L}^*$ which are in one to one correspondence (12) with the elements of Λ.

Consider the set of bonds $V \subset \mathbb{L}_1$ such that

$$\Lambda_0(V) = \left\{(x_1, x_2) \in \mathbb{Z}^2 : |x_i| \leq m,\ i = 1, 2\right\}$$

The dual set $[\Lambda(V)]^*$ is the union $[\Lambda(V)]^* = \cup_{p=0}^{2} \Lambda^*_p(V)$ where each p–cells $s^*_p \in \Lambda^*_p(V)$ is in one -to-one correspondence with a $2-p$ cell $s_{2-p} \in \Lambda_{2-p}(V)$. In particular

$$\left|\Lambda^*_p(V)\right| = |\Lambda_{2-p}(V)| \tag{13}$$

We let $\Omega = \Lambda(V) \cup f$ be the complex obtained from $\Lambda(V)$ by adding an extra face f whose boundary is the set of bonds of $B(V)$. Then Ω is homeomorphic to the sphere S^2. We let x^* be some point inside the face f.

We define the dual Ω^* of Ω as the set $[\Lambda(V)]^* \cup x^*$ except that now all the bonds of $\Lambda^*_1(V)$ that crosses $B(V)$ meet at the point x^*. We call faces of Ω^* the new 2–cells. Again the correspondence is one-to-one provided one replaces 2–cells by faces.

We now turn to Alexander's duality. Let $n \in \{0, 1\}^V$ and

$$V^* = \{b^* \in \Omega^* : b^* \text{ crosses } b\}$$

We define the A-dual configuration $\widehat{n} \in \{0, 1\}^{V^*}$of n by

$$\widehat{n}_{b^*} = 1 - n_b \tag{14}$$

where b^* is the bond of V^* intersecting b. In other words, if we let $X \subset V$ be the set of bonds b such that $n_b = 1$, $\Lambda(X)$ its associated cell-complex and $\widehat{\Lambda}(X) = [\Omega \setminus \Lambda(X)]^*$, then the bonds of $\widehat{\Lambda}(X)$ are those of Ω^* that do not intersect the bonds of $\Lambda(X)$. Obviously one has

$$N^1(n) + N^1(\widehat{n}) = |\Lambda_1(V)| = |\Lambda^*_1(V)| \tag{15}$$

$$N^2(n) + N^0(\widehat{n}) = |\Lambda_2(V)| + 1 = |\Lambda^*_0(V)| + 1 \tag{16}$$

where $N^p(\widehat{n})$ denote the number of p–cells of $\widehat{\Lambda}(X)$. Moreover by Alexander's duality theorem, see [A] (Theorem 4.10 p. 24), [Le] (p. 205), or [PV]:

$$\pi^0(n) = \pi^1(\widehat{n}) + 1 \tag{17}$$
$$\pi^0(\widehat{n}) = \pi^1(n) + 1 \tag{18}$$

where $\pi^p(\widehat{n})$ denotes the p–Betti number of the complex $\widehat{\Lambda}(X)$. Then, by using successively (15), (10) and Euler's formula (2) (3) we can write

$$\begin{aligned}
\left(e^\beta - 1\right)^{N^1(n)} q^{|\Lambda_0(V)| - N^0(n) + \pi^0(n)} &= \left(\frac{e^\beta - 1}{q}\right)^{N^1(n)} q^{|\Lambda_0(V)| - N^0(n) + \pi^0(n) + N^1(n)} \\
&= \left(\frac{e^\beta - 1}{q}\right)^{|\Lambda_1(V)|} \left(\frac{e^\beta - 1}{q}\right)^{-N^1(\widehat{n})} q^{|\Lambda_0(V)| - N^0(n) + \pi^0(n) + N^1(n)} \\
&= \left(\frac{e^\beta - 1}{q}\right)^{|\Lambda_1(V)|} q^{|\Lambda_0(V)|} (e^{\beta^*} - 1)^{N^1(\widehat{n})} q^{-N^0(n) + \pi^0(n) + N^1(n)} \\
&= \left(\frac{e^\beta - 1}{q}\right)^{|\Lambda_1(V)|} q^{|\Lambda_0(V)|} (e^{\beta^*} - 1)^{N^1(\widehat{n})} q^{N^2(n) + \pi^1(n)}
\end{aligned}$$

Setting $R = \left(\frac{e^\beta - 1}{q}\right)^{|\Lambda_1(V)|} q^{|\Lambda_0(V)|}$, we get from (16)(18)

$$R^{-1}\left(e^\beta - 1\right)^{N^1(n)} q^{|\Lambda_0(V)| - N^0(n) + \pi^0(n)} = (e^{\beta^*} - 1)^{N^1(\widehat{n})} q^{\left|\Lambda_0^*(V)\right| - N^0(\widehat{n}) + \pi^0(\widehat{n})} \tag{19}$$

This formula establishes the one-to-one correspondence between the FK measures with free and wired b.c.: indeed, the right-hand-side of (19) coincide up to a boundary term independent of $\widehat{n}$ with the Boltzman weight entering in the definition of the partition function (4) with wired boundary condition. From this formula, we get by (17)(18):

$$\langle \boldsymbol{\chi} \rangle^{\mathrm{f}} (V, \beta, q) = \langle 2 - \boldsymbol{\chi} \rangle^{\mathrm{w}} (V^*, \beta^*, q)$$

It then suffices to divide both terms of this equation by $|\Lambda_0(V)|$ and to take the thermodynamic limit $(m \to \infty)$ to get the result. ■

Remark 3 *Let us mention that for every* $\beta \neq \beta_t$

$$\chi^{\mathrm{f}}(\beta, q) = \chi^{\mathrm{w}}(\beta, q) \tag{20}$$

when $q = 2$ *and* $q \geq 4$

This is a consequence of the differentiability of the free energy with respect to β for the corresponding values of q combined with FKG inequality, (see [LM]).

Actually by FKG inequality, one has that

$$\begin{cases} \langle n_b \rangle^{\mathrm{f}}(V,\beta,q) \leq \langle n_b \rangle^{\mathrm{bc}}(V,\beta,q) \leq \langle n_b \rangle^{\mathrm{w}}(V,\beta,q) \\ \\ 0 \leq \langle \prod_{b\in A} n_b \rangle^{\mathrm{w}} - \langle \prod_{b\in A} n_b \rangle \leq \sum_{b\in A} \left(\langle n_b \rangle^{\mathrm{w}} - \langle n_b \rangle \right) \end{cases}$$

since obviously n_b is an increasing function and the same is also true for the function $f_A = \sum_{b\in A} n_b - \prod_{b\in A} n_b$, (see again [LM]).

Therefore the state is unique as soon as, in the thermodynamic limit, $\langle n_b \rangle^{\mathrm{f}}(\beta,q) = \langle n_b \rangle^{\mathrm{w}}(\beta,q)$. This property holds whenever the free energy is differentiable with respect to β.

Indeed, again by FKG, $\langle n_b \rangle^{\mathrm{f}}(\beta,q)$ and $\langle n_b \rangle^{\mathrm{w}}(\beta,q)$ are the respective left and right derivatives of the free energy $f(\beta,q) = \lim\limits_{V\to\mathbb{L}_1} \frac{1}{|\Lambda_0(V)|} \ln Z^{\mathrm{bc}}(V,\beta,q)$ with respect to β.

The proof of differentiability of $f(\beta,q)$ for $q=2$ and $q\geq 4$ may be found, e.g. in [B, MM, HKW].

Let us also mention that for the percolation model ($q=1$) the free energy is obviously differentiable for any (inverse) temperature β. Therefore in that case $\chi^{\mathrm{f}}(\beta,q=1) = \chi^{\mathrm{w}}(\beta,q=1) = \chi^{\mathrm{bc}}(\beta,q=1)$. This implies, by Theorem 2, that at the transition point $\beta_t = \ln 2$, one has $\chi^{\mathrm{bc}}(\beta_t,q=1) = 0$ (for any bc).

Theorem 4 *Assume q is large enough, then the mean local Euler characteristic is discontinuous at β_t*

$$\chi^{\mathrm{f}}(\beta_t,q) = -\chi^{\mathrm{w}}(\beta_t,q) = 2q^{-1/2} + O(q^{-7/12}) \tag{21}$$

Proof. The proof is mainly based on a contour representation of the model derived in [LMMRS]. To define these contours, we first introduce the co–boundary $C(X)$ of a set of bonds X as the set of bonds of $\mathbb{L}_1 \setminus X$ that contain 1 or two vertices in $\Lambda(X)$: $C(X) = C_1(X) \cup C_2(X)$ where

$$C_1(X) = \{b \in \mathbb{L}_1 \setminus X : b \text{ has 1 endpoint in } \Lambda(X)\}$$
$$C_2(X) = \{b \in \mathbb{L}_1 \setminus X : b \text{ has 2 endpoints in } \Lambda(X)\}$$

A subset $\gamma \subset \mathbb{L}_1$ is called contour if it is the co–boundary of a set of bonds X and if its dual is a connected set of bonds. The connectedness of the dual of γ means that it is the union of the bonds of a sequence $b_1, ..., b_n$ such that for any $i \leq n-1$, b_i and b_{i+1} belongs to the boundary of a same plaquette. When X is finite, γ is called contour of the free class and X will be denoted by $\mathrm{Int}\,\gamma$. The length of γ is defined as

$$\|\gamma\| = |C_1(X)| + |C_2(X)| = |C_1(\mathrm{Int}\,\gamma)| + |C_2(\mathrm{Int}\,\gamma)|$$

As a result of [LMMRS], one has

$$\begin{aligned} Z^{\mathrm{f}}(V, \beta_t, q) &= q^{|\Lambda_0(V)|+|\lambda(V)|} \mathcal{Z}(V|\varphi) \\ \mathcal{Z}(V|\varphi) &= \sum_{\{\gamma_1,...,\gamma_n\} \subset V \cup C(V)} \prod_{i=1}^{n} \varphi(\gamma_i) \end{aligned} \tag{22}$$

where the last sum runs over compatible families of contours, i.e. $\gamma_i \cap \gamma_j = \emptyset$, and $\lambda(V)$ is a boundary term such that $|\lambda(V)| \leq$ *cte* $|B(V)|$. The activities $\varphi(\gamma)$ of contours are given by

$$\varphi(\gamma) = q^{-\frac{\|\gamma\|}{4} + \pi^0(\mathrm{Int}\,\gamma)} e^{\Delta(\gamma)} \tag{23}$$

where $\Delta(\gamma)$ satisfies the bound :

$$|\Delta(\gamma)| \leq 6 q^{-\frac{1}{12}} \|\gamma\| \tag{24}$$

This implies that for large q the free energy $f(\varphi) = \lim_{V \to \mathbb{L}_1} \frac{1}{|\Lambda_0(V)|} \ln \mathcal{Z}(V|\varphi)$ of the contour model behaves like $2q^{-\frac{1}{2}} + O(q^{-1})$. We will show that the dominant contribution to the local Euler characteristic $\chi^{\mathrm{f}}(\beta, q)$ shares the same behaviour. This fact might be seen as follows. In the limit $q \to \infty$, there are only two ground state configurations where the bonds are either all empty or all occupied. The Euler characteristic of such configurations is 0. The first excitations of the empty ground state configuration consist of occupied bonds in the "sea" of empty ones. Each excitation gives a contribution to π^0 and the number of excitations scales like the volume, giving that the Euler characteristic behaves like $0(q^{-\frac{1}{2}})$. However, in order to compute the coefficient it is more convenient to consider the expression of the Euler characteristic in terms of the coefficient N_p and more precisely the expression (6) of the local Euler characteristic $\chi_x(n)$.

Notice that χ_x obviously depends only on the values of the bonds of the 4 plaquettes intersecting x. Define the distance between 2 vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ associated to the sup norm by $d((x_1, x_2), (y_1, y_2)) = \max_{i=1,2}\{|x_i - y_i|\}$. Let $d(x, S) = \min_{z\in S} d(x, z)$ the distance between a set S and a vertex $x \notin S$.

To compute $\langle\chi_0\rangle$ we will consider the following events

$$E_f = \big\{n \in \{0,1\}^V : \text{all the bonds of the co–boundary } \delta 0 \text{ of the origin } 0 \text{ are empty}\big\} \tag{25}$$

$$E = \big\{n \in \{0,1\}^V : \text{there is exactly one occupied bond in } \delta 0 \text{ and all the bonds of the co–boundary of sites at distance 1 from the end points of the occupied bond are empty}\big\} \tag{26}$$

and the complementary event

$$E' = \big\{n \in \{0,1\}^V \setminus (E \cup E_f)\big\} \tag{27}$$

Since for any configuration $n \in E_f$, $\chi_0(n) = 0$ and any configuration $n \in E$, $\chi_0(n) = \frac{1}{2}$, one gets

$$\langle\chi_0\rangle^{\mathrm{f}}(V, \beta, q) = \frac{1}{2}\Pr(E) + \sum_{n:n\in E'} [\chi_0(n)] \Pr(n) \tag{28}$$

It is easy to check that all configuration $n \in E'$ satisfy $|\chi_x(n)| \leq \frac{1}{2}$. Therefore

$$2\left|\sum_{n:n\in E'} [\chi_0(n)] \Pr(n)\right| \leq \Pr(E') \tag{29}$$

which is bounded by

$$\Pr(E') \leq \Pr(\exists\gamma : \gamma \ni 0; |\operatorname{Int}\gamma| \geq 2) \tag{30}$$

Actually, for every configuration $n \in E'$, there exists a contour $\gamma \ni 0$ such that $|\operatorname{Int}\gamma| \geq 2$. Taking into account that the minimal length of contours with $|\operatorname{Int}\gamma| \geq 2$ is 8 and that $\pi^0(\operatorname{Int}\gamma) \leq \frac{1}{6}\|\gamma\|$, this probability can be estimated by Peierls type arguments as follows

$$\begin{aligned}\Pr(E') &\leq \sum_{\gamma:\gamma\ni 0; |\operatorname{Int}\gamma|\geq 2} \Pr(\gamma) \leq \sum_{\gamma:\gamma\ni 0; \|\gamma\|\geq 8} \varphi(\gamma) \\ &\leq \sum_{m\geq 8} m^2 4^m \big(q^{-\frac{1}{4}} e^{6q^{-\frac{1}{12}}}\big)^m \leq 2\cdot 8^2 \cdot 4^8 \frac{q^{-\frac{3}{4}} e^{48q^{-\frac{1}{12}}}}{(1 - 4q^{-\frac{1}{4}} e^{6q^{-\frac{1}{12}}})^3}\end{aligned} \tag{31}$$

provided $q^{-\frac{1}{4}}e^{6q^{-\frac{1}{12}}} < \frac{1}{4}$. Here we used the equality (23) and the fact that the number of contours of length m that contain a given point is less than $m^2 4^m$.

On the other hand, in order to estimate the probability $\mathrm{Pr}(E)$ we let $\bar{\gamma}$ to be the set constituted by the six bonds containing one endpoint of the occupied bond as endpoint but not both of them. Then

$$\mathrm{Pr}(E) = 4\frac{\sum\limits_{\{\gamma_1,\ldots,\gamma_n\}\supset\bar{\gamma}} \prod\limits_{i=1}^{n} \varphi(\gamma_i)}{\mathcal{Z}(V|\varphi)} = 4\varphi(\bar{\gamma})\frac{\sum\limits_{\{\gamma_1,\ldots,\gamma_n\},\gamma_i\cap\bar{\gamma}=\emptyset} \prod\limits_{i=1}^{n} \varphi(\gamma_i)}{\mathcal{Z}(V|\varphi)} \tag{32}$$

The factor 4 stems from the fact that there are 4 bonds in the co–boundary of 0: we take V to be a box centered at the origin.

The last ratio can be estimated by cluster expansion [D, M]. For that purpose we introduce clusters X as multi-indexes defined on the sets of contours with integer values. The support $\mathrm{Supp}\, X$ of a cluster X is the set of contours γ for which $X(\gamma) \geq 1$. We also introduce the truncated functionals:

$$\phi^T(X) = \frac{a(X)}{\prod\limits_{\gamma} X(\gamma)!} \prod_{\gamma} \varphi(X)^{X(\gamma)} \tag{33}$$

The coefficient $a(X)$ is a combinatoric factor defined in terms of the connectivity properties of the graph $G(X)$ with vertices corresponding to $\gamma \in \mathrm{Supp}\, X$ (there are $X(\gamma)$ vertices for each $\gamma \in \mathrm{Supp}\, X$) being connected by an edge whenever the corresponding contours are not compatible.

As a result we have

$$\sum_{X:\mathrm{Supp}\, X\cap b\neq\emptyset} |\phi^T(X)| \leq (q^{-\frac{1}{4}}e^{6q^{-\frac{1}{12}}})^6 \tag{34}$$

for $4gq^{-\frac{1}{4}}e^{6q^{-\frac{1}{12}}} < 1$, where $g \approx e^{1.58}$ [KP, M]. This is actually a consequence of the theorem given in [M] using that $\varphi(\gamma) = (q^{-\frac{1}{4}}e^{6q^{-\frac{1}{12}}})^{\|\gamma\|}$ and that the minimal length of contours is 6.

Moreover both terms in the ratio of the r.h.s. of (32) can be exponentiated in terms of the truncated functionals leading to

$$\begin{aligned}
\Pr(E) &= 4\varphi(\bar{\gamma}) \exp\left\{ \sum_{X:\mathrm{Supp}\, X\cap\bar{\gamma}=\emptyset} \phi^T(X) - \sum_X \phi^T(X) \right\} \\
&= 4\varphi(\bar{\gamma}) \exp\left\{ - \sum_{X:\mathrm{Supp}\, X\cap\bar{\gamma}\neq\emptyset} \phi^T(X) \right\} \\
&= 4q^{-\frac{\|\bar{\gamma}\|}{4}+\pi^0(\mathrm{Int}\,\bar{\gamma})} \exp\left\{ \Delta(\bar{\gamma}) - \sum_{X:\mathrm{Supp}\, X\cap\bar{\gamma}\neq\emptyset} \phi^T(X) \right\} \\
&= 4q^{-\frac{1}{2}} \exp\left\{ \Delta(\bar{\gamma}) - \sum_{X:\mathrm{Supp}\, X\cap\bar{\gamma}\neq\emptyset} \phi^T(X) \right\} \qquad (35)
\end{aligned}$$

For the last two equalities, we have taken into account the expression (23) and the fact that $\|\bar{\gamma}\| = 6$ and $\pi^0(\mathrm{Int}\,\bar{\gamma}) = 1$. According to (34) the sum in the exponential satisfies

$$\begin{aligned}
\left| \sum_{X:\mathrm{Supp}\, X\cap\bar{\gamma}\neq\emptyset} \phi^T(X) \right| &\leq \sum_{b\in\bar{\gamma}} \sum_{X:\mathrm{Supp}\, X\cap b\neq\emptyset} |\phi^T(X)| \\
&\leq \|\bar{\gamma}\| \, (4g)^6 q^{-\frac{3}{2}} e^{36q^{-\frac{1}{12}}} \qquad (36)
\end{aligned}$$

This implies from (28), (29), (31) and (36)

$$\chi^{\mathrm{f}}(\beta_t, q) = 2q^{-1/2} + O(q^{-7/12}) \qquad (37)$$

giving the statement of the Theorem by taking into account the duality relation(11). ■

3 Euler Characteristic on $\mathbb{Z}^d, d \geq 3$

3.1 Definitions

In this section, we consider the hypercubic lattice

$$\mathbb{Z}^d = \{x = (x_1, \ldots, x_d) : x_i \in \mathbb{Z},\, i = 1, \ldots, d\}$$

We let $\mathbb{L}$, be the associated cell complex. Recall that $\mathbb{L} = \{\mathbb{L}_0, \ldots, \mathbb{L}_d\}$ where $\mathbb{L}_0 = \mathbb{Z}^d$ is the set of sites, $\mathbb{L}_1$ is the set of bonds and $\mathbb{L}_2$ is the set of plaquettes, etc... . The elements s^p of $\mathbb{L}_p$ are called p–cells and a p-cell may be represented as $(x; \sigma_1 e_{i_1}, ..., \sigma_p e_{i_p})$ where $x \in \mathbb{Z}^d, (e_1, ..., e_d)$ is an orthonormal basis of $\mathbb{R}^d$ and $\sigma_\alpha = \pm 1, \alpha = 1, ..., p$. We shall use $S^p(x)$ to denote the set of all such p-cells containing x. The boundary of a p-cell is denoted by ∂s^p and the co–boundary of a p-cell is denoted δs^p (see [DW]).

As in the previous section, to a set of bonds $X \subset \mathbb{L}_1$ we associate the subcomplex $\Lambda(X) \subset \mathbb{L}$ defined as the maximal closed subcomplex that contains X : this subcomplex is the union $\Lambda(X) = \Lambda_0(X) \cup \ldots \cup \Lambda_d(X)$ where

$$\Lambda_1(X) = X$$

is the set of bonds itself,

$$\Lambda_0(X) = \{x \in \mathbb{L}_0 : x \cap X \neq \emptyset\}$$

is the set of sites which are endpoints of bonds of X, and for $p = 2, \ldots, d$

$$\Lambda_p(X) = \{s^p \in \mathbb{L}_p : \text{ every cell in } \partial s^p \text{ belongs to } \Lambda_{p-1}(X)\}$$

For a set of bonds X, we define the boundary $B(X)$ of X as the set of bonds of X that belong to a plaquette of $\mathbb{L}_2 \setminus \Lambda_2(X)$.

We also associate to the set of bonds X the occupation number

$$n_b = \begin{cases} 1 & \text{if} \quad b \in X \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and let $n \equiv \{n_b\}_{b \in \mathbb{L}_1}$ be the associated configuration. We then use $N^p(n) \equiv |\Lambda_p(X)|$ to denote the number of p–cells of $\Lambda_p(X)$.

The Euler characteristic of $\Lambda(X)$ is defined by

$$\boldsymbol{\chi}(n) = \sum_{p=1}^{d} (-1)^p N^p(n) \tag{2}$$

It satisfies the Euler–Poincaré formula

$$\boldsymbol{\chi}(n) = \sum_{p=1}^{d} (-1)^p \pi^p(n) \tag{3}$$

where $\pi^p(n)$ are the Betti numbers of the cell complex $\Lambda(X)$.

We then define, for a "volume" $V \subset \mathbb{L}_1$, the Fortuin–Kasteleyn partition functions

$$Z^{\mathrm{bc}}(V,\beta,q) = \sum_{n\in\{0,1\}^V} (e^\beta - 1)^{N^1(n)} q^{|\Lambda_0(V)|-N^0(n)+\pi^0(n)}\, \kappa^{\mathrm{bc}}(n) \tag{4}$$

and the corresponding finite volume expectations of local functions f of the bonds variable $\{n_b\}$ by

$$\langle f\rangle^{\mathrm{bc}}(V,\beta,q) = \frac{1}{Z^{bc}(V,\beta,q)} \sum_{n\in\{0,1\}^V} f(e^\beta - 1)^{N^1(n)} q^{|\Lambda_0(V)|-N^0(n)+\pi^0(n)}\, \kappa^{\mathrm{bc}}(n) \tag{5}$$

Here $\kappa^{\mathrm{bc}}(n)$ refers to the boundary condition *bc*. In particular, we will be interested in the free boundary condition $\kappa^{\mathrm{f}}(n) = 1$ and in the ordered or wired boundary condition $\kappa^{\mathrm{w}}(n) = \prod_{b\in B(V)} n_b$.

Notice that for a configuration $n \in \{0,1\}^{|V|}$, one has

$$\begin{aligned} N^0(n) &= \Lambda_0(V) - \sum_{x\in\Lambda_0(V)} \prod_{b\in S^1(x)\cap V} (1-n_b) \\ N^1(n) &= \sum_{x\in\Lambda_0(V)} \frac{1}{2} \sum_{b\in S^1(x)\cap V} n_b \\ N^p(n) &= \sum_{x\in\Lambda_0(V)} \frac{1}{2^p} \sum_{s^p\in S^p(x)} \prod_{b\in s^p\cap V} n_b, \quad p = 2,\ldots,d \end{aligned}$$

where, with a slight abuse of notations, the last product is over the bonds of V that belong to the cells s^p. Accordingly we define the Euler characteristic per site as follows

$$\chi_x(n) \equiv 1 - \prod_{b\in S1(x)} (1-n_b) - \frac{1}{2}\sum_{b\in S1(x)} n_b + \sum_{p=2}^{d} \frac{(-1)^p}{2^p} \sum_{s^p\in S^p(x)} \prod_{b\in s^p} n_b \tag{6}$$

where the last product runs over the bonds that belong to the cells s^p.

3.2 Results

The following theorem shows existence results that can be proved in the same way as in the previous Section.

Theorem 5 *a) The following limits exist and coincide*

$$\lim_{V\to\mathbb{L}_1}\left\langle\frac{\boldsymbol{\chi}}{|\Lambda_0(V)|}\right\rangle^{\mathrm{f}}(V,\beta,q)=\lim_{V\to\mathbb{L}_1}\langle\chi_0\rangle^{\mathrm{f}}(V,\beta,q)\equiv\chi^{\mathrm{f}}(\beta,q)\tag{7}$$

and define the mean local Euler characteristic with free boundary condition $\chi^{\mathrm{f}}(\beta,q)$.

b) Analogously, the following limits exist and coincide

$$\lim_{V\to\mathbb{L}_1}\left\langle\frac{\boldsymbol{\chi}}{|\Lambda_0(V)|}\right\rangle^{\mathrm{w}}(V,\beta,q)=\lim_{V\to\mathbb{L}_1}\langle\chi_0\rangle^{\mathrm{w}}(V,\beta,q)\equiv\chi^{\mathrm{w}}(\beta,q)\tag{8}$$

and define the mean local Euler characteristic with wired boundary condition $\chi^{\mathrm{w}}(\beta,q)$.

It is known that, for large q and $d\geq 2$, the model exhibits a (temperature driven) first order phase transition at some inverse temperature β_t ($\approx\frac{1}{d}\ln q$) where the mean energy $\frac{d}{d\beta}f(\beta,q)$ is discontinuous [KS]. The following theorem shows that it is also the case for the mean local Euler characteristic.

Theorem 6 *Assume that q is large enough, then there exists a unique inverse temperature β_t where the mean local Euler characteristic is discontinuous.*

$$\chi^{\mathrm{f}}(\beta_t,q)=d\,q^{-\frac{d-1}{d}}+O(q^{-\frac{d-1}{d}[1+\frac{1}{4d-2}]})\tag{9}$$

$$\chi^{\mathrm{w}}(\beta_t,q)=(-1)^{d-1}d\,q^{-\frac{1}{d}}+O(q^{-\frac{1}{d}-\frac{1}{2}})\tag{10}$$

and for every $\beta\neq\beta_t$

$$\chi^{\mathrm{f}}(\beta,q)=\chi^{\mathrm{w}}(\beta,q)\tag{11}$$

Note that while the jump of the mean energy is $\Delta E=1-O(q^{-\frac{1}{d}})$ the jump $\Delta\chi=|\chi^f(\beta_t,q)-\chi^w(\beta_t,q)|=O(q^{-\frac{1}{d}})$ for even space dimensions and $\Delta\chi=O(q^{-\frac{d-1}{d}})$ for odd space dimensions.

Proof of Theorem 6. We begin with the proof of (9) which is based, as in Section 2, on a contour representation of the model [LMMRS]. We first introduce some geometrical definitions. Two bonds are said co–adjacent if there exists a d–cell containing these two bonds. A set of bonds, X, is said co–connected if for any b and b' in X, there exists a sequence of bonds in X $b = b_1, \ldots, b_n = b'$, such that b_i and b_{i+1} are co–adjacent. We note that to any p–cell in $\mathbb{L}$ we can associate the orthogonal $(d-p)$–cell of the dual cell complex $\mathbb{L}^*$ corresponding to the dual lattice $(\mathbb{Z}+1/2)^d$ which intersects it. The co–connectedness of a set of bonds means that its dual is a connected set of $\mathbb{R}^d$ (with the usual topology). Consider the co–boundary $C(X)$ of a set of bonds X : $C(X) = C_1(X) \cup C_2(X)$ where

$$\begin{aligned}C_1(X) &= \{b \in \mathbb{L}_1 \setminus X : b \text{ has 1 endpoint in } \Lambda(X)\}\\ C_2(X) &= \{b \in \mathbb{L}_1 \setminus X : b \text{ has 2 endpoints in } \Lambda(X)\}\end{aligned}$$

A subset $\gamma \subset \mathbb{L}_1$ is called contour if it is the co–boundary of a set of bonds X and if it is co–connected. When X is finite, γ is called contour of the free class (or f–contour) and X will be denoted by $\operatorname{Int}\gamma$. The length of γ is defined as

$$\|\gamma\| = |C_1(\operatorname{Int}\gamma)| + |C_2(\operatorname{Int}\gamma)| \quad \text{if } \gamma \text{ is a f–contour}$$

With these definitions, one has ([LMMRS]):

$$\begin{aligned}Z^{\mathrm{f}}(V, \beta_t, q) &= q^{|\Lambda_0(V)|+|\lambda(V)|}\mathcal{Z}(V|\varphi_f)\\ \mathcal{Z}(V|\varphi_f) &= \sum_{\{\gamma_1,\ldots,\gamma_n\}\subset V\cup C(V)} \prod_{i=1}^{n} \varphi_f(\gamma_i)\end{aligned} \tag{12}$$

where the last sum runs over compatibles families, i.e. $\gamma_i \cap \gamma_j = \emptyset$, and $\lambda(V)$ is a boundary term such that $|\lambda(V)| \leq$ *cte* $|B(V)|$. The activities $\varphi_f(\gamma)$ of contours are given by

$$\varphi_f(\gamma) = q^{-\frac{1}{2d}\|\gamma\| + \pi^0(\operatorname{Int}\gamma)} e^{\Delta(\gamma)} \tag{13}$$

where $\Delta(\gamma)$ satisfies the bound:

$$|\Delta(\gamma)| \leq 6q^{-\frac{d-1}{2d(2d-1)}}\, \|\gamma\| \tag{14}$$

This implies that for large q the free energy $f(\varphi_f) = \lim_{V\to\mathbb{L}_1} \frac{1}{|\Lambda_0(V)|} \ln \mathcal{Z}(V|\varphi_f)$ of contour model behaves like $d\, q^{-\frac{d-1}{d}} + O(q^{-\frac{2(d-1)}{d}})$. We will show that the dominant contribution to the local Euler characteristic $\chi^{\mathrm{f}}(\beta, q)$ shares the same behaviour.

For that purpose, we will follow the same strategy as in as in the proof of Theorem 4. Namely we consider the events E_f, E and E' defined in (25–27), again one has $\chi_0(n) = 0$ for $n \in E_f$, $\chi_0(n) = 1/2$ for $n \in E$ and $|\chi_0(n)| \leq 1/2$ for $n \in E'$. The rest of the proof is analogous to that of Theorem 4. Only some estimates have to be changed, namely (30) now reads

$$\begin{aligned} \Pr(E') &\leq \sum_{\gamma:\gamma\ni 0;|\operatorname{Int}\gamma|\geq 2} \Pr(\gamma) \leq \sum_{\gamma:\gamma\ni 0;\|\gamma\|\geq 6d-4} \varphi(\gamma) \\ &\leq \sum_{m\geq 6d-4} \nu_d^m \big(q^{-\frac{d-1}{(4d-2)d}} e^{6q^{-\frac{d-1}{(4d-2)d}}}\big)^m \\ &\leq \nu_d^{6d-4} \frac{q^{-(\frac{d-1}{d}+\frac{d-1}{2d-1})} e^{6(6d-4)q^{-\frac{d-1}{(4d-2)d}}}}{1-\nu_d q^{-\frac{d-1}{(4d-2)d}} e^{6q^{-\frac{d-1}{(4d-2)d}}}} \end{aligned} \tag{15}$$

Here we have used $\pi^0(\operatorname{Int}\gamma) \leq \frac{1}{4d-2}\|\gamma\|$ and that ν_d^m bounds the number of contours of length m that contain a given point: ν_d can be bounded by $\nu_d \leq d^2 2^{4(d-1)}$. To see it, we first observe that the number of p–cells $(x; \sigma_1 e_{i_1}, ..., \sigma_p e_{i_p})$ that share a same vertex x equals $2^p\binom{d}{p}$: 2^p stands for the choice for the signs of the σ's and the binomial coefficient $\binom{d}{p}$ stands for the choice of p vectors among $(e_1, ..., e_d)$. This implies (by duality) that a $(d-1)$–cell contains 2^{d-1} vertices and hence that the number of $(d-1)$–cells connected (i.e. sharing at least a vertex) with a given $(d-1)$–cell is less than $2^{(d-1)}(d2^{(d-1)}-1)$. Finally, one uses the fact that for any contour there exist a path going through the center of every $(d-1)$–cell of its dual at most twice [Or].

The cluster expansion now gives

$$\sum_{X:\operatorname{Supp} X\cap b\neq\emptyset} |\phi^T(X)| \leq \big(\nu_d g q^{-\frac{d-1}{(4d-2)d}} e^{6q^{-\frac{d-1}{(4d-2)d}}}\big)^{4d-2} \tag{16}$$

for $\nu_d g q^{-\frac{d-1}{(4d-2)d}} e^{6q^{-\frac{d-1}{(4d-2)d}}} < 1$, leading to

$$\begin{aligned} \Pr(E) &= 2dq^{-\frac{\|\bar\gamma\|}{2d}+\pi^0(\operatorname{Int}\bar\gamma)} \exp\left\{\Delta(\bar\gamma) - \sum_{X:\operatorname{Supp} X\cap\bar\gamma\neq\emptyset} \phi^T(X)\right\} \\ &= 2dq^{-\frac{d-1}{d}} \exp\left\{\Delta(\bar\gamma) - \sum_{X:\operatorname{Supp} X\cap\bar\gamma\neq\emptyset} \phi^T(X)\right\} \end{aligned} \tag{17}$$

and

$$\left|\sum_{X:\mathrm{Supp}\,X\cap\bar{\gamma}\neq\emptyset}\phi^T(X)\right| \leq \sum_{b\in\bar{\gamma}}\ \sum_{X:\mathrm{Supp}\,X\cap b\neq\emptyset}|\phi^T(X)|$$
$$\leq (4d-2)(\nu_d g)^{4d-2} q^{-\frac{d-1}{d}} e^{6(4d-2)q^{-\frac{d-1}{(4d-2)d}}} \qquad (18)$$

using $\|\bar{\gamma}\| = 4d-2$ and $\pi^0(\mathrm{Int}\,\bar{\gamma}) = 1$. This implies

$$\chi^{\mathrm{f}}(\beta_t, q) = d\,q^{-\frac{d-1}{d}} + O(q^{-\frac{d-1}{d}[1+\frac{1}{4d-2}]}) \qquad (19)$$

giving the first statement of the theorem.

We now turn to the proof of the second statement of the theorem. We first define the contours of the wired class. A subset $\gamma \subset \mathbb{L}_1$ is called contour if it is the co–boundary of a set of bonds X and if it is co–connected. When X is infinite, γ is called contour of the wired class (or w–contour). The unique infinite component of $\mathbb{L}_1 \setminus \gamma$ is denoted by $\mathrm{Ext}\,\gamma$. The length of γ is defined by

$$\|\gamma\| = |C_1(\mathrm{Ext}\,\gamma)| + |C_2(\mathrm{Ext}\,\gamma)| \quad \text{if } \gamma \text{ is a w–contour}$$

With these definitions, one has ([LMMRS]):

$$Z^{\mathrm{w}}(V, \beta_t, q) = q(e^{\beta-1})^{|V|}\mathcal{Z}(V|\varphi_w)$$
$$\mathcal{Z}(V|\varphi_w) = \sum_{\{\gamma_1,\ldots,\gamma_n\}\subset V}\prod_{i=1}^{n}\varphi_w(\gamma_i) \qquad (20)$$

where the last sum runs over compatibles families, i.e. $\gamma_i \cap \gamma_j = \emptyset$ and the activities $\varphi_w(\gamma)$ of contours are given by

$$\varphi_w(\gamma) = q^{-\frac{1}{2d}\|\gamma\|}e^{\Delta(\gamma)} \qquad (21)$$

where $\Delta(\gamma)$ satisfies the bound:

$$|\Delta(\gamma)| \leq 9q^{-\frac{1}{2}}\,\|\gamma\| \qquad (22)$$

This implies that for large q the free energy $f(\varphi_w) = \lim_{V\to\mathbb{L}_1}\frac{1}{|\Lambda_0(V)|}\ln\mathcal{Z}(V|\varphi_w)$ of the contour model behaves like $d\,q^{-\frac{1}{d}} + O(q^{-\frac{2}{d}})$. We will show that (up to the sign) the local Euler characteristic $\chi^{\mathrm{w}}(\beta, q)$ shares the same behaviour.

This fact might be seen as follows. The first excitations of the occupied ground state configuration consist of empty bonds in the "sea" of occupied ones. Each excitation gives a contribution to π^{d-1} and the number of excitations scales like the volume, giving that the Euler characteristic behaves like $(-1)^{d-1}O(q^{-\frac{1}{d}})$. To compute the coefficient we will consider the expression (6) of the local Euler characteristic $\chi_x(n)$.

Let us introduce the p–neighborhood of a site $x = (x_1, \ldots, x_d)$, $p = 1, \ldots, d$, as $\mathcal{N}_p(x) = \{y = (y_1, \ldots, y_d) : \sum_{i=1}^{d} |x_i - y_i| = p\}$ and recall that $S^d(x)$ is the set of the 2^d–cells that contain the site x.

To compute $\langle \chi_0 \rangle$ we will consider the following events

$$\begin{aligned} E_w = \big\{ n \in \{0,1\}^V :\ & \text{all the bonds of the } d\text{–cells belonging to } S^d(0) \\ & \text{are occupied} \big\} \end{aligned} \tag{23}$$

$$\begin{aligned} E'_w = \big\{ n \in \{0,1\}^V :\ & \text{there is exactly one empty bond in the } d\text{–cells} \\ & \text{belonging to } S^d(0) \text{ and this bond has no endpoint in } \mathcal{N}_d(0); \\ & \text{in addition, all other bonds of the } d\text{–cells that contain this} \\ & \text{bond are occupied} \big\} \end{aligned} \tag{24}$$

$$\begin{aligned} E = \big\{ n \in \{0,1\}^V :\ & \text{there is exactly one empty bond in the } d\text{–cells} \\ & \text{belonging to } S^d(0) \text{ and this bond has one endpoint in } \mathcal{N}_d(0); \\ & \text{in addition, all other bonds of the } d\text{–cells that contain this} \\ & \text{bond are occupied} \big\} \end{aligned} \tag{25}$$

and the complementary event

$$E' = \big\{ n \in \{0,1\}^V \setminus (E \cup E_w \cup E'_w) \big\} \tag{26}$$

We have

$$\chi_0(n) = 0 \qquad \text{if } n \in E_w \text{ or if } n \in E'_w \tag{27}$$

$$\chi_0(n) = \frac{(-1)^{d-1}}{2^d} \qquad \text{if } n \in E \tag{28}$$

$$|\chi_0(n)| \leq 1 \qquad \text{if } n \in E' \tag{29}$$

Then,

$$\langle \chi_0 \rangle^{\mathrm{w}} (V, \beta, q) = \frac{(-1)^{d-1}}{2^d} \Pr(E) + \sum_{n : n \in E'} [\chi_0(n)] \Pr(n) \tag{30}$$

with

$$\left|\sum_{n:n\in E'}[\chi_0(n)]\,\mathrm{Pr}(n)\right| \le \mathrm{Pr}(E') \tag{31}$$

Taking into account that for any configuration $n \in E'$ there is at least a contour γ with length $\|\gamma\| \ge 4$, we estimate this probability by Peierls type arguments as follows

$$\begin{aligned}\mathrm{Pr}(E') &\le \sum_{\gamma:\gamma\ni 0;\|\gamma\|\ge 4}\mathrm{Pr}(\gamma) \le \sum_{\gamma:\gamma\ni 0;\|\gamma\|\ge 4}\varphi_w(\gamma)\\ &\le \sum_{m\ge 4}{\nu_d}^m\big(q^{-\frac{1}{2d}}e^{9q^{-\frac{1}{2}}}\big)^m \le {\nu_d}^4\frac{q^{-\frac{2}{d}}e^{36q^{-\frac{1}{12}}}}{1-\nu_d q^{-\frac{1}{2d}}e^{9q^{-\frac{1}{2}}}}\end{aligned} \tag{32}$$

provided $\nu_d q^{-\frac{1}{2d}}e^{9q^{-\frac{1}{2}}} < 1$.

On the other hand, in order to estimate the probability $\mathrm{Pr}(E)$ we let $\tilde{\gamma}$ to be the dual of the occupied bond. Then

$$\begin{aligned}\mathrm{Pr}(E) =& \Big[\frac{(-1)^{d-1}}{2^d}(2^d d)\Big]\frac{\displaystyle\sum_{\{\gamma_1,\dots,\gamma_n\}\supset\tilde{\gamma}}\prod_{i=1}^{n}\varphi_w(\gamma_i)}{\mathcal{Z}(V|\varphi_w)}\\ =& \big[(-1)^{d-1}d\big]\,\varphi_w(\tilde{\gamma})\frac{\displaystyle\sum_{\{\gamma_1,\dots,\gamma_n\},\gamma_i\cap\tilde{\gamma}=\emptyset}\prod_{i=1}^{n}\varphi_w(\gamma_i)}{\mathcal{Z}(V|\varphi_w)}\end{aligned} \tag{33}$$

The factor $2^d d$ stems from the fact that there are $2^d d$ bonds in the d–cells of $S^d(0)$ that contain one endpoint in the d –neighbourhood $\mathcal{N}_d(0)$ of the origin.

We will compute this last ratio with cluster expansion by introducing clusters X as multi-indexes defined on the sets of wired contours and the corresponding truncated functionals:

$$\phi_w^T(X) = \frac{a(X)}{\prod_\gamma X(\gamma)!}\prod_\gamma \varphi_w(X)^{X(\gamma)} \tag{34}$$

By taking into account the expression (21) and that $\|\bar{\gamma}\| = 2$, we get as a result of cluster expansion

$$\begin{aligned}\Pr(E) &= \big[(-1)^{d-1}d\big]q^{-\frac{\|\bar{\gamma}\|}{2d}} \exp\left\{\Delta(\bar{\gamma}) - \sum_{X:\mathrm{Supp}\,X\cap\bar{\gamma}\neq\emptyset} \phi_w^T(X)\right\} \\ &= \big[(-1)^{d-1}d\big]q^{-\frac{1}{d}} \exp\left\{\Delta(\bar{\gamma}) - \sum_{X:\mathrm{Supp}\,X\cap\bar{\gamma}\neq\emptyset} \phi_w^T(X)\right\} \qquad (35)\end{aligned}$$

and

$$\begin{aligned}\left|\sum_{X:\mathrm{Supp}\,X\cup\bar{\gamma}\neq\emptyset} \phi_w^T(X)\right| &\leq \sum_{b\in\bar{\gamma}}\ \sum_{X:\mathrm{Supp}\,X\cap b\neq\emptyset} |\phi_w^T(X)| \\ &\leq 2(\mu_d g q^{-\frac{1}{2d}} e^{9q^{-\frac{1}{2}}})^2 \qquad (36)\end{aligned}$$

provided $\mu_d g q^{-\frac{1}{2d}} e^{9q^{-\frac{1}{2}}} < 1$. This implies

$$\chi^{\mathrm{w}}(\beta_t, q) = \big[(-1)^{d-1}d\big]q^{-\frac{1}{d}} + O(q^{-\frac{1}{d}-\frac{1}{2}}) \qquad (37)$$

giving the second statement of the theorem. The last statement follows again from the differentiability properties given in [LMMRS]. ■

Let us give some consequences of Theorem 6. Consider a generalized Fortuin–Kasteleyn measure (GFK) where the occupation variables are defined on the $(d-1)$–cells of the lattice:

$$\langle\,\cdot\,\rangle(\mathbb{V}, \beta, q) = \frac{1}{\Xi(\mathbb{V}, \beta, q)} \sum_{n\in\{0,1\}^{\mathbb{V}}} \cdot\, (e^{\beta}-1)^{N^{d-1}(n)} q^{\pi^{d-1}(n)}\, \kappa^{\mathrm{bc}}(n) \qquad (38)$$

where $\mathbb{V} \subset \mathbb{L}_{d-1}$ and $\kappa^{\mathrm{bc}}(n)$ refers to the boundary conditions. In dimension $d = 3$, the partition function $\Xi(\mathbb{V}, \beta, q)$ is actually the Fortuin–Kasteleyn representation of the gauge Potts model. In dimension $d \geq 4$, it is the one of the so–called hypergauge model. Potts and hypergauge Potts model are related by the natural extension ($\mathfrak{D}$) of the Kramers-Wannier duality, i.e. Fourier expansion together with Poincaré duality [LMeR, KLMR]

$$\text{Potts} \xleftrightarrow{\mathfrak{D}} \text{Hypergauge Potts}$$

Denoting by FK and GFK their Fortuin–Kasteleyn representations, the complete commutativity of the following diagram

$$\begin{tikzcd}\text{Potts} \arrow[r, leftrightarrow, "\mathfrak{D}"] \arrow[d] & \text{Hypergauge Potts} \arrow[d] \\ \text{FK} \arrow[r, leftrightarrow, "\mathfrak{A}"] & \text{GFK}\end{tikzcd}$$

actually holds. The correspondence FK $\overset{\mathfrak{A}}{\longleftrightarrow}$ GFK is the Alexander duality (used in Section 2 for $d = 2$) and defined as follows

$$\widehat{n}_{b^*} = 1 - n_b, \quad b \in V$$

where b^* is the $(d-1)$–cell that crosses b.

Under this transformation, the FK measure (5) at (inverse) temperature β is transformed in the GFK measure (38) at (inverse) temperature β^*, while free boundary conditions are transformed into the wired ones. Note that this duality actually holds at the level of measures. It implies, as a consequence of Theorem 6, that the (suitably defined) mean local Euler characteristic with respect to the generalized measures (38) is discontinuous at the dual temperature β_t^* given by $(e^{\beta_t^*} - 1)(e^{\beta_t} - 1) = q$.

4 Conclusion and outlooks

This work clearly shows that the mean Euler-Poincaré characteristic per site of the FK measure exhibits a non-trivial behaviour at the transition point. More precisely, we have proved that it satisfies, in dimension two, a duality property implying that at the transition point it is either zero or it is discontinuous.

For the percolation model ($q = 1$) it actually takes the value zero at the transition point.

Moreover, for large q, using an analysis of the order–disorder transition, we have shown that a jump of order $q^{-\frac{1}{2}}$ occurs. This is illustrated in Figure 2 for $q = 5000$.

In higher dimensions, the magnitude of the jump is of order $q^{-\frac{1}{d}}$ for even space dimensions and of order $q^{-\frac{d-1}{d}}$ for odd space dimensions.

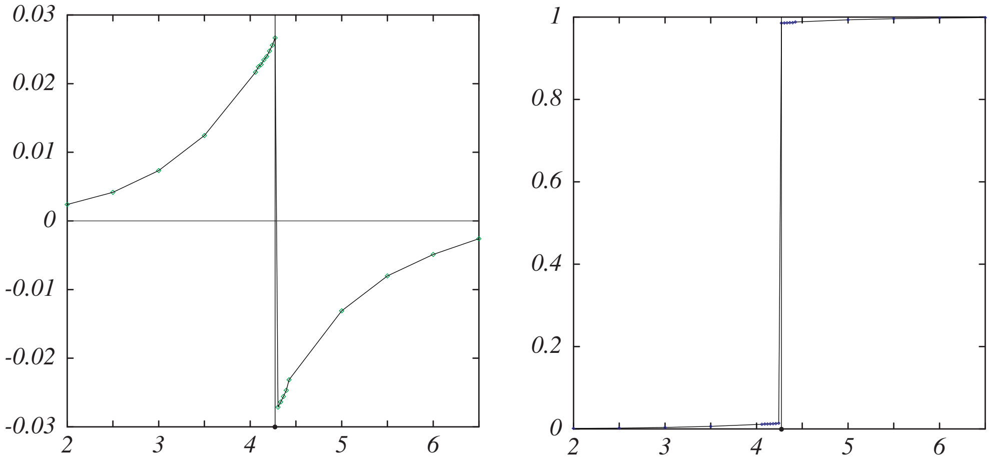

Figure 2: Mean Euler characteristic per site and mean bond occupation number as functions of β, for $q = 5000$ in dimension $d = 2$.

For small values of q (to which the above mentioned analysis does not apply) it would be, in particular, interesting to see if the mean local Euler-Poincaré characteristic is continuous and takes the value 0 when the transition is second order, as it is the case for the percolation model. Numerical simulations performed in [BFG] indicate that this is actually the case. We illustrate in Figure 3 when $d = 2$ and $q = 2$ the behaviour of the mean local Euler-Poincaré characteristic as a function of the temperature that confirms this prediction.

In the numerics we have directly sampled the FK distribution through the heat bath Monte Carlo method (Glauber dynamics). As we needed the number of connected components to compute the weights of this distribution, the Hoshen–Kopelman [HK] cluster algorithm has been used.

We think that the fact that the mean local Euler-Poincaré characteristic vanishes at the transition gives new non trivial informations on the topology of typical configurations at this point.

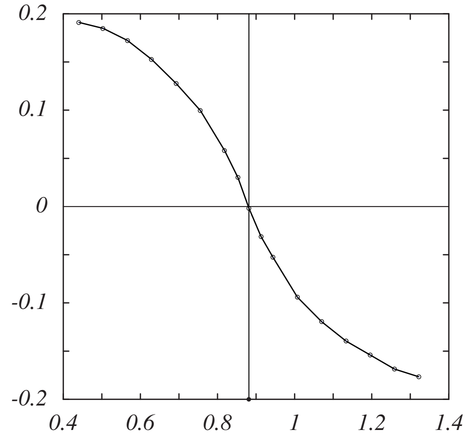

Figure 3: Mean Euler characteristic per site as function of β, for $q = 2$ in dimension $d = 2$.

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References

[ACCN] M. Aizenman, J.T. Chayes, L. Chayes, C.M. Newman, *Discontinuity of the magnetization in one–dimensional* $1/|x-y|^2$ *Ising and Potts models*, J. Stat. Phys. **50**, 1–40 (1988).

[A] P.S. Aleksandrov, *Combinatorial Topology*, vol 3, Graylock Press, Albany, 1960.

[B] R.J. Baxter, *Exactly solved models in statistical mechanics*, New-York, Academic Press, 1982.

[Be] C. Berge, *Graphes*, Paris: Gauthier-Villars, 1983.

[BFG] Ph. Blanchard, S. Fortunato, and D. Gandolfo, *Euler-Poincaré characteristic and phase transition in the Potts model on* $\mathbb{Z}^2$, Nucl. Phys. B **644**[FS], 495–508 (2002).

[D] R. L. Dobrushin, *Estimates of semi–invariants for the Ising model at low temperatures*, Amer. Math. Soc. Transl. **177**, 59–81 (1996).

[DW] K. Druhl and H. Wagner, *Algebraic formulation of duality transformation for abelian lattice model*, Ann. Phys. **141**, 225 (1982).

[F1] C.M. Fortuin, *On the random–cluster model II: The percolation model*, Physica **58**, 393–418 (1972).

[F2] C.M. Fortuin, *On the random–cluster model III: The simple random–cluster model*, Physica **59**, 545–570 (1972).

[FK] C.M. Fortuin, P.W. Kasteleyn, *On the random–cluster model I: Introduction and relation to other models*, Physica **57**, 536–564 (1972).

[FKG] C.M. Fortuin, P.W. Kasteleyn, J. Ginibre, *Correlation inequalities on some partially ordered sets*, Commun. Math. Phys. **22**, 89–103 (1971).

[G] R.B. Griffiths *Correlations in Ising ferromagnets II*, J. Math. Phys. **8** 484–489 (1967).

[Gr] G. Grimmett, *Potts models and random–cluster processes with many–body interactions*, J. Stat. Phys. **75**, 67–121 (1994).

[Gra] S. B. Gray, I.E.E.E. Trans. Comp. **20**, 551, (1971).

[Gri] G. Grimmett, *Percolation*, Springer, 2nd edition (1999).

[Ha] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, 1957.

[HK] J. Hoshen and R. Kopelman, *Percolation and cluster distribution. I. Cluster multiple labeling technique and critical concentration algorithm*, Phys. Rev. B **14**, 3438 (1976).

[HKW] A. Hinterman, H. Kunz, and F.Y. Wu, *Exact results for the Potts models in two dimension*, J. Stat. Phys. **19**, 621–632 (1978).

[KLMR] R. Koteckỳ, L. Laanait, A. Messager, and J. Ruiz, *The q–states Potts model in the standard Pirogov-Sinaï theory: Surface tension and Wilson loops*, J. Stat. Phys. **58**, 199 (1990).

[KP] R. Koteckỳ and D. Preiss, *Cluster expansion for abstract polymer models,* Commun. Math. Phys. **103**, 491 (1986).

[KS] R. Koteckỳ and S. Shlosman, *First order phase transitions in large entropy lattice systems,* Commun. Math. Phys. **83**, 493 (1982).

[LM] J. L. Lebowitz and A. Martin-Löf, *On the uniqueness of the equilibrium state for ising spin systems,* Commun. Math. Phys. **25**, 276 (1972).

[LMMRS] L. Laanait, S. Miracle-Solé, A. Messager, J. Ruiz, and S. Shlosman, *Interfaces in the Potts model I: Pirogov-Sinai theory of the Fortuin-Kasteleyn representation*, Commun. Math. Phys. **140**, 81–91 (1991).

[LMeR] L. Laanait, A. Messager, J. Ruiz, *Discontinuity of the Wilson string tension in the 4-dimensional lattice pure gauge Potts model*, Commun. Math. Phys. **126**, 103–131 (1989).

[Le] S. Lefschetz, *Introduction to Topology*, Princeton University Press, Princeton, 1942.

[M] S. Miracle-Solé, *On the convergence of cluster expansion*, Physica **A 279**, 244-249 (2000).

[MM] A. Messager and S. Miracle-Solé, *Equilibrium states of the two-dimensional Ising model in the two-phase region*, Commun. Math. Phys. **40**, 187–196 (1975).

[MW] K. R. Mecke and H. Wagner, *Euler Characteristic and Related Measures for Random Geometric Sets*, J. Stat. Phys. **64**, 843, (1991).

[M1] K. R. Mecke, *Applications of Minkowski Functionals in Statistical Physics*, in : *Statistical Physics and Spatial Statistics. The Art of Analyzing and Modeling Spatial Structures and Pattern Formation*, Lecture Notes in Physics **554**, K. R. Mecke and D. Stoyan Eds. Springer (2000).

[M2] K. R. Mecke, *Complete Family of Fractal Dimensions Based on Integral Geometry*, submitted to Physica A (2000).

[O] B. L. Okun, *Euler Characteristic in Percolation Theory*, J. Stat. Phys. **59**, 523, (1990).

[Or] O. Ore, *The Four-Color Problem*, Academic Press, New-York, London, 1967.

[PV] C.-E. Pfister and Y. Velenik, *Random cluster representation of the Ashkin-Teller model*, J. Stat. Phys. **88**, 1295–1331 (1997).

[S] L. A. Santalò, *Integral Geometry and Geometric Probability*, Addison-Wesley, (1976).

[Se] J. Serra, *Mathematical Morphology*, Academic Press, (1982).

[SyEs] M. F. Sykes and J. W. Essam, *Exact Critical Percolation Probabilities for Site and Bond Problems in Two Dimensions*, J. Math. Phys. **5**, 1117-1121, (1964).

[Wa1] H. Wagner, *Euler Characteristic for Archimedean Lattices*, (2000). (unpublished)

[Wa2] K. R. Mecke Th. Buchert and H. Wagner, Astron. Astrophys. **288**, 697, (1994).