MULTIPLICITY RESULTS FOR INTERFACES OF GINZBURG-LANDAU-ALLEN-CAHN EQUATIONS IN PERIODIC MEDIA.

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ABSTRACT. The Ginzburg-Landau-Allen-Cahn equation is a variational model for phase coexistence and for other physical problems. It contains a term given by a kinetic part of elliptic type plus a double-well potential. We assume that the functional depends on the space variables in a periodic way.

We show that given a plane with rational normal, there are at least two equilibrium solutions, which are asymptotic to the pure phases but are separated by an interface. The width of the interface is bounded above by a universal constant and these solutions satisfy some monotonicity properties with respect to integer translations. Also, such solutions approach the pure phases exponentially fast.

We then show that all the interfaces of the global periodic minimizers satisfy similar monotonicity and plane-like properties.

We also consider the case of irrationally oriented planes. In this case, we show that either there is a one parameter family of minimizers whose graphs provide a field of extremals or there are at least two solutions, one which is a minimizer and another one which is not. These solutions also have interfaces bounded by a universal constant and enjoy monotonicity properties with respect to integer translations.

> ``E infinite soluzioni, sicuro:
> non è vero che esistano problemi
> i quali ne ammettano due sole.''
>
> (Tommaso Landolfi, *Racconti impossibili*.)

CONTENTS

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1. Introduction

In this paper we will study critical points of the functional

$$\mathcal{E}_\Omega(u) = \int_\Omega \frac{1}{2}\, a_{ij}(x)\, \partial_i u(x)\, \partial_j u(x) \;+\; F(x, u(x))\, dx\,. \tag{1.1}$$

Here, $\Omega \subseteq \mathbf{R}^d$ is a (possibly unbounded) domain and $u \in W^{1,2}_{\rm loc}(\mathbf{R}^d, \mathbf{R})$. The summation over the indexes i, j is understood in (1.1). We will refer to this functional as the *"energy"*.

Our main goal is to study the existence and the geometric properties of the periodic minimizers, under the assumption that the functional has a periodic space-dependence, and then to construct critical points of the functional other then the minima. In particular, we are interested in the monotonicity properties of the solutions of the associated PDE with respect to integer translations and in the possibility of confining the level sets of these solutions between two planes which lie on a universal distance. A detailed description of the main results of this paper will be given in § 2.3 and § 3.1 below.

Following is a detailed list of assumptions we make. We assume that the coefficients a_{ij} are uniformly elliptic and that F is a "double-well potential". Also, we will deal with a periodic medium, that is, we assume the dependence on the space variable in $\mathcal{E}$ to be periodic with respect to integer translations. More precisely, we make the following hypotheses:

(H0) $a_{ij} \in C^{\bar n}(\mathbf{R}^d)$ for $1 \le i, j \le d$;
(H1) $\lambda|\zeta| \le a_{ij}(x)\,\zeta_i\zeta_j \le \Lambda|\zeta|$, for any $\zeta, x \in \mathbf{R}^d$;
(H2) $F \in C^{\bar n}(\mathbf{R}^d \times \mathbf{R})$, $F(x, \pm 1) = 0$, $0 \le F(x, \mu) \le \Lambda$, for any $x \in \mathbf{R}^d$, $\mu \in \mathbf{R}$;

(H3) For any $\theta \in [0,1)$, $\inf_{|\mu|\le\theta} F(x,\mu) \ge \gamma(\theta)$, where $\gamma(\theta)$ is a decreasing, strictly positive function in the interval $[0,1)$;

(H4) $\Lambda\,(1-|\mu|)^2 \ge F(x,\mu) \ge \lambda\,(1-|\mu|)^2$, if $|\mu| \in (1-\lambda,1)$;

(H5) $\partial_\mu F(x,-1+s) \ge \lambda\, s$ and $\partial_\mu F(x,1-s) \le -\lambda\, s$, for any $s \in (0,\lambda)$;

(H6) For a fixed $x \in \mathbf{R}^d$, $\partial_\mu F(x,\mu)$ is increasing for $\mu \in (1-\lambda,1)$ and for $\mu \in (-1,-1+\lambda)$;

(H7) $a_{ij}(x+k) = a_{ij}(x)$ and $F(x+k,\mu) = F(x,\mu)$ for any $k \in \mathbf{Z}^d$, $x \in \mathbf{R}^d$, $\mu \in \mathbf{R}$.

Here above and in what follows, we assume $0 < \lambda \le \Lambda$ and $\bar n \in \mathbf{N}$ suitably large. Since the main results of this paper are of geometric type, we did not try to minimize regularity assumptions. Taking $\bar n = 4$ is certainly enough.

Condition **(H1)** is a standard uniform ellipticity assumption. Conditions **(H2)**–**(H6)** state what we mean here by "double-well" potential, an important example being given by $F = Q(x)\,(1-u^2)^2$, with Q positive, bounded and invariant under integer translations. Condition **(H7)** is a periodicity assumption. Constants depending only on d and on the quantities introduced in **(H1)**–**(H7)** will be referred to as "universal constants".

Functionals of the type considered here arise in the Ginzburg-Landau-Allen-Cahn theory of phase transitions (see [Row79]), and in this setting the constant solutions ± 1 are seen as "pure phases". Similar models also arise in the study of "super-fluids" and "super-conductors" (see [GL50] and [GP58]).

In the mathematics literature, this problem appears also as a regularization of the study of minimal hypersurfaces. It was shown in [Mod87] and [CC95] that suitably scaled level sets of minimizers of $\mathcal{E}$ converge to minimal hyper-surfaces. The intuition for this fact is that minimizers of the scaled functional will try to sit into the two minima of the double-well potential (which correspond to the two physical phases of the medium) as much as they can, paying the least possible price in the interface.

2. Notation and statement of results

2.1. **Notation and some standard definitions.** Since we will be dealing with periodic functions with several periods, it will be useful to develop some notation that will handle this comfortably.

Given $v \in \mathbf{R}^d$, we denote

$$v^\perp := \{w \in \mathbf{R}^d \;:\; v \cdot w = 0\}\,.$$

Given a direction $\omega \in \mathbf{R}^d - \{0\}$, we define the following equivalence relation $\sim_\omega$ on $\mathbf{R}^d$: we say that $x \sim_\omega y$ if and only if there exists $k \in \mathbf{Z}^d \cap \omega^\perp$ such that $x - y = k$.

We define

$$\mathbf{K}^d_\omega := \mathbf{R}^d / \sim_\omega\ .$$

Notice that if $\omega \in \mathbf{Q}^d$, then $\mathbf{K}^d_\omega$ is topologically equivalent to the d-dimensional cylinder $\mathbf{T}^{d-1} \times \mathbf{R}$. In the general case, $\mathbf{K}^d_\omega = \mathbf{T}^{d-r} \times \mathbf{R}^r$, where r is the number of *independent resonances*, that is, the dimension of the module

$$\mathcal{R}_\omega \;:=\; \{k \in \omega^\perp \cup \mathbf{Z}^d \;\big|\; \omega \cdot k = 0\}.$$

For almost all $\omega \in \mathbf{R}^d$, we have that $r = 0$ and therefore $\mathbf{K}^d_\omega = \mathbf{R}^d$.

We will be looking for functions that have the periodicities given by $\mathcal{R}_\omega$. Hence, it is natural to consider the functional

$$\mathcal{E}_\omega := \mathcal{E}_{\mathbf{K}^d_\omega}$$

defined on functions $u : \mathbf{K}^d_\omega \to \mathbf{R}$. (Later on, we will specify other properties of the functions on which the functional is defined such as regularity, decay, integrability, etc.) We thus define the functional by the formula (1.1) but we extend the integration only to $\mathbf{K}^d_\omega$.

Notice that, as it is, $\mathcal{E}_\omega$ is not a bona-fide functional – the domain of integration is unbounded – but rather it is just a variational principle. Later, we will find several regularizations that make it into a well defined functional in appropriate function spaces.

The easiest case is $\omega \in \mathbf{Q}^d - \{0\}$. In this case, there is only one unbounded direction, along which, as we will see, we have uniform decay estimates. In certain sense, the case which is hardest is when ω does not satisfy any relations, hence $\mathbf{K}^d_\omega = \mathbf{R}^d$.

Even if the functional (1.1) is not, in general meant as a convergent integral, we recall that u is said to be a *local minimizer* (also called *class-A* minimizers in calculus of variations or *ground states* in phase transition theory) whenever, for every ball $B \subset \mathbf{R}^d$,

$$\mathcal{E}_B(u) \le \mathcal{E}_B(u + \phi)$$

for all ϕ of compact support contained in B.

Note that the definition of local minimizer does not require that the expression (1.1) is a functional when extended over the whole space.

In the cases that the functional is indeed convergent, we will see that u is a *global minimizer*, i.e. $\mathcal{E}(u)$ is the smallest possible value that it can take in the space considered. Global minimizers will be considered in § 4.1.

We denote

$$f(x, \mu) = \frac{\partial}{\partial \mu} F(x, \mu) \, .$$

We say that u is a critical point when it satisfies

$$\partial_i(a_{ij}(x)\partial_j u(x)) = f(x, u(x)) \tag{2.1}$$

in the weak sense (and, in fact, classically, thanks to elliptic regularity theory).

It is well known that as soon as the space of functions we consider is large enough, then all class-A minimizers are critical points, but the converse is, in general, not true.

The solutions of (2.1) sometimes are called *metastable states* in the physics literature.

2.2. The Birkhoff property.

Definition 2.1 (Birkhoff property)**.** *We say that the function $u \in C(\mathbf{R}^d)$ enjoys the Birkhoff property with respect to ω if, for any $k \in \mathbf{Z}^d$ so that $k \cdot \omega \ge 0$, we have that $u(x) \le u(x + k)$ for any $x \in \mathbf{R}^d$.*

We remark that, even if we have formulated the Birkhoff property only for continuous functions (and this is what we use in this paper), the property makes sense for measurable functions too, with obvious modifications. Properties of inclusion related to the Birkhoff property play an important rôle both in dynamical systems and PDEs (see, for instance, [MF94], [KdlLR97], [CdlL01] and [Val04]). Also, such property can be used to avoid some problems as the Hedlund-type counterexamples in the geodesic setting (see [Hed32]). Indeed, the Birkhoff property implies the *doubling property* (also known in the literature as *no-symmetry-breaking-property*), i.e.: a function satisfying the Birkhoff property and having period multiple of the original one, is always periodic with the original period. More precisely, we have:

Proposition 2.2. *Let v satisfy the Birkhoff property with respect to ω. Then $v(x+k)=v(x)$, for any $k\in\mathbf{Z}^d\cap\omega^\perp$.*

Proof. Since $\pm k\in\omega^\perp$, we deduce that $v(x+k)\ge v(x)$ and $v(x-k)\ge v(x)$ for any $x\in\mathbf{R}^d$. Therefore, for any $x\in\mathbf{R}^d$,

$$v(x)=v((x+k)-k)\ge v(x+k)\ge v(x)\,,$$

hence $v(x)=v(x+k)$. □

2.3. **Results on periodic minimizers for arbitrary frequencies.** The following result is a slight strengthening of the results in [Val04]. The main difference is that we produce uniform decay properties along the direction ω. This will be useful later in controlling several limits of minimizers.

Theorem 2.3. *Let $\omega\in\mathbf{Q}^d-\{0\}$. Then, there exists $u\in C^2(\mathbf{K}^d_\omega)$, $|u|\le 1$, and positive universal constants c_1, c_2 and c_3, such that*

1) $\mathcal{E}_\omega(u)<+\infty$
2) $\mathcal{E}_\omega(u)\le\mathcal{E}_\omega(u+\varphi)\,,\qquad\forall\varphi\in W^{1,2}(\mathbf{K}^d_\omega)\cap L^\infty(\mathbf{K}^d_\omega)$.
3) $u(x+k)\ge u(x)$ *for any* $k\in\mathbf{Z}^d$ *with* $\omega\cdot k\ge 0$.
4)
$$|u(x)-1|\le c_2 e^{-c_3\frac{\omega}{|\omega|}\cdot x}\,,\qquad \textit{if}\quad \omega\cdot x\ge c_1$$
and
$$|u(x)+1|\le c_2 e^{c_3\frac{\omega}{|\omega|}\cdot x}\,,\qquad \textit{if}\quad \omega\cdot x\le -c_1.$$

In particular, fixed any $\theta\in(0,1)$, there exists $M>0$, depending only on θ and on universal quantities, such that $u(x)\in[\theta,1]$ provided that $\omega\cdot x\ge M|\omega|$ and $u(x)\in[-1,-\theta]$ provided that $\omega\cdot x\le -M|\omega|$.

The last claim in Theorem 2.3 may be summarized by saying that the interface (that is, the level sets “close to zero”) of u “looks like a hyperplane” or, briefly, that it is “plane-like”.

As straightforward consequence of Theorem 2.3 we deduce the existence of heteroclinic orbits in a wide class of ordinary differential equations. Namely, applying Theorem 2.3 for $d=1$, one obtains immediately:

Corollary 2.4. *Let $F\in C^{\bar n}(\mathbf{T}\times\mathbf{R})$, $F(t,\pm1)=0$, $F(t,\mu)\ge 0$ for any $\mu\in\mathbf{R}$. Assume conditions **(H3)–(H6)** on F. Then, the ordinary differential equation $\ddot q(t)=\partial_\mu F(t,q(t))$, has a solution $q_1:\mathbf{R}\longrightarrow\mathbf{R}$ so that*

$$\lim_{t\to\pm\infty} q_1(t)=\pm1 \tag{2.2}$$

and a solution $q_2:\mathbf{R}\longrightarrow\mathbf{R}$ so that

$$\lim_{t\to\pm\infty} q_2(t)=\mp1\,. \tag{2.3}$$

Furthermore, $|q_i(t)|\le 1$, $q_1(t+k)\ge q_1(t)$ and $q_2(t-k)\ge q_2(t)$, for any $k\in\mathbf{N}$ and any $t\in\mathbf{R}$.

We will also show that all periodic minimizers have level sets contained in a strip of universally large width. Loosely speaking, this says that all periodic minimizers "are plane-like". More precisely, we will prove the following result:

Theorem 2.5. *Let $\omega \in \mathbf{Q}^d - \{0\}$. Fix $\theta_0 \in (0,1)$. Let $u \in W^{1,2}_{\mathrm{loc}}(\mathbf{K}^d_\omega)$, be such that*

1) $\mathcal{E}_\omega(u) < +\infty$.
2) $\mathcal{E}_\omega(u) \le \mathcal{E}_\omega(u+\varphi)$ *for any* $\varphi \in W^{1,2}(\mathbf{K}^d_\omega) \cap L^\infty(\mathbf{K}^d_\omega)$.
3) *Moreover, there exists $M_0 > 0$, depending only on universal constants and on θ_0, such that*

$$\{|u| < \theta_0\} \subseteq \left\{ \frac{\omega}{|\omega|} \cdot x \in [p, p+M_0] \right\} ,$$

for a suitable $p \in \mathbf{R}$.

The above results for the minima of Ginzburg-Landau-Allen-Cahn-type functionals fit in the theory of plane-like structures for phase transitions in periodic media, which has been recently addressed in [Val04].

Seee also [Mos86], [Ban89], [Ban90], [CdlL01], [RS03], [RS04], [Tor04], [PV05a], [PV05b], [Bes05] and [CdlL05] for related results in different contexts. We remark that in the elliptic integrand framework (see, e.g., [Mos86], [Ban89], [Ban90], [RS03] and [RS04]) one is interested in finding solutions whose graphs, when "seen from far", behave "like hyperplanes" and the solutions are thus expected to grow kind of linearly at infinity. In the phase transition setting we deal with, all the solutions are bounded, thus, when seen from far in $\mathbf{R}^{d+1}$, their graphs are obviously close to horizontal d-dimensional planes: the target is then to show that also the level sets of these solutions (i.e., the "interfaces"), when seen from far in $\mathbf{R}^d$, behave like $(d-1)$-dimensional planes. For the important rôle played by flat interfaces, see also [DG79].

3. Minimizers for all frequencies

It is well known that the limit (understood in many senses, e.g. locally C^0) of local minimizers is a local minimizer. See Lemma 3.1 below for a detailed statement and proof of this result.

By the elliptic regularity theory, we see that the set of minimizers for every frequency has uniformly bounded derivatives.

Hence, given a sequence $\omega_n \in \mathbf{Q}^d$ so that $\omega_n \to \omega \in \mathbf{R}^d$, if we consider the minimizers u_{ω_n} produced in Theorem 2.3 and translated so that the interface is at a uniformly bounded distance from the origin, we see that we can pass to a subsequence and obtain something that converges in local C^0. Hence, if in Theorem 2.3 we exchange the hypothesis that $\omega \in \mathbf{Q}^d - \{0\}$ for $\omega \in \mathbf{R} - \{0\}$, we obtain a function u_ω that satisfies 2), 3), 4) of Theorem 2.3. The conclusion 1) – i.e., that the total energy of the minimizer was finite – may fail because the domain is unbounded. Similarly, by passing to limits we get analogues of Theorem 2.5 for any frequency except that we cannot guarantee that the functional is finite.

The fact that the variational principle $\mathcal{E}_\omega$ is not a bona-fide functional is what makes it impossible to apply straightforwardly the direct methods of the calculus of variations.

It will be useful to remark for future purposes that there are several interesting geometric features of local minimizers that satisfy the Birkhoff property.

First, we note that if u is a local minimizer, by the periodicity assumption **(H7)**, so is $u^{(k)}$ defined for $k \in \mathbf{Z}^d$ by:

$$u^{(k)}(x) = u(x+k).$$

If u is Birkhoff, we know by the maximum principle (see Lemma A.1 below) that, given $u^{(k)}$ and $u^{(\ell)}$, they are either identical or there is a strict comparison between them (that is, either $u^{(k)}(x) < u^{(\ell)}(x)$ for any x or $u^{(\ell)}(x) < u^{(k)}(x)$ for any x).

Similarly, if we consider any accumulation point (under pointwise convergence, which is equivalent to accumulation under locally uniform C^0 convergence because the $u^{(k)}$, being solutions of (2.1) have uniformly bounded second derivatives), we obtain also a local minimizer and, hence a critical point. Again by the maximum principle, any two of these functions are identical or there is a strict comparison between them.

In geometric language, we have just verified that the closure of

$$\mathcal{L}_u := \bigcup_{k \in \mathbf{Z}^d} \mathrm{Graph}(u^{(k)})$$

is a *lamination* in $\mathbf{K}^d_\omega \times [-1,1]$. We recall, indeed, that laminations are just closed sets in which we can define leaves that either do not intersect or are identical. In our case, the leaves are the graphs of each of the minimizers.

In the rational case, the set of minimizers is discrete, but in the irrational case, there are complicated accumulations of translations of minimizers.

It can happen (e.g., in the case that the $a_{i,j} = \delta_{i,j}$, that F does not depend on x and that u is a one-dimensional minimizers) that $\mathcal{L}_u = \mathbf{K}^d_\omega \times [-1,1]$. In such a case, following a standard geometric terminology, we say that $\mathcal{L}_u$ is a *foliation*. It can also happen even in the irrational case that $\mathcal{L}_u \neq \mathbf{K}^d_\omega \times [-1,1]$ and that there are *gaps* in the lamination, that is, connected sets in the complement of $\mathcal{L}_u$ whose boundaries are just two leaves of $\mathcal{L}_u$. We will not present here explicit examples of this phenomenon, but they are constructed for similar models in [Ban89].

In the irrational case, if we identify a gap in $\mathcal{L}_u$, this implies that there is also a corresponding gap in $\mathcal{L}_{u^{(k)}}$, $k \in \mathbf{Z}^d$. Since $\mathcal{L}_{u^{(k)}} = \mathcal{L}_u$, we see that there have to be an infinite number of gaps. This is reminiscent of the existence of gaps in the Cantor sets in the Aubry-Mather theory of dynamical systems.

Lemma 3.1. *Let u_n be a sequence of continuous local minimizers of the variational problem* (1.1) *converging pointwise to a function u. Then u is a local minimizer.*

Proof. Because the local minimizers satisfy (2.1), we have, by the elliptic regularity theory that the second derivatives are uniformly bounded. Therefore, by Ascoli-Arzelà theorem we obtain that u_n converges locally C^1 to u.

Assume, by contradiction, that there exists a smooth function ϕ of support contained in a ball B such that

$$\mathcal{E}_B(u+\phi) - \mathcal{E}_B(u) =: -\delta < 0.$$

Then, for large enough n, we have that

$$|\mathcal{E}_B(u_n+\phi) - \mathcal{E}_B(u+\phi)| \le \delta/10$$
$$|\mathcal{E}_B(u_n) - \mathcal{E}_B(u)| \le \delta/10$$

and therefore

$$\mathcal{E}_B(u_n+\phi) - \mathcal{E}_B(u_n) \le -\frac{4}{5}\delta < 0$$

which is a contradiction with u_n being a minimizer. $\square$

3.1. **Results on the existence critical points other than minimizers.** Given the above discussion, the best that one can hope along the lines of producing a new solution is to show that, if there are gaps in $\mathcal{L}$, a lamination whose leaves are Birkhoff minimizers, then there is a solution inside the gaps. Clearly, for the rational case, since the set of translations is discrete, there will always be gaps, and hence, there will always be another solution besides the translations of the Birkhoff minimizer.

In general, we will consider a lamination of minimizers. We will not need to assume that the lamination is the closure of the translations of a single minimizer, even if this is what we had discussed before. As we will see, it is quite possible that there are laminations generated by the translation of several minimizers.

Hence, our result is:

Theorem 3.2. *Let $u^{(0)}$ and $u^{(1)}$ be two local minimizers satisfying the conclusions in Theorem 2.3 except 1), so that*

$$u^{(1)}(x) > u^{(0)}(x)$$

for any $x \in \mathbf{K}^d_\omega$. Then, there exists $u \in C^2(\mathbf{K}^d_\omega)$ satisfying:

- *u is a critical point.*
- *For any $k \in \mathbf{Z}^d$ so that $k \cdot \omega \geq 0$, $u(x) \leq u(x+k)$, for any $x \in \mathbf{K}^d_\omega$;*
- *$u^{(0)}(x) < u(x) < u^{(1)}(x)$.*

Later, when we have introduced more notation, we will state some more precise results. As it turns out, the solutions thus produced could have an infinite energy. Nevertheless, we will show that they have a finite *renormalized energy* defined in (5.8). This renormalized energy will allow us to formulate a criterion for the existence of a foliation by minimizers or not.

The proof of Theorem 3.2 does not depend on the exponential bounds concluded in Theorem 2.3. Of course, when the $u^{(0)}$ and $u^{(1)}$ we consider are limits of rational minimizers, they satisfy the exponential bounds and therefore, since $u^{(0)} < u < u^{(1)}$ so does u. In the case that $u^{(0)}$ and $u^{(1)}$ are obtained as limits of rational minimizers, they inherit the property that the interface is contained in a strip whose width is bounded by a universal number. Therefore, the solutions u we construct also have a width bounded by a universal constant.

One question we have not settled in this paper is whether all minimizers can be approximated by periodic ones. Closely related questions are whether all minimizers are Birkhoff, whether all minimizers satisfy converge to free phases exponentially fast. The later question is also connected to whether the structure of the interfaces is simple enough at infinity so that one can construct barrier functions as is done in the proof of Proposition 4.3.

3.2. **Remarks on the literature, sketch of the proofs and organization of the paper.**

3.2.1. *Remarks on the literature.* In the elliptic integrand setting, a result related to Theorem 3.2 for the rational case has been announced by Paul Rabinowitz on the occasion of the International Symposium on Variational Methods and Nonlinear Differential Equations, Rome, 2005. The study of the irrational case has been recently started in [Bes05] for elliptic integrands, under an additional hypothesis introduced by [Ban87]. To the best of our knowledge, a full treatment of the irrational case was not available yet.

We also point out that, for $d = 1$, we obtain from Theorem 3.2 the following multiplicity result for heteroclinics:

Corollary 3.3. *In the setting of Corollary 2.4, the ordinary differential equation $\ddot{q}(t) = \partial_\mu F(t, q(t))$, has at least two solutions satisfying* (2.2) *and at least two solutions satisfying* (2.3)*. Also, if q is any of such solutions, we have that $|q(t)| \le 1$, and either $q(t+k) \ge q(t)$ or $q(t-k) \ge q(t)$, for any $k \in \mathbf{N}$ and any $t \in \mathbf{R}$.*

There are a variety of results on asymptotic trajectories of Hamiltonian systems which are related to the ones in Corollary 2.4 and Corollary 3.3 here above: see, for instance, [CZES90], [Bol95] and references therein. In this sense, our results may also been considered as PDE versions of some results in dynamical systems.

There seems also to be a close connection with the Aubry-Mather theory. In particular, in the case of irrational frequency, one can considered the existence of gaps as being very similar to the existence of the Aubry-Mather Cantor sets and the case that the minimizers form a foliation corresponds to the existence of an invariant circle. For twist mappings, the fact that when there are gaps in the Aubry-Mather sets there are other Cantor sets homoclinic to the previous one was proved in [Mat86]. In § 5.9, we discuss some relations of the reduced energy that we introduce in (5.8) with the Peierls barrier introduced in [Mat86] extending ideas of [ALD83]. A more detailed comparison between the results for PDE's and those for Aubry-Mather theory can be found in [Mos86].

3.2.2. *Sketch of the proof.* The main ideas for proving the above results are the following. Theorems 2.3 and 2.5 follow from some results of [Val04] and some estimates on the exponential approach to pure phases of the solutions found there.

The proof of Theorem 3.2 uses some properties of the associated heat flow. The heat flow is our tool to overcome the intrinsic lack of compactness of the problems (namely, its translation invariance and the fact that the domain of the functional is unbounded), which makes the standard nonlinear analysis methods not directly applicable. Namely, the proof of Theorem 3.2 is by contradiction: if no other critical points existed, the heat flow would gain compactness and produce a new solution. Arguments of this type have been used in [KdlLR97]. In our case, the heat flow is the gradient of a functional, which we call the renormalized energy (5.8). The proof is somewhat reminiscent of the the Ljusternik-Schnirelmann theory (see [LS34] and [Szu89]) but we have to overcome some difficulties arising from the unbounded nature of the problem so that the choice of topologies is not so obvious, so we find it advantageous to consider the gradient flow (which in our context is a parabolic equation that we call the heat flow) directly rather than to use some of the standard versions of the theory as in [Sch64] and [Bro65]. In our context, the heat flow is particularly useful since it preserves the order, which is an important part of our conclusions. The order properties will also give some integral a-priori bounds that supplement the usual local regularity gains to give strong compactness properties.

In order to apply the heat flow effectively, we have to use that it decreases some well defined renormalized energy, defined in (5.8), and we have to provide a very simple a-priori estimate that is obtained directly from the fact that the translations form a lamination (see Lemma 5.1 here below).

We mention that the use of the renormalized energy and the conclusion of existence of homoclinic orbits in the gaps is somewhat reminiscent of the results on existence of homoclinic Cantor sets in [Mat86], which uses very different methods than those in this paper. We think that it would be very interesting to undertake a more systematic comparison between the methods of this paper and those in [Mat86]. The relation of gradient flow methods and Aubry-Mather theory was pointed out in [Ang90a].

3.2.3. *Organization of the paper.* The paper is organized as follows. In § 4, we prove Theorems 2.3 and 2.5. The proofs will make use of the exponential decay of the solutions and of some results in [Val04]. In § 5, we introduce a "renormalized" energy. This is needed because the "regular" energy $\mathcal{E}$ may become infinite on unbounded domains. We then consider the heat flow associated to this renormalized energy. The basin of attraction of such a heat flow is thus discussed, under the assumption that the claim of Theorem 3.2 is false. This will lead to the construction of a further critical point, by heat flowing a suitable path of initial data, thus proving Theorem 3.2. Some features of the energy barriers are also discussed, in relation with analogous phenomena arising in Aubry-Mather theory.

The paper ends with an appendix. The aim of § A.1 and § A.2 (which may be skipped by expert readers) is to state some standard elliptic and parabolic results in a way that fits our purposes. In § A.3, we collect some results of [Val04] which are used in the proofs of the main results of this paper.

4. Exponential decay for the solutions and the plane-like minimizers

We deduce from the comparison principle Lemma A.1 an exponential bound on the solutions of our equation in the rational frequency case.

Corollary 4.1. *Let $u \in W^{1,2}_{\rm loc}(\mathbf{K}^d_\omega)$ satisfy weakly*

$$\partial_i(a_{i,j}\partial_j u) = \partial_\mu F(x,u)\,.$$

*Let the coefficients a_{ij} be Lipschitz continuous and satisfy the uniform ellipticity condition given in **(H1)**. Let F satisfy the assumptions in **(H0)**–**(H7)** and let λ be the quantity introduced there. Assume that $1 \ge u(x) \ge 1-\lambda$ for any $x \in \mathbf{R}^d$ for which $\frac{\omega}{|\omega|}\cdot x \ge M$ and that*

$$\lim_{s\longrightarrow+\infty} u\Big(x+s\frac{\omega}{|\omega|}\Big) = 1\,. \tag{4.1}$$

Then, there exists a universal constant $a>0$ so that

$$u(x) \ge 1-\lambda e^{a(M-\frac{\omega}{|\omega|}\cdot x)}\,.$$

Proof. Let $\beta(x) = 1-\lambda e^{a(M-\frac{\omega}{|\omega|}\cdot x)}$ if $\frac{\omega}{|\omega|}\cdot x \ge M$, with $a\in(0,1)$ to be chosen in the sequel. Then,

$$\partial_i(a_{i,j}\partial_j\beta) \ge -C^\star a(1-\beta)\,,$$

for a suitable $C^\star>0$. Define $\psi = \beta-u$. Thus, from **(H5)**,

$$\begin{aligned}
\partial_i(a_{i,j}\partial_j\psi) - C^\star a\psi &\ge -C^\star a(1-\beta) - \partial_\mu F(x,u) - C^\star a\beta + C^\star a u \ge \\
&\ge -C^\star a + \lambda(1-u) + C^\star a u = \\
&= (\lambda - C^\star a)(1-u) \ge 0\,,
\end{aligned}$$

if $\frac{\omega}{|\omega|}\cdot x \ge M$, provided that $a < C^\star/\lambda$. Assume, by contradiction, that there exists $x^\star$ so that $\frac{\omega}{|\omega|}\cdot x^\star \ge M$ and $u(x^\star) < \beta(x^\star)$. Then, there exists $\varepsilon_0>0$ so that

$\psi(x^\star) > \varepsilon_0$. Notice that the set

$$\Omega := \{\psi > \varepsilon_0\} \cap \left\{ \frac{\omega}{|\omega|} \cdot x > M \right\}$$

must be bounded in the direction of ω, since

$$\lim_{s \longrightarrow +\infty} \psi\Big(x + s\frac{\omega}{|\omega|}\Big) = 0\,.$$

Then, a contradiction easily follows from Lemma A.1. $\square$

4.1. **Existence and qualitative properties of global minima when $\omega \in \mathbf{Q}^d - \{0\}$.** In this section, we show the existence of a global minimum for $\mathcal{E}_\omega$ and we point out that it will approach the boundary values exponentially fast. We will also point out that all global minima satisfy the Birkhoff property.

First, we prove the exponential convergence at infinity for the local minima found in [Val04]:

Corollary 4.2. *Let u be the local minimizer given by Theorem A.8. Then, there exist positive universal constants c_1, c_2 and c_3, so that*

$$\begin{aligned} |u(x) - 1| &\le c_2 e^{-c_3 \frac{\omega}{|\omega|}\cdot x}\,, \qquad if \quad \omega \cdot x \ge c_1 \\ \text{and} \quad |u(x) + 1| &\le c_2 e^{c_3 \frac{\omega}{|\omega|}\cdot x}\,, \qquad if \quad \omega \cdot x \le -c_1\,. \end{aligned}$$

Proof. Since $\mathcal{E}_\omega(u) < +\infty$, it follows that $\lim_{s \longrightarrow \pm\infty} u(x + s\frac{\omega}{|\omega|}) = \pm 1$ for any $x \in \mathbf{K}^d_\omega$: if not, using that u is uniformly Hölder continuous (see [GG82]), there would be a family of disjoint balls of uniform radius on which $|u| \le 1 - \delta$, for some $\delta > 0$, and this would provide an infinite amount of energy. Thus, the claim follows from Corollary 4.1. $\square$

Analogous decays for higher derivatives are obtained from Corollary 4.1 and elliptic bootstrap. Namely, if $v := u \pm 1$, then

$$\partial_i(a_{ij}\partial_j v) = f(x, v(x) \mp 1) =: g(x)\,,$$

thus, from (8.86) of [GT83] (recalling also (4.3)–(4.17) and (6.10) for notations there), one has that

$$\begin{aligned} |\nabla u(x)| &\le \|v\|_{C^{1,\alpha}(B_1(x))} \le \text{const}\Big(\|v\|_{L^\infty(B_2(x))} + \|g\|_{L^\infty(B_2(x))}\Big) \le \\ &\le \text{const}\,\|v\|_{L^\infty(B_2(x))} \le \\ &\le \text{const}\, e^{-\text{const}\left|\frac{\omega}{|\omega|}\cdot x\right|}\,. \end{aligned} \tag{4.2}$$

And then, by Schauder Estimates (see, e.g., (6.23) in [GT83]),

$$\begin{aligned} |D^2 u(x)| &\le \|v\|_{C^{2,\alpha}(B_{1/2}(x))} \le \text{const}\Big(\|v\|_{L^\infty(B_1(x))} + \|g\|_{C^\alpha(B_1(x))}\Big) \le \\ &\le \text{const}\,\|v\|_{C^\alpha(B_1(x))} \le \\ &\le \text{const}\, e^{-\text{const}\left|\frac{\omega}{|\omega|}\cdot x\right|}\,. \end{aligned} \tag{4.3}$$

We now show that all periodic minimizers satisfy the Birkhoff property (up to a sign change):

Proposition 4.3. *Let $u \in W^{1,2}_{\mathrm{loc}}(\mathbf{K}^d_\omega) \cap L^\infty(\mathbf{K}^d_\omega)$ be such that $\mathcal{E}_\omega(u) < +\infty$ and*

$$\mathcal{E}_\omega(u) \le \mathcal{E}_\omega(u+\varphi) \tag{4.4}$$

for any $\varphi \in W^{1,2}(\mathbf{K}^d_\omega) \cap L^\infty(\mathbf{K}^d_\omega)$. Then, either u or $-u$ is Birkhoff with respect to ω (see Definition 2.1).

Proof. We first prove that, for any $k \in \mathbf{Z}^d$,

$$\text{either } u(x+k) \ge u(x) \text{ for any } x \in \mathbf{K}^d_\omega \text{ or } u(x+k) \le u(x) \text{ for any } x \in \mathbf{K}^d_\omega. \tag{4.5}$$

The proof of (4.5) is by contradiction. Let $k \in \mathbf{Z}^d$. Assume that there exists x_1 and x_2 so that

$$u(x_1+k) > u(x_1) \qquad \text{and} \qquad u(x_2+k) < u(x_2)\,. \tag{4.6}$$

Then, there exists x^* so that

$$u(x^*+k) = u(x^*)\,. \tag{4.7}$$

Let

$$u^*(x) := \max\{u(x),\, u(x+k)\}\,, \qquad u_*(x) := \min\{u(x),\, u(x+k)\}\,.$$

Since $\mathcal{E}_\omega(u) < +\infty$, we have that $u \in W^{1,2}(\mathbf{K}^d_\omega)$, therefore

$$u^* - u,\, u_* - u \in W^{1,2}(\mathbf{K}^d_\omega) \cap L^\infty(\mathbf{K}^d_\omega)\,, \tag{4.8}$$

and so, from (4.4),

$$\mathcal{E}_\omega(u^*) \ge \mathcal{E}_\omega(u) \;\text{ and }\; \mathcal{E}_\omega(u_*) \ge \mathcal{E}_\omega(u)\,. \tag{4.9}$$

Also, by splitting the domains of integration (using the so called Rellich Lemma, see, e.g., page 50 on [KS00]), one gets that

$$\mathcal{E}_\omega(u^*) + \mathcal{E}_\omega(u_*) = 2\mathcal{E}_\omega(u)\,,$$

hence, by (4.9),

$$\mathcal{E}_\omega(u^*) = \mathcal{E}_\omega(u_*) = \mathcal{E}_\omega(u)\,. \tag{4.10}$$

In particular, from (4.8), if $\phi \in C^\infty_0(\mathbf{K}^d_\omega)$, then

$$u^* + \phi - u \in W^{1,2}(\mathbf{K}^d_\omega) \cap L^\infty(\mathbf{K}^d_\omega)$$

and therefore, by (4.4) and (4.10), we gather that

$$\mathcal{E}_\omega(u^*+\phi) \ge \mathcal{E}_\omega(u) = \mathcal{E}_\omega(u^*)\,.$$

Analogously, $\mathcal{E}_\omega(u_*+\phi) \ge \mathcal{E}_\omega(u_*)$, for any $\phi \in C^\infty_0(\mathbf{K}^d_\omega)$. Thus u^*, u_* are critical for $\mathcal{E}_\omega$. By Corollary A.3 and the fact that $u^* \ge u_*$, we gather that either $u_* = u^*$ or $u_* < u^*$. The first possibility is ruled out by (4.6). The second by (4.7). This ends the proof of (4.5).

With this, we now prove the desired result. Take $\bar k \in \mathbf{Z}^d$ with $\bar k = \alpha\omega$ for some $\alpha > 0$ and with α as small as possible. Due to (4.5), we have that either $u(\cdot + \bar k) \ge u(\cdot)$ or $u(\cdot + \bar k) \le u(\cdot)$. We assume that the first possibility holds (the other case being analogous) and we then show that $u(\cdot + k) \ge u(\cdot)$ for any $k \in \mathbf{Z}^d$ so that $\omega \cdot k \ge 0$.

Indeed, take such a k. Assume by contradiction that $u(\bar x + k) < u(\bar x)$ for some $\bar x$. Then, by (4.5) and Corollary A.3,

$$u(\cdot + k) < u(\cdot)\,. \tag{4.11}$$

Also, there exist $a \in \mathbf{N} - \{0\}$, $b \in \mathbf{Z}$, $\alpha \in \mathbf{R}$ and $\hat{k} \in \mathbf{Z}^d \cap \omega^\perp$ so that

$$ak = b\bar{k} + \hat{k}\,. \tag{4.12}$$

Explicitly, to confirm (4.12), given

$$k = \sum_{j=1}^{d} k_j e_j \in \mathbf{Z}^d\,,$$

one may take

$$\begin{aligned} a &:= \sum_{i=1}^{d} \bar{k}_i^2 \in \mathbf{N} - \{0\}\,, \\ h^{(j)} &:= ae_j - (\bar{k}\cdot e_j)\bar{k} \in \mathbf{Z}^d \cap \bar{k}^\perp = \mathbf{Z}^d \cap \omega^\perp\,, \text{ for } j = 1,\dots,d, \\ \hat{k} &:= \sum_{j=1}^{d} k_j h^{(j)} \in \mathbf{Z}^d \cap \omega^\perp \\ \text{and } b &:= k\cdot\bar{k} \in \mathbf{Z} \end{aligned}$$

and these choices easily give (4.12).

Then, by (4.12),

$$0 \le a\omega\cdot k = b\bar{k}\cdot\omega = \alpha b\,|\omega|^2$$

and so

$$b \ge 0\,. \tag{4.13}$$

Also, by the periodicity of u,

$$u(x+ak) = u(x+b\bar{k}+\hat{k}) = u(x+b\bar{k})$$

for any x. From this, (4.13) and our assumptions on $\bar{k}$, we get that

$$u(\cdot + ak) \ge u(\cdot)\,. \tag{4.14}$$

On the other hand, from the fact that $a \in \mathbf{N} - \{0\}$ (and so $a \ge 1$) and (4.11), we deduce that

$$u(\cdot + ak) < \cdots < u(\cdot + k) < u(\cdot)\,,$$

in contradiction with (4.14). □

One of the first consequences of the Birkhoff property and of the no-symmetry-breaking is that global minimizers (when their domain is unfolded to the whole space $\mathbf{R}^d$) are local minimizers in any domain of $\mathbf{R}^d$. More precisely, the following result easily follows from Corollary 4.3 and Proposition 2.2:

Proposition 4.4. *Let $u \in W^{1,2}_{\mathrm{loc}}(\mathbf{K}^d_\omega) \cap L^\infty(\mathbf{K}^d_\omega)$ be such that $\mathcal{E}_\omega(u) < +\infty$ and $\mathcal{E}_\omega(u) \le \mathcal{E}_\omega(u+\varphi)$ for any $\varphi \in W^{1,2}(\mathbf{K}^d_\omega) \cap L^\infty(\mathbf{K}^d_\omega)$. Let $\tilde{u} \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}^d)$ be the $\mathbf{K}^d$-periodic extension of u to the whole $\mathbf{R}^d$. Then*

$$\mathcal{E}_B(\tilde{u}+\phi) \ge \mathcal{E}_B(\tilde{u})\,, \tag{4.15}$$

for any ball $B \subset \mathbf{R}^d$ and any $\phi \in C_0^\infty(B)$.

4.2. **Proof of Theorem 2.3.** Let u be as in Theorem A.8 (and recall also Corollary 4.2). The last property of u needed for proving Theorem 2.3 is the minimizing property of u under perturbations in $W^{1,2}(\mathbf{K}^d_\omega)\cap L^\infty(\mathbf{K}^d_\omega)$. For that, let $\phi\in W^{1,2}(\mathbf{K}^d_\omega)\cap L^\infty(\mathbf{K}^d_\omega)$ and let us show that

$$\mathcal{E}_\omega(u)\le\mathcal{E}_\omega(u+\phi)\,. \tag{4.16}$$

Let ρ_n be a standard mollifier (see, e.g., Theorem 1.6.1 in [Zie89]) and $\phi_n:=\phi*\rho_n$. Then, up to subsequences, ϕ_n converges to ϕ almost everywhere and in $W^{1,2}(\mathbf{K}^d_\omega)$.

For any $R>0$, let

$$G_R:=\Big\{x\in\mathbf{K}^d_\omega \text{ s.t. } \Big|\frac{\omega}{|\omega|}\cdot x\Big|\le R\Big\}$$

and let $\chi_R\in C^\infty_0(G_{R+1})$ with $\|\nabla\chi_R\|\le 10$, $0\le\chi_R\le 1$ and $\chi_R(x)=1$ for any $x\in G_R$. Set also $\phi_n^{(R)}:=\phi_n\chi_R$ and $\phi^{(R)}:=\phi\chi_R$. Notice that, by construction,

$$\|\phi_n^{(R)}\|_{L^\infty(\mathbf{K}^d_\omega)}\le\|\phi_n\|_{L^\infty(\mathbf{K}^d_\omega)}\le\|\phi\|_{L^\infty(\mathbf{K}^d_\omega)}\,. \tag{4.17}$$

Let us now make some elementary observations. First of all, for any a, $b\in\mathbf{R}$,

$$|a+b|-1\le|a|+|b|-1\le\Big||a|-1\Big|+|b|$$

and

$$|a+b|-1\ge|a|-|b|-1\ge-\Big||a|-1\Big|-|b|\,,$$

that is,

$$\Big|1-|a+b|\Big|\le\Big|1-|a|\Big|+|b|\,. \tag{4.18}$$

Also, since $F\in C^2(\mathbf{R}^d\times\mathbf{R})$ and $\partial_\mu F(x,\pm1)=0$, we have that there exists $M>0$, which may depend on $\|\phi\|_{L^\infty(\mathbf{K}^d_\omega)}$ so that

$$|\partial_\mu F(x,\mu)|\le M\Big|1-|\mu|\Big|\,, \tag{4.19}$$

for any $x\in\mathbf{K}^d_\omega$ and any $|\mu|\le 1+\|\phi\|_{L^\infty(\mathbf{K}^d_\omega)}$. Fixed $t\in[0,1]$, let now

$$U_R(x,t):=u(x)+t\phi^{(R)}(x)+(1-t)\phi(x)\,.$$

By construction, $|U_R|\le 1+\|\phi\|_{L^\infty(\mathbf{K}^d_\omega)}$, therefore, from (4.19) and (4.18),

$$\begin{aligned}\Big|\partial_\mu F\Big(x,U_R(x,t)\Big)\Big| &\le M\Big|1-|U_R(x,t)|\Big|\le\\ &\le M\Big(\Big|1-|u(x)|\Big|+|t(\phi^{(R)}(x)-\phi(x))+\phi(x)|\Big)\le\\ &\le M\Big(\Big|1-|u(x)|\Big|+|\phi^{(R)}(x)-\phi(x)|+|\phi(x)|\Big)\,.\end{aligned}$$

Since, by Theorem A.8 and Corollary 4.2,

$$|1-|u(x)||\le C_1\,e^{-C_2\left|\frac{\omega}{|\omega|}\cdot x\right|}\,,$$

for suitable constants C_1 and C_2, we gather that

$$\Big|\partial_\mu F\Big(x,U_R(x,t)\Big)\Big|\le C_3\Big(e^{-C_2\left|\frac{\omega}{|\omega|}\cdot x\right|}+|\phi^{(R)}(x)-\phi(x)|+|\phi(x)|\Big)\,,$$

for some C_3, which may depend on $\|\phi\|_{L^\infty(\mathbf{K}^d_\omega)}$. Therefore,

$$\begin{aligned}
&\left|\int_{\mathbf{K}^d_\omega} F(x,u+\phi) \,-\, F(x,u+\phi^{(R)})\,dx\right| \leq \\
\leq\;& \int_0^1 \int_{\mathbf{K}^d_\omega} |\partial_\mu F(x,U_R(x,t))|\,|\phi^{(R)}(x)-\phi(x)|\,dx\,dt \leq \\
\leq\;& C_3 \int_{\mathbf{K}^d_\omega} \left(e^{-C_2\left|\frac{\omega}{|\omega|}\cdot x\right|} + |\phi^{(R)}(x)-\phi(x)| + |\phi(x)|\right) |\phi^{(R)}(x)-\phi(x)|\,dx \leq \\
\leq\;& C_4\,\|\phi^{(R)}-\phi\|_{L^2(\mathbf{K}^d_\omega)} \\
\leq\;& 2C_4\,\|\phi\|_{L^2(\mathbf{K}^d_\omega - G_R)}\,,
\end{aligned}$$

with C_4 possibly depending on $\|\phi\|_{L^\infty(\mathbf{K}^d_\omega)}$. In particular, since the above quantity tends to zero for $R \longrightarrow +\infty$, we get that

$$\lim_{R\to+\infty}\int_{\mathbf{K}^d_\omega} F(x,u+\phi^{(R)})\,dx \;=\; \int_{\mathbf{K}^d_\omega} F(x,u+\phi)\,dx\,. \tag{4.20}$$

Also, by inspection, one sees that

$$\|\nabla(\phi_n^{(R)}-\phi^{(R)})\|_{L^2(\mathbf{K}^d_\omega)} \leq C\,\|\phi_n-\phi\|_{W^{1,2}(\mathbf{K}^d_\omega)}\,, \tag{4.21}$$

for a suitable positive universal constant C, and the latter quantity goes to zero when $n \longrightarrow +\infty$. Moreover, from (4.17) and the Bounded Convergence Theorem,

$$\lim_{n\to+\infty}\int_{G_{R+1}} F(x,u+\phi_n^{(R)}) = \int_{G_{R+1}} F(x,u+\phi^{(R)})\,.$$

Since both $\phi^{(R)}$ and $\phi_n^{(R)}$ vanish outside G_{R+1}, the above reads

$$\lim_{n\to+\infty}\int_{\mathbf{K}^d_\omega} F(x,u+\phi_n^{(R)}) = \int_{\mathbf{K}^d_\omega} F(x,u+\phi^{(R)})\,. \tag{4.22}$$

Since $\phi_n^{(R)} \in C_0^\infty(\mathbf{K}^d_\omega)$, we know from Theorem A.8 that $\mathcal{E}_\omega(u) \leq \mathcal{E}_\omega(u+\phi_n^{(R)})$. Thus, taking the limit as $n \longrightarrow +\infty$ and exploiting (4.21) and (4.22), one sees that

$$\mathcal{E}_\omega(u) \leq \mathcal{E}_\omega(u+\phi^{(R)})\,. \tag{4.23}$$

Furthermore, for some $C>0$,

$$\|\nabla(\phi^{(R)}-\phi)\|_{L^2(\mathbf{K}^d_\omega)} \leq C\,\|\phi\|_{W^{1,2}(\mathbf{K}^d_\omega - G_R)}\,, \tag{4.24}$$

which tends to zero as $R \longrightarrow +\infty$. Hence, taking the limit as $R \longrightarrow +\infty$ in (4.23) and using (4.24) and (4.20), we have that (4.16) follows, thus ending the proof of Theorem 2.3. ☐

4.3. **Proof of Theorem 2.5.** We now address the proof of Theorem 2.5. Take a point $\bar{x}$ so that $|u(\bar{x})| < \theta_0$. From Theorem A.7, we obtain two balls $B_\geq$ and $B_\leq$ of radius ρ, one contained in $\{u \geq \theta_0\}$ and the other in $\{u \leq -\theta_0\}$, provided that $\kappa^{-1}\rho \geq r_0$. Both balls are also contained in $B_{\kappa^{-1}\rho}$. Let us denote by T_k the translation by a vector $k \in \mathbf{Z}^d$. By the Birkhoff property of minimizers (recall Corollary 4.3), we deduce that

$$\begin{aligned} H_{\geq} &:= \bigcup_{k\cdot\omega\geq 0} T_k(B_{\geq}) \subseteq \{u \geq \theta_0\} \\ \text{and} \quad H_{\leq} &:= \bigcup_{k\cdot\omega\leq 0} T_k(B_{\leq}) \subseteq \{u \leq -\theta_0\}\,. \end{aligned}$$

Assuming ρ suitably big with respect to d, it follows that $H_{\geq}$ and $H_{\leq}$ contain a half-space. Also, the slab left outside these half-spaces is parallel to the plane $\{\omega \cdot x = 0\}$, has a width depending only on κ, r_0 and d (and so it is a universal constant) and contains the set $\{|u| < \theta_0\}$. This ends the proof of Theorem 2.5. □

5. Construction of other critical points. Proof of Theorem 3.2

The proof we will carry out will have the same steps whether ω is rational or not. Some of the steps will require more delicate arguments when ω is irrational.

5.1. **Notation and preliminaries.** We recall that our starting point is two local minimizers $u^{(0)} < u^{(1)}$ which are in the same lamination and are at the edges of a gap.

We consider the convex combination of $u^{(0)}$ and $u^{(1)}$, that is, for any $s \in [0,1]$, we set

$$u^{(s)} := su^{(1)} + (1-s)u^{(0)}\,.$$

Let also

$$v^{(s)} := s(u^{(1)} - u^{(0)}) = u^{(s)} - u^{(0)}\,. \tag{5.1}$$

Notice that $u^{(s)} \in C^2(\mathbf{K}^d_\omega)$, $|u| \leq 1$ and, from Theorem 2.3, we gather the decay estimate

$$|v^{(s)}(x)| \leq C_1 e^{-C_2\left|\frac{\omega}{|\omega|}\cdot x\right|}, \tag{5.2}$$

for any $s \in [0,1]$, for suitable universal constants C_1 and C_2.

We consider the set of functions trapped between $u^{(0)}$ and $u^{(1)}$: for this, we define

$$\begin{aligned} \mathcal{C} \;:=\; &\{v \in C(\mathbf{K}^d_\omega) \text{ s.t. } u^{(0)} + v \text{ enjoys the Birkhoff property and} \\ &u^{(0)} \leq u^{(0)} + v \leq u^{(1)}\}\,. \end{aligned} \tag{5.3}$$

Notice that, due to (5.2), $v^{(s)} \in \mathcal{C}$ for any $s \in [0,1]$. Note that the functions $u^{(s)}$ are Birkhoff if $u^{(0)}, u^{(1)}$ are.

Some words may be needed to justify the notation chosen in (5.3): in the following arguments, our objects of primary interest will be the increments from $u^{(0)}$ rather than the function describing the state. Hence, we will develop functionals etc. adapted to this notation. This simplifies some of the calculations later.

5.2. **An integral a-priori bound for Birkhoff functions.** The following a-priori integral bound will be crucial for our work.

Lemma 5.1. *Let $u^{(0)}$ and $u^{(1)}$ be Birkhoff functions at the edge of the gaps of a lamination invariant under integer translations. Then,*

$$\int_{\mathbf{K}^d_\omega} |u^{(1)} - u^{(0)}| \le 2 \tag{5.4}$$

Note that an immediate consequence of Lemma 5.1 is that for all functions $v \in \mathcal{C}$ we have

$$\int_{\mathbf{K}^d_\omega} v \le 2.$$

Proof. The idea of the proof is very simple. Because of the Birkhoff property, the translations of the gaps to a fixed unit cube times $[-1,1]$ cannot overlap, so that the total volume should be less than the volume of $[0,1]^d \times [-1,1]$ (and thence 2).

Let $\mathcal{Q} := [0,1]^d$ and let $\mathcal{Z} \subset \mathbf{Z}^d$ be a minimal set of integer vectors so that

$$\bigcup_{k\in\mathcal{Z}} (k + \mathcal{Q})$$

covers the lift of $\mathbf{K}^d_\omega$ to $\mathbf{R}^d$. For any $k \in \mathbf{Z}^d$, let

$$\mathcal{A}_k := \Big\{(x,y) \in \mathcal{Q} \times \mathbf{R} \text{ s.t. } u^{(0)}(x+k) < y \le u^{(1)}(x+k)\Big\}.$$

Thanks to the Birkhoff property, we have that

$$\mathcal{A}_h \cap \mathcal{A}_k = \emptyset,$$

if $h, k \in \mathcal{Z}$ with $h \neq k$. Therefore,

$$\begin{aligned}\int_{\mathbf{K}^d_\omega} |u^{(1)} - u^{(0)}| &= \sum_{k\in\mathcal{Z}} \int_{k+\mathcal{Q}} u^{(1)}(x) - u^{(0)}(x)\,dx \\ &= \sum_{k\in\mathcal{Z}} |\mathcal{A}_k| \\ &= |\bigcup_{k\in\mathcal{Z}} \mathcal{A}_k| .\end{aligned}$$

Since $\mathcal{A}_k \subseteq \mathcal{Q} \times [-1,1]$ we thus deduce that

$$\begin{aligned}\int_{\mathbf{K}^d_\omega} |u^{(1)} - u^{(0)}| &\le \Big|\mathcal{Q} \times [-1,1]\Big| \\ &= 2 .\end{aligned}$$

∎

We note that the proof of (5.4) only uses that the $u^{(0)}$ and $u^{(1)}$ are Birkhoff and are inside a lamination. The minimization properties do not play any role. In the case that the u's are periodic, we obtain equality in (5.4).

If $u^{(0)}$ and $u^{(1)}$ satisfy the elliptic equation (2.1), recalling that they are bounded, we can obtain from (5.1) the following result:

Lemma 5.2. *For $r = 0, 1, \ldots$ as large as we wish, there exists a positive constant C, possibly depending on r, so that*

$$\int_{\mathbf{K}^d_\omega} |D^r(u^{(0)} - u^{(1)})| \le C\,. \tag{5.5}$$

Proof. We note that $u^{(1)} - u^{(0)}$ satisfies

$$L(u^{(0)} - u^{(1)}) = f(x, u^{(0)}(x)) - f(x, u^{(1)}(x))\,. \tag{5.6}$$

Furthermore,

$$|f(x, u^{(0)}(x)) - f(x, u^{(1)}(x))| \le \text{ const}\, |u^{(1)}(x) - u^{(0)}(x)|\,. \tag{5.7}$$

Thus, the RHS of (5.6) is in $L^1 \cap L^\infty$. By Schauder estimates (see, e.g., formula (6.23) in [GT83]), we obtain that

$$\|D^r(u^{(1)} - u^{(0)})\|_{L^\infty(\mathbf{K}^d_\omega)} \le \text{ const}\,.$$

Combining this with (5.6), (5.7) and the Calderón-Zygmund inequality (see, e.g., Theorem 9.13 of [GT83]), we obtain that the L^2-norm of $D^r(u^{(1)} - u^{(0)})$ in a ball of radius 1 centered at x_0 is bounded by the L^2 norm of $f(x, u^{(0)}(x)) - f(x, u^{(1)}(x))$ in a ball centered at x_0 and of radius 2.

Consequently, by Hölder's inequality and by the fact that $|u| \le 1$,

$$\begin{aligned}
&\left(\int_{B_1(x_0)} |D^r(u^{(0)} - u^{(1)})|\right)^2 \\
&= \left(\int_{B_1(x_0)} |D^r(u^{(0)} - u^{(1)})|^{1/2} \cdot |D^r(u^{(0)} - u^{(1)})|^{1/2}\right)^2 \\
&\le \left(\int_{B_1(x_0)} |D^r(u^{(0)} - u^{(1)})|^{2/3}\right)^{3/2} \cdot \left(\int_{B_1(x_0)} |D^r(u^{(0)} - u^{(1)})|^2\right)^{1/2} \\
&\le \text{ const}\, \|D^r(u^{(0)} - u^{(1)})\|_{L^2(B_1(x_0))} \\
&\le \text{ const}\, \|u^{(0)} - u^{(1)}\|_{L^2(B_2(x_0))} \\
&\le \text{ const}\, \|u^{(0)} - u^{(1)}\|_{L^1(B_2(x_0))}\,.
\end{aligned}$$

Hence, the desired result follows because we can estimate $\int_{\mathbf{K}^d_\omega} |D^r(u^{(1)} - u^{(0)})|$ by the sum of the integral over balls of radius 1 centered at $d^{-1/2}\mathbf{Z}^d$. Then, following the argument above, we estimate each one of those by the integral of $u^{(1)} - u^{(0)}$ over a ball of radius 2 centered at an integer point. Since the number of such balls that cover one point is finite, the sum of the integrals can be bounded by the integral over $\mathbf{K}^d_\omega$. ∎

5.3. **The renormalized energy.** Given that we have a local minimizer $u^{(0)}$ for the energy $\mathcal{E}$, we now seek another critical point. To do so, we will find it very convenient to define the following renormalized energy:

$$
\begin{aligned}
\tilde{\mathcal{E}}_\omega(v) = \int_{\mathbf{K}^d_\omega} \Big[\frac{1}{2}\Big(& a_{ij}(x)\,\partial_i(u^{(0)}+v)(x)\,\partial_j(u^{(0)}+v)(x) \\
& - a_{ij}(x)\,\partial_i u^{(0)}(x)\,\partial_j u^{(0)}(x)\Big) \\
& + F(x, u^{(0)}(x)+v(x)) - F(x, u^{(0)}(x))\Big]\,dx\,.
\end{aligned} \tag{5.8}
$$

Notice that in the case that $\omega \in \mathbf{Q}^d - \{0\}$, this renormalized energy differs from the "regular" energy $\mathcal{E}_\omega$ by a constant, namely the energy of the minimizer $u^{(0)}$. When ω is irrational and the domain of integration is unbounded, the regular energy of $u^{(0)}$ is infinite but the renormalized energy can be defined as a functional for all functions that are close – in some appropriate sense – to $u^{(0)}$. We emphasize that the Euler-Lagrange equations for (5.8) are just the same as the Euler-Lagrange equations for the standard variational principle. Hence, finding critical points of (5.8) will lead to critical points of the original problem.

Proposition 5.3. *If $u^{(0)}$ and $u^{(1)}$ are local minimizers, then*

$$
\tilde{\mathcal{E}}_\omega(u^{(1)} - u^{(0)}) = 0 \tag{5.9}
$$

Proof. We note that, because (5.5) and the fact that, by elliptic regularity theory, we can bound the derivatives in L^∞, we obtain that the integral in the definition of $\tilde{\mathcal{E}}_\omega(u^{(1)} - u^{(0)})$ converges.

If $g_n(x) = \Psi(|x| - n)$ where Ψ is a C^∞ function taking the value 1 in $(-\infty, 1]$ and the value 0 in $[2, \infty)$, we have that

$$
\tilde{\mathcal{E}}_\omega(u^{(1)} - u^{(0)}) = \lim_{n\to\infty} \tilde{\mathcal{E}}_\omega\Big(g_n \cdot (u^{(1)} - u^{(0)})\Big)
$$

by the Dominated Convergence Theorem. On the other hand, we have that $\tilde{\mathcal{E}}_\omega(g_n \cdot (u^{(1)} - u^{(0)})) \geq 0$ because of the fact that $u^{(0)}$ is a local minimizer and $g_n \cdot (u^{(1)} - u^{(0)})$ is a perturbation of $u^{(0)}$ with compact support. Hence, by the Dominated Convergence Theorem,

$$
\tilde{\mathcal{E}}_\omega(u^{(1)} - u^{(0)}) \geq 0\,. \tag{5.10}
$$

Analogously, by Dominated Convergence Theorem,

$$
0 = \tilde{\mathcal{E}}_\omega(0) = \lim_{n\to\infty} \tilde{\mathcal{E}}_\omega\Big((1-g_n)\cdot(u^{(1)} - u^{(0)})\Big)\,. \tag{5.11}
$$

Using that $u^{(1)}$ is a local minimizer and that $(1-g_n)\cdot(u^{(1)} - u^{(0)})$ is a compactly supported perturbation for $u^{(1)}$, we thence obtain that

$$
\tilde{\mathcal{E}}_\omega\Big((1-g_n)\cdot(u^{(1)} - u^{(0)})\Big) \geq \tilde{\mathcal{E}}_\omega(u^{(1)} - u^{(0)})\,. \tag{5.12}
$$

By collecting (5.11) and (5.12), we gather that

$$
\tilde{\mathcal{E}}_\omega(u^{(1)} - u^{(0)}) \leq 0\,. \tag{5.13}
$$

The proof of the desired result then follows from (5.10) and (5.13). ☐

A similar proof will give the following

Proposition 5.4. *Let $v \in W^{1,2}(\mathbf{K}^d_\omega) \cap L^1(\mathbf{K}^d_\omega)$, $|v| \leq 1$. Then $\tilde{\mathcal{E}}_\omega(v) \geq 0$. Moreover, $\tilde{\mathcal{E}}_\omega(v) = 0$ if and only if $u^{(0)} + v$ is a local minimizer.*

Proof. With the same g_n as in the previous proof, we have

$$\tilde{\mathcal{E}}_\omega(v) = \lim_{n\to\infty} \tilde{\mathcal{E}}_\omega(g_n \cdot v)$$

Using that $u^{(0)}$ is a local minimizer we have that $\tilde{\mathcal{E}}_\omega(g_n \cdot v) \geq 0$, which establishes the first part of the claim.

To show that if $u^{(0)} + v$ is a local minimizer $\tilde{\mathcal{E}}_\omega(v) = 0$, we proceed in a similar way. We have:

$$0 = \tilde{\mathcal{E}}_\omega(0) = \lim_{n\to\infty} \tilde{\mathcal{E}}_\omega\Big((1-g_n)\cdot v\Big).$$

But, if $u^{(0)} + v$ is a local minimizer, then

$$\tilde{\mathcal{E}}_\omega((1-g_n)\cdot v) \geq \tilde{\mathcal{E}}_\omega(v)$$

and so, by the first part of the claim, $\tilde{\mathcal{E}}_\omega(v) = 0$, as desired.

To show that if $\tilde{\mathcal{E}}_\omega(v) = 0$, then $u^{(0)} + v$ is a local minimizer, we note that for a function ϕ with support inside a ball B, we have:

$$\mathcal{E}_B(u^{(0)} + v + \phi) - \mathcal{E}_B(u^{(0)} + v) = \tilde{\mathcal{E}}_\omega(v+\phi) - \tilde{\mathcal{E}}_\omega(v).$$

Combining this with the assumption that $\tilde{\mathcal{E}}_\omega(v) = 0$ and using the first part of the claim, we deduce that

$$\mathcal{E}_B(u^{(0)} + v + \phi) - \mathcal{E}_B(u^{(0)} + v) = \tilde{\mathcal{E}}_\omega(v+\phi) \geq 0,$$

as desired. □

Even if we will not use it, we note that the above calculations show that if we define the renormalized energy by subtracting the energy density of any other local minimizer remaining at a finite distance from $u^{(0)}$ we obtain the same functional.

5.4. **The heat equation.** The heat flow is the gradient flow of our energy and it approaches critical points, as we will discuss in detail. A detailed study of the heat flow properties will be the tool to overcome the difficulties caused by the lack of compactness of the standard Ljusternik-Schnirelmann minimax methods (see, for instance, [LS34] or [Szu89]).

We define the second-order uniformly elliptic operator L by

$$Lu := \partial_i(a_{ij}\partial_j u). \tag{5.14}$$

and we consider the semilinear equation

$$\begin{aligned} &\partial_t V = LV - f(x, u^{(0)} + V) + f(x, u^{(0)}) \quad \forall (x,t) \in \mathbf{K}^d_\omega \times (0,+\infty) \\ &V(x,0) = v(x), \quad \forall x \in \mathbf{K}^d_\omega. \end{aligned} \tag{5.15}$$

We note that, formally, (5.15) is the gradient flow of the renormalized energy. That is, (5.15) can be written $V_t = -\nabla\tilde{\mathcal{E}}_\omega(V)$ where ∇ denotes the Euler-Lagrange derivative. Hence, one has formally that

$$\frac{d}{dt}\tilde{\mathcal{E}}_\omega(\Phi^t(v)) = -\|\nabla\tilde{\mathcal{E}}_\omega(\Phi^t(v))\|^2_{L^2(\mathbf{K}^d_\omega)}.$$

We will give precise meaning to the above formal calculations in Lemma 5.13.

Standard theory of semigroups (see Section A.2) gives that, if $V \in L^2(\mathbf{K}^d_\omega)$, there is a unique solution for $t \leq T$, we denote such solution by $\Phi^t(V)$.

Notice that, since $u^{(0)}$ and $u^{(1)}$ are critical for $\tilde{\mathcal{E}}_\omega$, we have that

$$\Phi^t(0) = 0 \qquad \text{and} \qquad \Phi^t(u^{(1)} - u^{(0)}) = u^{(1)} - u^{(0)}. \tag{5.16}$$

Our next goal is to extend the flow Φ^t, in principle defined only for short times $t \in [0,T]$, to all times $t \in [0,+\infty)$. This will be accomplished in Corollary 5.7. We first establish two consequences of the comparison principle Proposition A.6, which are of interest by themselves:

Corollary 5.5. *If the function $u^{(0)} + v$ is Birkhoff, then $u^{(0)} + \Phi^t(v)$ is Birkhoff.*

Corollary 5.6. *$\Phi^t(\mathcal{C}) \subseteq \mathcal{C}$, for any $t \in [0,T]$.*

We just note that both Corollary 5.5 and 5.6 follow because of the fact that the comparisons that are true for the initial data remain true for subsequent times.

Corollary 5.7. *For any $v \in \mathcal{C}$, $\Phi^t(v)$ is well defined for any $t \geq 0$.*

Proof. Let $w := \Phi^T(v)$. Then, by Corollary 5.6, we may define $z := \Phi^t(w)$, for any $t \in [0,T]$. By the uniqueness property, $z = \Phi^{t+T}(v)$, hence we have defined $\Phi^t(v)$ for any $v \in [0,2T]$. Repeating the argument, we define $\Phi^t(v)$ for any $t \geq 0$. □

Using Corollary 5.6 and 5.4, we have that

$$\|\Phi^t(v)\|_{L^2(\mathbf{K}^d_\omega)} \leq \text{const}\,, \tag{5.17}$$

for any $v \in \mathcal{C}$ and for any $t \geq 0$. We now give some further bounds on the Sobolev norms of the heat flow:

Lemma 5.8. *$\|\Phi^t(v)\|_{L^2(\mathbf{K}^d_\omega)} \leq C\|v\|_{L^2(\mathbf{K}^d_\omega)}$, for any $v \in \mathcal{C}$ and any $t \in [0,1]$, for a suitable constant $C \geq 1$.*

Proof. We recall the following well known triangle inequality (see, e.g., page 650 in [Eva98] and references therein): given a function $\phi(x,t)$, one has that

$$\Big\| \int_0^t \phi(\cdot,s)\,ds \Big\|_{W^{k,2}(\mathbf{K}^d_\omega)} \leq \int_0^t \|\phi(\cdot,s)\|_{W^{k,2}(\mathbf{K}^d_\omega)}\,ds\,, \tag{5.18}$$

for any $k \in \mathbf{N}$.

Let $g(t) := \|\Phi^t(v)\|_{L^2(\mathbf{K}^d_\omega)}$, we deduce from (A.4), (A.2) and (5.18) that

$$g(t) \leq \text{const}\left(\|v\|_{L^2(\mathbf{K}^d_\omega)} + \int_0^t g(s)\,ds\right)$$

and so the claim follows from the standard Gronwall inequality (see, e.g., [Tay96] on page 26). □

Our next result is a Sobolev bound uniform in t:

Lemma 5.9. *Fix $t_0 > 0$ $k \in \mathbf{N}$ and $v \in \mathcal{C}$. Then, there exists $C = C(t_0,k)$ so that $\|\Phi^t(v)\|_{W^{k,2}(\mathbf{K}^d_\omega)} \leq C$, for any $t \geq t_0$.*

Proof. We will prove only the case $k = 1$, the others follow by bootstrapping one further derivative via (A.2). Also, we will assume $t \in \mathbf{N}$ (the general case following by a time scaling). Observe that, in the light of (A.2), applied here with $r := 0$ and $s := 1$, we have that

$$\|\Phi^1(v)\|_{W^{1,2}(\mathbf{K}^d_\omega)} \leq \text{const}\left(\frac{\|v\|_{L^2(\mathbf{K}^d_\omega)}}{\sqrt{t_0}} + \int_0^1 \frac{\|\Phi^s(v)\|_{L^2(\mathbf{K}^d_\omega)}}{\sqrt{1-s}}\,ds\right).$$

Lemma 5.8 thus yields that

$$\|\Phi^1(v)\|_{W^{1,2}(\mathbf{K}^d_\omega)} \leq \hat{C}\|v\|_{L^2(\mathbf{K}^d_\omega)}\,,$$

for a suitable $\hat{C}$, which may depend on t_0. By a time shift, we thus have that

$$\|\Phi^j(v)\|_{W^{1,2}(\mathbf{K}^d_\omega)} \le \hat{C}\|\Phi^{j-1}(v)\|_{L^2(\mathbf{K}^d_\omega)}\,,$$

for any $j \in \mathbf{N}$, $j \ge 1$. By (5.17), we thus get that

$$\|\Phi^j(v)\|_{W^{1,2}(\mathbf{K}^d_\omega)} \le \tilde{C}\,,$$

for a suitable $\tilde{C}$, which may depend on t_0. □

Corollary 5.10. *Fix $t_0 > 0$. Then, for any $t \ge t_0$, $\Phi^t(\mathcal{C})$ is pre-compact in $W^{2,2}(\mathbf{K}^d_\omega)$.*

Proof. If $\omega \in \mathbf{Q}^d - \{0\}$, the proof is an easy consequence of Corollary 5.6, Lemma 5.9 and the decay estimates (4.2)–(4.3). In the general case, the proof becomes more technical, since there are more space directions to bound, and it is based on the following ideas:

- Lemma 5.9 provides local convergence;
- Lemma 5.1 provides $L^1(\mathbf{K}^d_\omega)$-convergence and so $L^2(\mathbf{K}^d_\omega)$-convergence;
- the parabolic regularity theory in (A.2) then gives $W^{2,2}(\mathbf{K}^d_\omega)$-convergence.

Let us now discuss the details. We fix $t_0 > 0$ and take

$$v_n \in \Phi^{t_0}(\mathcal{C})\,, \tag{5.19}$$

for any $n \in \mathbf{N}$. We would like to prove that, up to subsequences, v_n converges in the $W^{2,2}(\mathbf{K}^d_\omega)$-topology. Note that, thanks to Corollary 5.6, by possibly replacing t_0 with $\min\{t_0, 1\}$, we may and do assume that $t_0 \in (0, 1]$. By (5.19), there exists $\tilde{v}_n \in \mathcal{C}$ so that

$$v_n = \Phi^{t_0}(\tilde{v}_n)\,. \tag{5.20}$$

Let also

$$\bar{v}_n := \Phi^{t_0/2}(\tilde{v}_n)\,. \tag{5.21}$$

Making use of Lemma 5.9, one gets that there exists a sequence $n_j \longrightarrow +\infty$ and a suitable function $\bar{v}$ so that

$$\bar{v} = L^\infty_{\mathrm{loc}}(\mathbf{K}^d_\omega) - \lim_{j\to+\infty} \bar{v}_{n_j}\,. \tag{5.22}$$

We now show that

$$\bar{v} = L^1(\mathbf{K}^d_\omega) - \lim_{j\to+\infty} \bar{v}_{n_j}\,. \tag{5.23}$$

For this, fix $\varepsilon > 0$. By Lemma 5.1, we have that there exists $R_\varepsilon > 0$ so that

$$\int_{\mathbf{K}^d_\omega - B_R} (u^{(1)} - u^{(0)}) \le \frac{\varepsilon}{2}\,, \tag{5.24}$$

so long as $R \ge R_\varepsilon$. Moreover, since $\bar{v}_n \in \mathcal{C}$, we deduce from (5.22) that

$$0 \le \bar{v}_{n_j},\ \bar{v} \le u^{(1)} - u^{(0)}$$

and so, from (5.24), that

$$\begin{aligned}
& \int_{\mathbf{K}^d_\omega - B_R} |\bar v_{n_j} - \bar v| \\
\leq\ & \int_{\mathbf{K}^d_\omega - B_R} (|\bar v_{n_j}| + |\bar v|) \\
\leq\ & 2 \int_{\mathbf{K}^d_\omega - B_R} (u^{(1)} - u^{(0)}) \\
\leq\ & \varepsilon \,,
\end{aligned}$$

provided that $R \geq R_\varepsilon$.

The above inequality, together with (5.22), implies that

$$\begin{aligned}
& \lim_{j\to+\infty} \int_{\mathbf{K}^d_\omega} |\bar v_{n_j} - \bar v| \\
=\ & \lim_{j\to+\infty} \int_{B_R} |\bar v_{n_j} - \bar v| + \lim_{j\to+\infty} \int_{\mathbf{K}^d_\omega - B_R} |\bar v_{n_j} - \bar v| \\
\leq\ & \varepsilon \,.
\end{aligned}$$

Since ε may be taken as small as we wish, (5.23) follows.

Since functions in $\mathcal{C}$ have sup-norm bounded by 2, we gather from (5.23) that

$$\bar v = L^2(\mathbf{K}^d_\omega) - \lim_{j\to+\infty} \bar v_{n_j} \,. \tag{5.25}$$

We now make some observation on the parabolic regularity theory. Given $w,\ z \in \mathcal{C}$ and $t \in [0,1]$, by (5.18), (A.4) and (A.2), we have that

$$\begin{aligned}
& \|\Phi^t(w) - \Phi^t(z)\|_{L^2(\mathbf{K}^d_\omega)} \\
\leq\ & \|e^{tL}(w-z)\|_{L^2(\mathbf{K}^d_\omega)} \\
& + \int_0^t \left\| e^{(t-s)L} \Big[f\Big(\cdot, u^{(0)} + \Phi^s(w)\Big) - f\Big(\cdot, u^{(0)} + \Phi^s(z)\Big)\Big] \right\|_{L^2(\mathbf{K}^d_\omega)} ds \\
\leq\ & C_0 \left(\|w-z\|_{L^2(\mathbf{K}^d_\omega)} + \int_0^t \|\Phi^s(w) - \Phi^s(z)\|_{L^2(\mathbf{K}^d_\omega)}\, ds \right) \,,
\end{aligned}$$

for a suitable constant $C_0 > 0$. This and the standard Gronwall inequality (see, e.g., [Tay96] on page 26) give that

$$\|\Phi^t(w) - \Phi^t(z)\|_{L^2(\mathbf{K}^d_\omega)} \leq C_1 \|w-z\|_{L^2(\mathbf{K}^d_\omega)} \,, \tag{5.26}$$

for any $w,\ z \in \mathcal{C}$ and any $t \in [0,1]$, for a suitable constant $C_1 > 0$.

Also, given any $\hat{w},\ \hat{z} \in \Phi^{t_0/4}(\mathcal{C})$, exploiting (5.18), (A.4), (A.2) and Lemma 5.9, one gathers that

$$\begin{aligned}
& \|\Phi^t(\hat{w}) - \Phi^t(\hat{z})\|_{W^{2,2}(\mathbf{K}^d_\omega)} \\
\le\ & \|e^{tL}(\hat{w}-\hat{z})\|_{W^{2,2}(\mathbf{K}^d_\omega)} \\
& + \int_0^t \Big\| e^{(t-s)L}\Big[f\Big(\cdot, u^{(0)} + \Phi^s(\hat{w})\Big) - f\Big(\cdot, u^{(0)} + \Phi^s(\hat{z})\Big)\Big]\Big\|_{W^{2,2}(\mathbf{K}^d_\omega)} ds \\
\le\ & C_2 \Big(\frac{1}{t} \|\hat{w}-\hat{z}\|_{L^2(\mathbf{K}^d_\omega)} \\
& + \int_0^t \Big\| f\Big(\cdot, u^{(0)} + \Phi^s(\hat{w})\Big) - f\Big(\cdot, u^{(0)} + \Phi^s(\hat{z})\Big)\Big\|_{W^{2,2}(\mathbf{K}^d_\omega)} ds\Big) \qquad (5.27) \\
\le\ & C_3 \Big(\frac{1}{t} \|\hat{w}-\hat{z}\|_{L^2(\mathbf{K}^d_\omega)} + t \Big),
\end{aligned}$$

for any $t \in (0,1]$, where the C_i's here above are positive constants possibly depending on t_0.

We now apply these estimates in order to prove that v_{n_i} converges in $W^{2,2}(\mathbf{K}^d_\omega)$.

For this, fix $\varepsilon > 0$ and let

$$t_\varepsilon := \min \left\{ \frac{t_0}{4}\,,\ \frac{\varepsilon}{2C_3} \right\} . \tag{5.28}$$

Let also

$$\tau_\varepsilon := \frac{t_0}{2} - t_\varepsilon\,. \tag{5.29}$$

It follows from (5.28) that

$$\tau_\varepsilon \in \left[\frac{t_0}{4},\ 1 \right]$$

and so

$$\Phi^{\tau_\varepsilon}(\bar{v}_n) \in \Phi^{t_0/4}(\mathcal{C})\,,$$

for any $n \in \mathbf{N}$. Consequently, making use of (5.27), we get that

$$\begin{aligned}
& \|\Phi^t(\Phi^{\tau_\varepsilon}(\bar{v}_{n_i})) - \Phi^t(\Phi^{\tau_\varepsilon}(\bar{v}_{n_j}))\|_{W^{2,2}(\mathbf{K}^d_\omega)} \qquad (5.30) \\
\le\ & C_3 \Big(\frac{1}{t} \|\Phi^{\tau_\varepsilon}(\bar{v}_{n_i}) - \Phi^{\tau_\varepsilon}(\bar{v}_{n_j})\|_{L^2(\mathbf{K}^d_\omega)} + t \Big),
\end{aligned}$$

for any $t \in (0,1]$.

Also, due to (5.25), there exists $n_\varepsilon \in \mathbf{N}$ so that

$$\|\bar{v}_{n_i} - \bar{v}_{n_j}\|_{L^2(\mathbf{K}^d_\omega)} \le \frac{t_\varepsilon \varepsilon}{2C_1C_3}$$

so long as $n_i,\ n_j \ge n_\varepsilon$. Therefore, by (5.26),

$$\|\Phi^t(\bar{v}_{n_i}) - \Phi^t(\bar{v}_{n_j})\|_{L^2(\mathbf{K}^d_\omega)} \le \frac{t_\varepsilon \varepsilon}{2C_3}\,, \tag{5.31}$$

if $n_i,\ n_j \ge n_\varepsilon$ and $t \in [0,1]$.

Thence, using (5.20), (5.21), (5.29), (5.30) and (5.31), we deduce that

$$\begin{aligned} & \|v_{n_i} - v_{n_j}\|_{W^{2,2}(\mathbf{K}^d_\omega)} \\ = \; & \|\Phi^{t_0/2}(\bar v_{n_i}) - \Phi^{t_0/2}(\bar v_{n_j})\|_{W^{2,2}(\mathbf{K}^d_\omega)} \\ = \; & \|\Phi^{t_\varepsilon}(\Phi^{\tau_\varepsilon}(\bar v_{n_i})) - \Phi^{t_\varepsilon}(\Phi^{\tau_\varepsilon}(\bar v_{n_j}))\|_{W^{2,2}(\mathbf{K}^d_\omega)} \\ \leq \; & C_3\Big(\frac{1}{t_\varepsilon}\|\Phi^{\tau_\varepsilon}(\bar v_{n_i}) - \Phi^{\tau_\varepsilon}(\bar v_{n_j})\|_{L^2(\mathbf{K}^d_\omega)} + t_\varepsilon\Big) \\ \leq \; & C_3\Big(\frac{\varepsilon}{2C_3} + t_\varepsilon\Big), \end{aligned}$$

if $n_i,\ n_j \geq n_\varepsilon$. Therefore, from (5.28),

$$\|v_{n_i} - v_{n_j}\|_{W^{2,2}(\mathbf{K}^d_\omega)} \leq \varepsilon$$

provided that $n_i,\ n_j \geq n_\varepsilon$. That is, v_{n_i} is a Cauchy sequence (and thus converges) in $W^{2,2}(\mathbf{K}^d_\omega)$. □

As an immediate corollary of Lemma 5.9, we obtain:

Corollary 5.11. *For any $t \geq t_0 > 0$ and any $v \in \mathcal{C}$, we have that*

$$0 \leq \tilde{\mathcal{E}}_\omega(\Phi^t(v)) \leq C \tag{5.32}$$

We now show the continuity from initial data of the heat flow, which will will be needed in the sequel.

Proposition 5.12. *Fixed $T > 0$ and $\epsilon > 0$, there exists $\delta_0(T,\epsilon) > 0$ so that if $v,\, w \in \mathcal{C}$ and $\|v - w\|_{W^{2,2}(\mathbf{K}^d_\omega)} \leq \delta_0(T,\epsilon)$, then*

$$\|\Phi^t(v) - \Phi^t(w)\|_{W^{2,2}(\mathbf{K}^d_\omega)} \leq \epsilon,$$

for any $t \in [0,T]$.

Proof. First, we deal with the case $t \in [0,1]$. Let $U(x,t) := \Phi^t(v) - \Phi^t(w)$. Then, by (A.4),

$$\begin{aligned} U(x,t) \; = \; & e^{tL}(v-w) \\ & -\int_0^t e^{(t-s)L}\Big[f\Big(x, u^{(0)}(x) + \Phi^t(w) + U(x,s)\Big) \\ & -f\Big(x, u^{(0)}(x) + \Phi^t(w)\Big)\Big]\,ds. \end{aligned} \tag{5.33}$$

Define also

$$g(t) := \|U(\cdot,t)\|_{W^{2,2}(\mathbf{K}^d_\omega)}.$$

Then, by (A.2), (5.18), (5.33) and Lemma 5.9, we have that

$$\begin{aligned} g(t) \leq \mathrm{const}\,\Big(& \|v-w\|_{W^{2,2}(\mathbf{K}^d_\omega)} \\ & + \int_0^t \|f(\cdot, u^{(0)} + \Phi^t(w) + U(\cdot,s)) \\ & - f(\cdot, u^{(0)} + \Phi^t(w))\|_{W^{2,2}(\mathbf{K}^d_\omega)}\,ds\Big) \leq \\ \leq \mathrm{const}\,\Big(& \|v-w\|_{W^{2,2}(\mathbf{K}^d_\omega)} + \int_0^t g(s)\,ds\Big), \end{aligned} \tag{5.34}$$

for any $t \in [0,1]$. Therefore, exploiting the standard Gronwall inequality (see, e.g., [Tay96] on page 26),

$$g(t) \le \text{const}\, \|v-w\|_{W^{2,2}(\mathbf{K}^d_\omega)}\,, \tag{5.35}$$

for any $t \in [0,1]$, which implies the desired claim, for $T \le 1$. If, on the other hand, $T > 1$, by iterating (5.35), we deduce that

$$\|\Phi^t(v)-\Phi^t(w)\|_{W^{2,2}(\mathbf{K}^d_\omega)} \le \text{const}\, e^{\text{const}\, T}\|v-w\|_{W^{2,2}(\mathbf{K}^d_\omega)}\,,$$

for any $t \in [0,T]$, whence the claim. □

5.5. **Convergence of the heat flow.** We now investigate some convergence properties for heat flow on (the $W^{2,2}$-closure of) $\mathcal{C}$. From Corollary 5.10, we already know that, up to subsequences, the heat flow converges in the $W^{2,2}$-closure of $\mathcal{C}$; we will show in Proposition 5.15 here below that the *all* the limits that we may obtain by taking subsequences are critical points of $\mathcal{E}_\omega$.

Fixed $t_0 > 0$ as in Lemma 5.9, given $v \in \mathcal{C}$, we define, for any $t > t_0$

$$E_v(t) := \tilde{\mathcal{E}}_\omega(\Phi^t(v))\,.$$

Notice that, by Proposition 5.4,

$$E_v(t) \ge 0\,. \tag{5.36}$$

Now, we turn to estimate the derivatives of E_v.

Lemma 5.13. *With the notations above, we have that $E_v(t)$ is twice differentiable and, moreover:*

$$\frac{dE_v}{dt}(t) = -\left\|\partial_i\left(a_{ij}\partial_j\left(\Phi^t(v)\right)\right) - f\left(\cdot, u^{(0)}+\Phi^t(v)\right) + f\left(\cdot, u^{(0)}\right)\right\|^2_{L^2(\mathbf{K}^d_\omega)}\,. \tag{5.37}$$

There exists a positive universal constant C so that

$$\left|\frac{d^2E_v}{dt^2}(t)\right| \le C\,, \qquad \forall v \in \mathcal{C}, \quad \forall t > 0\,. \tag{5.38}$$

An immediate consequence of (5.37) is that

$$\frac{dE_v}{dt}(t) \le 0\,, \tag{5.39}$$

which implies that

$$\mathcal{E}_\omega(u^{(0)}+\Phi^T(v)) \le \mathcal{E}_\omega(u^{(0)}+\Phi^t(v))\,, \tag{5.40}$$

for any $T \ge t \ge 0$.

Proof. The fact that the energy is twice differentiable follows from the theory of semigroups. It suffices to study equation (A.4) to obtain that the flow gives a twice differentiable curve in $W^{1,2}(\mathbf{K}^d_\omega)$. Since the energy is differentiable as a function on $W^{1,2}(\mathbf{K}^d_\omega)$, we obtain the desired result (see Lemma A.4). We also note that the integration by parts needed to write (5.37) is bona-fide since

$$D_t\Phi^t(v) = \partial_i(a_{ij}\partial_j\Phi^t(v)) - f(\cdot, u^{(0)}+\Phi^t(v)) + f(\cdot, u^{(0)})\,,$$

which belongs to $W^{k,2}(\mathbf{K}^d_\omega)$ by Lemma 5.9. □

The next result shows that $\Phi^t(v)$ gets closer and closer to being a solution, as t increases.

Corollary 5.14.

$$\lim_{t\longrightarrow+\infty}\left\|\partial_i\left(a_{ij}\partial_j\left(\Phi^t(v)\right)\right)-f\left(\cdot,u^{(0)}+\Phi^t(v)\right)+f\left(\cdot,u^{(0)}\right)\right\|_{L^2(\mathbf{K}^d_\omega)}=0\,.$$

Proof. Given the previous results (5.37) and (5.38) this is just an elementary real analysis argument.

By (5.37)) it is enough to show that

$$\lim_{t\longrightarrow+\infty}\frac{dE_v}{dt}(t)=0\,.$$

Assume, by contradiction, that this is not true. Then, by (5.39), there exist $\delta>0$ and a sequence $t_k\longrightarrow+\infty$ so that

$$\frac{dE_v}{dt}(t_k)\le-\delta\,.$$

By (5.36) and (5.39), we can also set

$$\ell:=\lim_{t\longrightarrow+\infty}E_v(t)\in[0,+\infty)\,.$$

Let C be as in Lemma 5.13. Take $t:=\delta/(2C)$ and $\varepsilon:=\delta^2/(5C)$. Then, by taking k large enough, and by using again (5.39), we gather that

$$\ell\le E_v(t_k+t)=E_v(t_k)+\frac{dE_v}{dt}(t_k)\,t+\int_{t_k}^{t_k+t}\left(\frac{dE_v}{dt}(\tau)-\frac{dE_v}{dt}(t_k)\right)\,d\tau\le$$

$$\le\ell+\varepsilon-\delta t+Ct^2<\ell\,,$$

which is, of course, a contradiction. □

We now show that *all* the $W^{2,2}$-limit points of the heat flow are critical points for the phase transition equation; more precisely, we have:

Proposition 5.15. *Let $v\in\mathcal{C}$. Assume that, for $t_n\longrightarrow+\infty$,*

$$\Phi^{t_n}(v)\longrightarrow v^\star\ \text{in}\ W^{2,2}(\mathbf{K}^d_\omega)\,.$$

Then,

$$\partial_i\left(a_{ij}\partial_j v^\star\right)-f\left(x,u^{(0)}+v^\star\right)+f\left(x,u^{(0)}\right)\;=\;0\,.\tag{5.41}$$

Proof. Notice that, since v and $v^\star$ are in the $W^{2,2}$-closure of $\mathcal{C}$, we have that

$$|u^{(0)}+v|\,,\ |u^{(0)}+v^\star|\le1\,.$$

Hence, if $C \geq 1$ is large enough, we deduce from Corollary 5.14 that

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \left\| \partial_i \left(a_{ij}\partial_j \left(\Phi^{t_n}(v)\right)\right) - f\left(\cdot, u^{(0)} + \Phi^{t_n}(v)\right) + f\left(\cdot, u^{(0)}\right) \right\|_{L^2(\mathbf{K}^d_\omega)} \\
&\geq \lim_{n\to+\infty} \left\| \partial_i \left(a_{ij}\partial_j v^\star\right) - f\left(\cdot, u^{(0)} + v^\star\right) + f\left(\cdot, u^{(0)}\right) \right\|_{L^2(\mathbf{K}^d_\omega)} \\
&\quad - \left\| \partial_i \left(a_{ij}\partial_j \left(\Phi^{t_n}(v) - v^\star\right)\right) \right\|_{L^2(\mathbf{K}^d_\omega)} \\
&\quad - \left\| f\left(\cdot, u^{(0)} + \Phi^{t_n}(v)\right) - f\left(\cdot, u^{(0)} + v^\star\right) \right\|_{L^2(\mathbf{K}^d_\omega)} \\
&\geq \lim_{n\to+\infty} \left\| \partial_i \left(a_{ij}\partial_j v^\star\right) - f\left(\cdot, u^{(0)} + v^\star\right) + f\left(\cdot, u^{(0)}\right) \right\|_{L^2(\mathbf{K}^d_\omega)} \\
&\quad -(1+C) \left\| \Phi^{t_n}(v) - v^\star \right\|_{W^{2,2}(\mathbf{K}^d_\omega)} \\
&= \left\| \partial_i \left(a_{ij}\partial_j v^\star\right) - f\left(\cdot, u^{(0)} + v^\star\right) + f\left(\cdot, u^{(0)}\right) \right\|_{L^2(\mathbf{K}^d_\omega)},
\end{aligned}$$

which proves the desired claim. □

5.6. **Existence of another critical point.** We have developed tools about the convergence of the heat flow and its relation with the renormalized energy. In this section, we will put them together to show the existence of another critical point in (the $W^{2,2}$-closure of) $\mathcal{C}$, under a suitable assumption on the basin of attraction of the heat flow. This assumption will then be established of in § 5.7.

Theorem 5.16. *Let $v^{(s)}$ be as defined in (5.1). For $i = 0, 1$ define*[1]

$$\mathbf{B}_i := \{s \in [0,1] \ s.t. \ W^{2,2}(\mathbf{K}^d_\omega) - \lim_{t\to+\infty} \Phi^t(v^{(s)}) = v^{(i)}\}. \tag{5.42}$$

Let us assume that $[0,1] \setminus (\mathbf{B}_0 \cup \mathbf{B}_1) \neq \emptyset$. Then, there exists $u^\star \in C^2(\mathbf{K}^d_\omega)$ such that:

- *$\partial_i(a_{ij}\partial_j u^\star) = f(x, u^\star)$;*
- *$u^\star$ enjoys the Birkhoff property;*
- *$u^{(0)}(x) < u^\star(x) < u^{(1)}(x)$.*

Proof. Notice that, if we prove the existence of $v^\star \in \mathcal{C} - \{0, u^{(1)} - u^{(0)}\}$ satisfying (5.41), then we may define $u^\star := v^\star + u^{(0)}$ and we are done (recall again Corollary A.3). For finding such a $v^\star$, thanks to Proposition 5.15, it is enough to find $\bar v \in \mathcal{C}$ and some sequence $T_n \longrightarrow +\infty$, such that

$$W^{2,2}(\mathbf{K}^d_\omega) - \lim_{n\to+\infty} \Phi^{T_n}(\bar v) \notin \{0, u^{(1)} - u^{(0)}\}. \tag{5.43}$$

Therefore, we will now show the existence of a function $\bar v$ for which (5.43) holds.

Let $\bar s \in [0,1] - (\mathbf{B}_0 \cup \mathbf{B}_1)$ and $\bar v := v^{(\bar s)}$. Then, from Corollary 5.10, there exist $V \in \mathcal{C}$ and a sequence $t_n \longrightarrow +\infty$ such that

$$W^{2,2}(\mathbf{K}^d_\omega) - \lim_{n\to+\infty} \Phi^{t_n}(\bar v) = V.$$

If $V \neq 0$, $u^{(1)} - u^{(0)}$, we are done. On the other hand, if, say $V = u^{(1)} - u^{(0)}$, since $\bar s \notin \mathbf{B}_1$, there must exist another sequence $\tau_n \longrightarrow +\infty$ and a suitable $\eta > 0$ such

[1] In what follows, we will use the notation $W^{2,2} - \lim$ to denote the limit in the $W^{2,2}$ norm. This is to avoid confusion with other types of convergence.

that

$$\left\|\Phi^{\tau_n}(\bar{v}) - (u^{(1)} - u^{(0)})\right\|_{W^{2,2}(\mathbf{K}_d^\omega)} \geq \eta \,. \tag{5.44}$$

In the light of Corollary 5.10, possibly extracting a subsequence, we may assume that

$$W^{2,2}(\mathbf{K}_\omega^d) - \lim_{n\to+\infty} \Phi^{\tau_n}(\bar{v}) = W \,,$$

for some $W \in \mathcal{C}$. Furthermore, from (5.44), $W \neq u^{(1)} - u^{(0)}$. If also $W \neq 0$, the proof is complete. If, on the other hand, $W = 0$, from the construction above we have the existence of two sequences, t_n and τ_n, for which

$$\begin{aligned} W^{2,2}(\mathbf{K}_\omega^d) - \lim_{n\to+\infty} \Phi^{t_n}(\bar{v}) &= u^{(1)} - u^{(0)} \qquad \text{and} \\ W^{2,2}(\mathbf{K}_\omega^d) - \lim_{n\to+\infty} \Phi^{\tau_n}(\bar{v}) &= 0 \,. \end{aligned}$$

Possibly taking subsequences, we may assume also that $t_n < \tau_n$. Let us define

$$\begin{aligned} \kappa &:= \|u^{(1)} - u^{(0)}\|_{W^{2,2}(\mathbf{K}_d^\omega)} \qquad \text{and} \\ g(t) &:= \left\|\Phi^t(\bar{v})\right\|_{W^{2,2}(\mathbf{K}_d^\omega)} \,. \end{aligned}$$

Clearly from its definition, $\kappa > 0$. For sufficiently large n, we have $g(t_n) \geq \kappa/2$ and $g(\tau_n) \leq \kappa/8$. Using the continuity properties of the heat flow (A.4) and (A.2), one sees that g is a continuous function. Therefore, there exists $T_n \in [t_n, \tau_n]$ for which $g(T_n) = \kappa/4$, that is

$$\left\|\Phi^{T_n}(\bar{v})\right\|_{W^{2,2}(\mathbf{K}_d^\omega)} = \frac{1}{4}\,\|u^{(1)} - u^{(0)}\|_{W^{2,2}(\mathbf{K}_d^\omega)} \,. \tag{5.45}$$

By Corollary 5.10, up to subsequence, we may assume that there exists $v^\sharp \in \mathcal{C}$ so that

$$v^\sharp = W^{2,2}(\mathbf{K}_\omega^d) - \lim_{n\to+\infty} \Phi^{T_n}(\bar{v}) \,.$$

From (5.45), we have that $v^\sharp \neq 0$, $u^{(1)} - u^{(0)}$, hence (5.43) follows, thus concluding the proof of the desired result. □

Since two disjoint open sets cannot cover an interval, we obtain from Theorem 5.16 that:

Corollary 5.17. *If* $\mathbf{B}_0$ *and* $\mathbf{B}_1$ *are open in the standard topology of* $[0,1]$*, then, the results in Theorem 5.16 hold true.*

Our next goal will be to show that, indeed, $\mathbf{B}_0$ and $\mathbf{B}_1$ are open.

5.7. **The basin of attraction of the fixed points of the heat flow.** We now investigate the basin of attraction of the heat flow, with the aim of proving that the hypothesis of Corollary 5.17 holds true. The idea of gaining compactness from the assumption that no critical points (but trivial ones) exist has been recently used by several authors (see, e.g., [CZES90], and in the context of Aubry-Mather theory [KdlLR97] and [CdlL98]).

For any $r > 0$, we introduce the "energy ball"

$$\mathcal{B}_r := \{v \in \mathcal{C} \text{ s.t. } \tilde{\mathcal{E}}_\omega(v) \leq r\} \,.$$

Also, given a norm $\|\cdot\|_X$ on some space X, we denote by B^X the standard (closed) ball. That is, we set

$$B_r^X(x_0) := \{x \in X \text{ s.t. } \|x - x_0\|_X \leq r\} \,.$$

We also set $B_r^X := B_r^X(0)$. We now point out an inclusion of balls, which is needed in what follows:

Lemma 5.18. *There exists a positive universal constant c for which*

$$\mathcal{C} \cap B_{c\sqrt{r}}^{W^{1,2}(\mathbf{K}_\omega^d)} \subseteq \mathcal{B}_r \,,$$

for any $r > 0$.

Proof. Take $v \in \mathcal{C}$ and let $C \geq 1$ be suitably large. Then,

$$\begin{aligned} \tilde{\mathcal{E}}_\omega(v) = & \\ = & \int_{\mathbf{K}_\omega^d} \frac{1}{2} a_{i,j} \partial_i v \partial_j v + F(x, u^{(0)} + v) - F(x, u^{(0)}) - \partial_\mu F(x, u^{(0)})\, v \, dx \leq \\ \leq & (\Lambda + C) \|v\|^2_{W^{1,2}(\mathbf{K}_\omega^d)} \,, \end{aligned}$$

proving that the desired result holds. □

Let us now study the connected components of the energy ball. Here and in the sequel, "connected components" is short for "*path-connected components in the $W^{2,2}(\mathbf{K}_\omega^d)$–topology*".

Lemma 5.19. *Let us assume that 0 and $u^{(1)} - u^{(0)}$ are the only functions v contained in $\mathcal{C}$ for which $\tilde{\mathcal{E}}_\omega(v) = 0$.*

Then, there exists $r_0 > 0$ so that, for any $r \in (0, r_0]$, 0 and $u^{(1)} - u^{(0)}$ are in different connected components of $\mathcal{B}_r$.

Proof. The argument is by contradiction.

Suppose that, contrary to the conclusions of the Lemma, for any $h \in \mathbf{N}$, there exists a path $\gamma_h \in C([0,1], W^{2,2}(\mathbf{K}_\omega^d))$ so that

$$\begin{aligned} &\gamma_h(0) = 0 \,, \\ &\gamma_h(1) = u^{(1)} - u^{(0)} \qquad \text{and} \\ &\tilde{\mathcal{E}}_\omega(\gamma_h(\sigma)) \leq \frac{1}{h} \,, \qquad \forall \sigma \in [0,1] \,. \end{aligned} \tag{5.46}$$

Let $\Gamma_h(\sigma) := \Phi^1(\gamma_h(\sigma))$, for any $\sigma \in [0,1]$. Notice that, by (5.46) and (5.40), we have that

$$\tilde{\mathcal{E}}_\omega(\Gamma_h(\sigma)) \leq \tilde{\mathcal{E}}_\omega(\gamma_h(\sigma)) \leq \frac{1}{h} \,. \tag{5.47}$$

Exploiting (5.46) and (5.16), one sees that

$$\Gamma_h(0) = 0 \text{ and } \Gamma_h(1) = u^{(1)} - u^{(0)} \,.$$

Also, since $\gamma_h \in C([0,1], W^{2,2}(\mathbf{K}_\omega^d))$, Proposition 5.12 implies that

$$\Gamma_h \in C([0,1], W^{2,2}(\mathbf{K}_\omega^d)) \,. \tag{5.48}$$

Let

$$\begin{aligned} \kappa &:= \|u^{(1)} - u^{(0)}\|_{W^{2,2}(\mathbf{K}_\omega^d)} \qquad \text{and} \\ g(\sigma) &:= \|\Gamma_h(\sigma)\|_{W^{2,2}(\mathbf{K}_\omega^d)} \,. \end{aligned}$$

We have that $\kappa > 0$, $g(0) = 0$ and $g(1) = \kappa$. Also, from (5.48), it follows that $g \in C([0,1], \mathbf{R})$, therefore there exists $\sigma_h \in [0,1]$, so that $g(\sigma_h) = \kappa/2$, that is

$$\|\Gamma_h(\sigma_h)\|_{W^{2,2}(\mathbf{K}_\omega^d)} = \frac{\|u^{(1)} - u^{(0)}\|_{W^{2,2}(\mathbf{K}_\omega^d)}}{2} \,. \tag{5.49}$$

Moreover, $\Gamma_h(\sigma_h) \in \Phi^1(\mathcal{C})$, which is compact in $W^{2,2}(\mathbf{K}^d_\omega)$, due to Corollary 5.10. Therefore, possibly taking subsequences, we may assume that there exists $\hat{v} \in \mathcal{C}$ so that

$$\hat{v} = W^{2,2}(\mathbf{K}^d_\omega) - \lim_{h\to+\infty} \Gamma_h(\sigma_h)\,.$$

Recalling (5.49), we get that $\hat{v} \neq 0$, $u^{(1)} - u^{(0)}$. Therefore, by hypothesis,

$$\mathcal{E}_\omega(\hat{v} + u^{(0)}) > \mathcal{E}_\omega(u^{(0)})\,. \tag{5.50}$$

On the other hand, passing to the limit in (5.47), we gather that

$$\mathcal{E}_\omega(\hat{v} + u^{(0)}) \le \mathcal{E}_\omega(u^{(0)})\,,$$

which contradicts (5.50). □

We can now show that, if there are no critical points of the phase transition equation with energy close to the one of the minima (except, of course, the minima themselves), then there is a $W^{2,2}$-ball around the minima which is contained in their basin of attraction under the heat equation. More precisely, we have that:

Proposition 5.20. *Let us assume that* 0 *and* $u^{(1)} - u^{(0)}$ *are the only functions* v *contained in* $\mathcal{C}$ *for which* $\mathcal{E}_\omega(u^{(0)} + v) = \mathcal{E}_\omega(u^{(0)})$ *holds. Let* r_0 *be as in Lemma 5.19 and* $r \in (0, r_0]$. *Let us assume that: if* $V \in \mathcal{C} \cap \mathcal{B}_r$ *satisfies*

$$\partial_i\left(a_{ij}\partial_j V\right) - f\left(x, u^{(0)} + V\right) + f\left(x, u^{(0)}\right) \;=\; 0\,,$$

then $V \in \{0, u^{(1)} - u^{(0)}\}$. *Then, there exists a universal constant* c *such that the following holds:*

- *if* $v \in \mathcal{C} \cap B^{W^{2,2}(\mathbf{K}^d_\omega)}_{c\sqrt{r}}$, *then* $W^{2,2}(\mathbf{K}^d_\omega) - \lim_{t\to+\infty} \Phi^t(v) = 0$*;*
- *if* $v \in \mathcal{C} \cap B^{W^{2,2}(\mathbf{K}^d_\omega)}_{c\sqrt{r}}(u^{(1)} - u^{(0)})$, *then* $W^{2,2}(\mathbf{K}^d_\omega) - \lim_{t\to+\infty} \Phi^t(v) = u^{(1)} - u^{(0)}$.

Proof. We will focus on the proof of the first claim, the proof of the second one being identical Recalling Lemma 5.19, we denote by $\mathcal{B}^\star_r$ be the connected component of $\mathcal{B}_r$ containing 0 and not containing $u^{(1)} - u^{(0)}$. If c is suitably small, Lemma 5.18 implies that

$$\mathcal{C} \cap B^{W^{2,2}(\mathbf{K}^d_\omega)}_{c\sqrt{r}} \subseteq \mathcal{B}_r\,.$$

Also, notice that $0 \in \mathcal{C} \cap B^{W^{2,2}(\mathbf{K}^d_\omega)}_{c\sqrt{r}}$, which is a convex (and hence connected) set in $W^{2,2}(\mathbf{K}^d_\omega)$, therefore, from the above inclusion, we get that

$$\mathcal{C} \cap B^{W^{2,2}(\mathbf{K}^d_\omega)}_{c\sqrt{r}} \subseteq \mathcal{B}^\star_r\,.$$

Take now $v \in \mathcal{C} \cap B^{W^{2,2}(\mathbf{K}^d_\omega)}_{c\sqrt{r}}$. By Corollary 5.10 and Proposition 5.15, we have that there exist a sequence $t_n \longrightarrow +\infty$ and $v^* \in \mathcal{C}$, such that

$$W^{2,2}(\mathbf{K}^d_\omega) - \lim_{n\to+\infty} \Phi^{t_n}(v) \;=\; v^* \qquad \text{and} \tag{5.51}$$

$$\partial_i\left(a_{ij}\partial_j v^*\right) - f\left(x, u^{(0)} + v^*\right) + f\left(x, u^{(0)}\right) \;=\; 0\,. \tag{5.52}$$

Notice also that, from Corollary 5.6 and (5.40),

$$\Phi^t(\mathcal{B}^\star_r) \subseteq \mathcal{B}^\star_r\,,$$

thus $v^* \in \mathcal{C} \cap \mathcal{B}_r^\star$. Consequently, from (5.52), our hypotheses imply that

$$v^* \in \{0, u^{(1)} - u^{(0)}\}.$$

Since $u^{(1)} - u^{(0)} \notin \mathcal{B}_r^\star$ by construction, we deduce that $v^* = 0$, that is

$$W^{2,2}(\mathbf{K}_\omega^d) - \lim_{n \to +\infty} \Phi^{t_n}(v) = 0.$$

We now show that, actually,

$$W^{2,2}(\mathbf{K}_\omega^d) - \lim_{t \to +\infty} \Phi^{t}(v) = 0, \tag{5.53}$$

which will indeed prove the first claim. If (5.53) were false, there would exist some $\eta > 0$ and another diverging sequence, say T_n, for which

$$\|\Phi^{T_n}(v)\|_{W^{2,2}(\mathbf{K}_\omega^d)} \geq \eta. \tag{5.54}$$

But now, as before, we can extract a subsequence T_{n_k} for which $\Phi^{T_{n_k}}(v)$ converges and, exactly as done here above, we can show that it must converge to 0, contradicting (5.54). This proves (5.53) and completes the proof of the (first) claim. □

We can now show that the hypotheses of Corollary 5.17 are fulfilled in the case that there are no critical points of the phase transition equation with energy close to the one of the minima:

Proposition 5.21. *Let us assume that* 0 *and* $u^{(1)} - u^{(0)}$ *are the only functions* v *contained in* $\mathcal{C}$ *for which* $\mathcal{E}_\omega(u^{(0)} + v) = \mathcal{E}_\omega(u^{(0)})$ *holds. Let* $r > 0$ *and let us assume that: if* $V \in \mathcal{B}_r$ *satisfies*

$$\partial_i (a_{ij} \partial_j V) - f\left(x, u^{(0)} + V\right) + f\left(x, u^{(0)}\right) = 0,$$

then $V \in \{0, u^{(1)} - u^{(0)}\}$. *Let* $\mathbf{B}_0$ *and* $\mathbf{B}_1$ *be as defined in* (5.42). *Then,* $\mathbf{B}_0$ *and* $\mathbf{B}_1$ *are open in the standard topology of* $[0,1]$.

Proof. We will only deal with $\mathbf{B}_0$, the case of $\mathbf{B}_1$ being analogous. Let $\bar{s} \in [0,1]$ be such that

$$W^{2,2}(\mathbf{K}_\omega^d) - \lim_{t \to +\infty} \Phi^{t}(v^{(\bar{s})}) = 0,$$

and let $\bar{t}$ be so that

$$\Phi^{\bar{t}}(v^{(\bar{s})}) \in B_{c\sqrt{r}/2}^{W^{2,2}(\mathbf{K}_\omega^d)},$$

with c as in Lemma 5.18. Then, from Proposition 5.12, and Corollary 5.6, we have that

$$\Phi^{\bar{t}}(v^{(s)}) \in \mathcal{C} \cap B_{c\sqrt{r}}^{W^{2,2}(\mathbf{K}_\omega^d)},$$

for any $s \in (\bar{s} - \epsilon, \bar{s} + \epsilon)$, for a suitable $\epsilon > 0$. Then, in the light of Proposition 5.20,

$$W^{2,2}(\mathbf{K}_\omega^d) - \lim_{t \to +\infty} \Phi^{t}(v^{(\bar{s})}) = 0,$$

for any $s \in (\bar{s} - \epsilon, \bar{s} + \epsilon)$, hence $\mathbf{B}_0$ is open. □

In particular, from Corollary A.3, Proposition 5.21, Corollary 5.17 and Theorem 5.16, we gather:

Corollary 5.22. *Let us assume that* 0 *and* $u^{(1)} - u^{(0)}$ *are the only functions* v *contained in* $\mathcal{C}$ *for which* $\mathcal{E}_\omega(u^{(0)} + v) = \mathcal{E}_\omega(u^{(0)})$ *holds. Let* $r > 0$ *be suitably small and let us assume that: if* $V \in \mathcal{C} \cap \mathcal{B}_r$ *satisfies*

$$\partial_i\left(a_{ij}\partial_j V\right) - f\left(x, u^{(0)} + V\right) + f\left(x, u^{(0)}\right) = 0\,,$$

then $V \in \{0, u^{(1)} - u^{(0)}\}$*. Then, there exists* $u^\star \in C^2(\mathbf{K}^d_\omega)$ *such that:*

- $\partial_i(a_{ij}\partial_j u^\star) = f(x, u^\star)$*;*
- $u^\star$ *enjoys the Birkhoff property;*
- $u^{(0)}(x) < u^\star(x) < u^{(1)}(x)$*.*

5.8. **Proof of Theorem 3.2.** If there exists $v \in \mathcal{C} - \{0, u^{(1)} - u^{(0)}\}$, for which $\tilde{\mathcal{E}}_\omega(v) = 0$, then we are done. The reason is that, we have shown in Lemma 5.4 that in such a case $u^{(0)} + v$ is a local minimizer and, hence, it satisfies (2.1).

Hence, we will assume that the only $v \in \mathcal{C}$ for which the renormalized energy vanishes are $0, u^{(1)} - u^{(0)}$.

Also, if, for any small $r > 0$, there exists $V^{(r)} \in \mathcal{C} \cap \mathcal{B}_r - \{0, u^{(1)} - u^{(0)}\}$ so that

$$\partial_i\left(a_{ij}\partial_j V^{(r)}\right) - f\left(x, u^{(0)} + V^{(r)}\right) + f\left(x, u^{(0)}\right) = 0\,,$$

then we set $U^{(r)} := u^{(0)} + V^{(r)}$ and $U^{(r)}$ satisfies the claims of Theorem 3.2. Therefore, we may assume that there exists $r > 0$ small, so that, if $V \in \mathcal{C} \cap \mathcal{B}_r$ satisfies

$$\partial_i\left(a_{ij}\partial_j V\right) - f\left(x, u^{(0)} + V\right) + f\left(x, u^{(0)}\right) = 0\,,$$

then $V \in \{0, u^{(1)} - u^{(0)}\}$. Under these assumptions, Corollary 5.22 provides the desired solution. □

5.9. **Some remarks about energy barriers.** We have shown that, given a gap in a lamination by Birkhoff minimizers, we can find a critical function v inside the gap.

This function has renormalized energy $\tilde{\mathcal{E}}_\omega(v) \geq 0$. If $\tilde{\mathcal{E}}_\omega(v) = 0$, we have shown in Proposition 5.4 that v is another minimizer. We can consider now the lamination generated by our previous lamination and $\mathcal{L}_{u^{(0)}+v}$. If it has gaps, we can apply the procedure again.

The conclusion is that *either there is a critical point with strictly positive renormalized energy or there is a foliation of Birkhoff minimizers.*

Hence, if we define the *energy gap* of the normal ω as the supremum of the renormalized energies of all the critical points with these frequency, we obtain that *there is a foliation by minimizers which are Birkhoff with respect to* ω *if and only if the energy gap is zero.*

This seems very reminiscent of the criterion in [Mat86] for the existence of an invariant circle for twist maps. Indeed, the energy gap defined above has the same flavor as the energy gap defined in [Mat86] since both are the supremum of the difference of energies in critical points. In [Mat86] it is shown that the energy gap depends continuously with respect to ω. We think that it would be interesting to investigate the continuity of the energy gap defined here.

We also note that the proof here constructs critical points by following the heat flow with initial data in a specific family $v^{(s)}$. Hence, we can define a simplified energy gap as the supremum of the renormalized energies for the critical points that are obtained by taking the heat flow on points in the above family. Our results show that there is a foliation by minimizers if and only if the simplified energy gap is zero. We do not know whether the energy gap and the simplified energy gap are the same. The simplified energy gap seems more amenable to computation because the range over which we are taken the supremum is explicitly given.

One would also expect that, when there is a gap in the lamination of minimizers, one can also construct many other critical points, which however could fail to be Birkhoff or plane-like. This is what happens in dynamical systems, when the existence of a barrier allows to construct a symbolic dynamics.

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Appendix A. Recalling some technical results

In this Appendix, we collect some technical results that are used in the main text and we give references to the literature or indicate the small modifications needed to get them.

A.1. Results from the theory of elliptic equations.

We now state some standard elliptic PDE results in a form which is convenient for our applications. First of all, we recall the following elliptic strong maximum principle for weak subsolutions, for the proof of which we refer to § 8.7 of [GT83]:

Lemma A.1. *Let Ω be a bounded connected domain in $\mathbf{R}^d$ and let a_{ij}, $c \in L^\infty(\Omega)$, for $1 \le i,j \le d$. Assume that the coefficients a_{ij} satisfy the uniform ellipticity condition given in **(H1)** and that $c \le 0$. Let $\psi \in W^{1,2}(\Omega)$ satisfy weakly*

$$\partial_i \Big(a_{ij}(x)\, \partial_j \psi(x) \Big) + c(x)\, \psi(x) \ge 0\,,$$

for any $x \in \Omega$. Assume that, for some ball $B \subset \Omega$ we have

$$\sup_B \psi = \sup_\Omega \psi \ge 0\,.$$

Then ψ is constant in Ω.

We now deduce some useful consequences from Lemma A.1. First of all (see Corollary A.2 here below), in the case of negative subsolutions, we remove the assumption on the sign of c from Lemma A.1. Then, we use this to obtain a strong comparison principle (see Corollary A.3).

Corollary A.2. *Let Ω be a bounded connected domain in $\mathbf{R}^d$ and let a_{ij}, $c \in L^\infty(\Omega)$, for $1 \le i,j \le d$. Assume that the coefficients a_{ij} satisfy the uniform ellipticity condition given in **(H1)**. Let $\phi \in W^{1,2}(\Omega)$ satisfy weakly*

$$\partial_i \Big(a_{ij}(x)\, \partial_j \phi(x) \Big) + c(x)\, \phi(x) \ge 0\,, \tag{A.1}$$

for any $x \in \Omega$. If $\phi \le 0$ in Ω, then either $\phi < 0$ in Ω or $\phi = 0$ in Ω.

Proof. Assume that there is a point $\bar{x} \in \Omega$ such that $\phi(\bar{x}) = 0$. Let $B \subset \Omega$ be a small ball centered at $\bar{x}$. By construction,

$$\sup_B \phi = \sup_\Omega \phi = 0\,.$$

Let $c(x) = c^+(x) - c^-(x)$, where c^+ and c^- are the positive and negative part of c, respectively. Hence, ϕ satisfy weakly

$$\partial_i \Big(a_{ij}(x)\, \partial_j \phi(x) \Big) - c^-(x)\, \phi(x) \geq -c^+(x)\, \phi(x) \geq 0\,.$$

Thus, from Lemma A.1, ϕ is constant in Ω and so, by construction, constantly equal to zero. □

Corollary A.3. *Let Ω be a bounded connected domain in $\mathbf{R}^d$ and let $a_{ij} \in L^\infty(\Omega)$, for $1 \leq i, j \leq d$, satisfy the uniform ellipticity condition given in **(H1)**. Let $f \in W^{1,\infty}(\mathbf{R}^{d+1})$. Let u, $v \in W^{1,2}(\Omega)$ satisfy weakly*

$$\partial_i \Big(a_{ij}(x)\, \partial_j u(x) \Big) + f(x, u(x)) \geq 0 \geq \partial_i \Big(a_{ij}(x)\, \partial_j v(x) \Big) + f(x, v(x))\,,$$

for any $x \in \Omega$. If $u \leq v$ in Ω, then either $u < v$ in Ω or $u = v$ in Ω.

Proof. The function $\phi := u - v$ satisfies weakly (A.1) with

$$c(x) := \int_0^1 f'(x, tu(x) + (1-t)v(x))\, dt\,.$$

Hence, the claim follows from Corollary A.2. □

A.2. **Results from the theory of parabolic equations.** We consider the operator L defined in (5.14). As a matter of fact, up to a $\sqrt{\det(a_{ij})}$–factor, L can be seen as the Laplacian operator acting on the Riemannian manifold $\mathbf{R}^d$ endowed with a suitable metric $g^{ij} = \det(a_{ij})\, a_{ij}$ (see, e.g., [Tay96], formula (4.4) on page 137)

The theory of semigroups with monotone generators (see, for instance, [Sho97] and [Tay96]) shows that there is an operator e^{tL} for positive t defined by the condition that $e^{tL} V_0$ to is unique function $V(x, t)$ defined for $x \in \mathbf{K}^d_\omega$ and $t \in [0, +\infty)$, so that the map $t \mapsto V(t, \cdot)$ belongs to $C^1\Big((0, +\infty), L^2(\mathbf{K}^d_\omega)\Big)$ and V solves the linear equation

$$\begin{aligned} \frac{\partial V}{\partial t}(x,t) &= LV(x,t)\,, \qquad \forall (x,t) \in \mathbf{K}^d_\omega \times (0, +\infty) \\ V(x, 0) &= V_0(x)\,, \qquad \forall x \in \mathbf{K}^d_\omega\,. \end{aligned}$$

Of course, if $a_{ij} = \delta_{ij}$, i.e., if L is the standard Laplacian operator, and the domain is the whole $\mathbf{R}^d$, then e^{tL} is simply the convolution with a Gaussian kernel (see, e.g., [Eva98]).

Explicit bounds on the linear operator norm of e^{tL} in different spaces can be found, for instance, on pages 273–275 of [Tay97]. In particular, denoting by $\mathcal{L}(X, Y)$ the space of the linear operators from X to Y (endowed with the standard operator norm), we will use here below that

$$\|e^{tL}\|_{\mathcal{L}(W^{r,2}(\mathbf{K}^d_\omega), W^{s,2}(\mathbf{K}^d_\omega))} \leq C t^{\frac{r-s}{2}}\,, \tag{A.2}$$

for any $0 < t \leq 1$, $s \geq r \geq 0$ and some constant C, possibly depending on r and s. Moreover, in Proposition 1.1 on page 273 of [Tay97], it is shown that, for a suitable

$T>0$, for any $v\in\mathcal{C}$, there exists a $V(x,t)$ so that the map $t\mapsto V(t,\cdot)$ belongs to $C\Big([0,T],C^1(\mathbf{K}^d_\omega)\Big)$ and V solves the semilinear equation

$$\begin{aligned}&\partial_t V=\partial_i(a_{ij}\partial_j V)-f(x,u^{(0)}+V)+f(x,u^{(0)})\,,\quad \forall(x,t)\in\mathbf{K}^d_\omega\times(0,+\infty)\\ &V(x,0)=v(x)\,,\quad \forall x\in\mathbf{K}^d_\omega\,.\end{aligned}\tag{A.3}$$

We will denote by $\Phi^t(v)$ such solution and we will sometimes refer to it as the "heat flow".

The flow Φ^t can be constructed by fixed point arguments as the solution of the following equation:

$$\Phi^t(v)\;=\;e^{tL}v-\int_0^t e^{(t-s)L}\Big[f\Big(x,u^{(0)}+\Phi^s(v)\Big)-f\Big(x,u^{(0)}\Big)\Big]\,ds\,,\tag{A.4}$$

for details, see again [Tay97] pages 272–274. For further properties on the heat flow in a more general setting see, e.g., [Ang90b] and [Hen81].

In particular, it is possible to show:

Lemma A.4. *There is one and only one solution of (A.3) in $L^2(\mathbf{K}^d_\omega)$. Furthermore, this solution is in $W^{k,2}(\mathbf{K}^d_\omega)$ for $t>t_0$. The map that applies v to $\Phi^t(v)$ is twice differentiable in t for $t>0$ and continuous in v with respect to the $W^{k,2}(\mathbf{K}^d_\omega)$-topology.*

Proof. On pages 272–274 of [Tay97], the reader may find the details of the fixed point argument in in $C([0,T],L^2)$ which solves (A.3). The fact that the solution is in $W^{k,2}(\mathbf{K}^d_\omega)$ was established in Lemma 5.9. For the continuity of the map $v\mapsto\Phi^t(v)$, recall Proposition 5.12. The only thing that remains to be done is to prove the differentiability with respect to t for $t>0$. For this it suffices to consider (A.4) and note that the RHS is indeed differentiable for $t>0$.

We note that for $t>0$, we have $\frac{d}{dt}e^{tL}v=Le^{tL}v$, and the bounds on its operator norm follow from (A.2). This controls the first term in (A.4). Similarly, we can see that the second term in (A.4) can also be differentiated, provided that $\Phi^t(v)$ is continuous. In this case, we have

$$\begin{aligned}&\frac{d}{dt}\int_0^t e^{(t-s)L}\Big[f\Big(x,u^{(0)}+\Phi^s(v)\Big)-f\Big(x,u^{(0)}\Big)\Big]\,ds\\ =\;&f\Big(x,u^{(0)}+\Phi^t(v)\Big)-f\Big(x,u^{(0)}\Big)\\ &+\int_0^t e^{(t-s)L}L\Big[f\Big(x,u^{(0)}+\Phi^s(v)\Big)-f\Big(x,u^{(0)}\Big)\Big]\,ds\,.\end{aligned}\tag{A.5}$$

Given the bounds in (A.2) we obtain:

$$\|e^{tL}L\|_{\mathcal{L}(W^{r,2}(\mathbf{K}^d_\omega),W^{s,2}(\mathbf{K}^d_\omega))}\leq Ct^{\frac{r-s-2}{2}}$$

for any $0<t\leq 1$, $s\geq r\geq 0$ and some constant C, possibly depending on r and s.

The last term in (A.5) is therefore a convergent integral in $W^{k,2}(\mathbf{K}^d_\omega)$ if we have that $\Phi^t(v)$ in $W^{k+1,2}(\mathbf{K}^d_\omega)$.

The second derivative can be established in the same way. We just need to check that the formulas for the derivative with respect to t we have just established can be differentiated again with respect to t. The procedure is the same as before and we thus leave the details to the reader. ∎

Following is the version of the parabolic maximum principle which is convenient for our applications. Though its proof is quite standard and it is inspired by the classical results (see, e.g., [Eva98]), we give it in full detail for the sake of completeness (and similarly for other standard proofs here below).

Proposition A.5. *Let $T>0$, $c \in L^\infty(\mathbf{R}^d \times (0,T))$ and assume that $w(x,t)$ solves*

$$\partial_t w(x,t) + c(x,t)\, w(x,t) = (Lw)(x,t)$$

for any $t \in (0,T)$ and $x \in \mathbf{R}^d$. Suppose $w(x,0) \geq 0$ for any $x \in \mathbf{R}^d$. Then, $w(x,t) \geq 0$ for any $x \in \mathbf{R}^d$ and any $t \in [0,T)$.

Proof. Let $M \geq 1 + (1/T)$ so that $|c(x,t)| \leq M$ for any $x \in \mathbf{R}^d$ and any $t \in (0,T)$. We prove the claim for $t \in (0,\, 1/(2M)]$, the general case following by flowing the solution iteratively.

Assume, by contradiction, that $w(\bar{x},\bar{t}) < 0$, for some $\bar{x} \in \mathbf{R}^d$ and $\bar{t} \in (0,\, 1/(2M)]$. Let us define

$$\begin{aligned} \varepsilon &:= \min\left\{ \frac{1}{4M},\, |w(\bar{x},\bar{t})| \right\} \qquad \text{and} \\ W(x,t) &:= w(x,t) + \varepsilon t + \varepsilon^2 . \end{aligned}$$

Then, $W(x,0) > 0$ for any $x \in \mathbf{R}^d$ and $W(\bar{x},\bar{t}) < 0$, due to our choice of ε. Then, there exists $x^\star \in \mathbf{R}^d$ and $t^\star \in [0,\bar{t}]$ so that $W(x,t) > 0$ for any $x \in \mathbf{R}^d$ and $t \in [0,\, t^\star)$, and $W(x^\star,t^\star) = 0$. In particular,

$$W(x,t^\star) \geq 0 = W(x^\star,t^\star)$$

for any $x \in \mathbf{R}^d$, and so

$$\partial_x W(x^\star,t^\star) = 0 \qquad \text{and} \qquad D_x^2 W(x^\star,t^\star) \geq 0 . \tag{A.6}$$

Analogously,

$$W(x^\star,t) \geq 0 = W(x^\star,t^\star)$$

for any $t \in [0,t^\star)$ and so

$$\partial_t W(x^\star,t^\star) \leq 0 . \tag{A.7}$$

Thence, if $z^\star := (x^\star,t^\star)$, combining (A.6) with (A.7), we deduce that

$$\Big((L-\partial_t)W\Big)(z^\star) \geq 0 . \tag{A.8}$$

On the other hand, by a direct computation,

$$\begin{aligned} \Big((L-\partial_t)W\Big)(z^\star) &= \Big((L-\partial_t)w\Big)(z^\star) - \varepsilon = \\ &= c(z^\star)\, w(z^\star) - \varepsilon = \\ &= c(z^\star)\, W(z^\star) - c(z^\star)\, (\varepsilon t^\star + \varepsilon^2) - \varepsilon = \\ &= -c(z^\star)\, (\varepsilon t^\star + \varepsilon^2) - \varepsilon \leq \\ &\leq M\varepsilon \left(\frac{1}{2M} + \varepsilon \right) - \varepsilon \,<\, 0 , \end{aligned}$$

due to our choice of ε, in contradiction with (A.8). □

A consequence of the above maximum principle is the following parabolic comparison principle:

Proposition A.6. *Let $v_0 \geq w_0 \in \mathcal{C}$. Then, $\Phi^t(v_0) \geq \Phi^t(w_0)$, for any $t \geq 0$.*

Proof. Denote $\zeta(x,t) := \Phi^t(v_0) - \Phi^t(w_0)$ and

$$c(x,t) := \int_0^1 \partial_\mu f\Big(x, u^{(0)}(x) + \sigma\Phi^t(v_0)(x) + (1-\sigma)\,\Phi^t(w_0)(x)\Big)\,d\sigma\,.$$

Then,

$$\begin{aligned} \partial_t \zeta &= L\zeta + f(x, u^{(0)} + \Phi^t(w_0)) - f(x, u^{(0)} + \Phi^t(v_0)) \\ &= L\zeta - c\zeta\,. \end{aligned}$$

Therefore, by Proposition A.5, $\zeta \geq 0$. □

A.3. **Summary of some results in** [Val04]**.** We now recall two results, namely Theorems A.7–A.8 here below, for the proof of which we refer to [Val04], concerning the existence and the geometric properties of local minimizers for the functional $\mathcal{E}_\omega$. These results are used during the proofs of the main theorems of this paper.

First, we recall a density estimate (see also [CC95]):

Theorem A.7. *Let $\Omega \subset \mathbf{R}^d$ be a bounded domain. Fix $\theta_0 \in (0,1)$ and $r_1 > 0$. Let $u \in W^{1,2}(\Omega)$. Assume that u is a local minimizer in Ω, i.e., that*

$$\mathcal{E}_\Omega(u+\phi) \geq \mathcal{E}_\Omega(u) \qquad \forall \phi \in C_0^\infty(\Omega)\,.$$

Assume also that $|u(\bar{x})| < \theta_0$. Then, there exist $\kappa, r_0 > 0$, depending only on universal constants and on θ_0 and r_1, and x_1, $x_2 \in \mathbf{R}^d$, such that

$$B_{\kappa r}(x_1) \subseteq \{u \geq \theta_0\} \cap B_r(\bar{x}) \qquad \text{and} \qquad B_{\kappa r}(x_2) \subseteq \{u \leq -\theta_0\} \cap B_r(\bar{x})\,,$$

provided that $r \geq r_0$ and $B_{2r+r_1}(\bar{x}) \subseteq \Omega$.

Now, we recall the existence of a plane-like, local minimizer for $\mathcal{E}_\omega$, with bounded energy, satisfying the Birkhoff property:

Theorem A.8. *Fix $\theta \in (0,1)$. Then, there exists $u \in C^2(\mathbf{K}^d_\omega)$ such that*

- *$\mathcal{E}_\Omega(u+\phi) \geq \mathcal{E}_\Omega(u)$, for all smooth functions ϕ supported in any bounded domain Ω;*
- *$\mathcal{E}_\omega(u) < +\infty$, $|u| \leq 1$ and u satisfies the Birkhoff property;*
- *There exists $M > 0$, depending only on θ and on universal quantities, such that $u(x) \in [\theta, 1]$ provided that $\omega \cdot x \geq M\,|\omega|$ and $u(x) \in [-1, -\theta]$ provided that $\omega \cdot x \leq -M\,|\omega|$.*

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