On the destruction of resonant Lagrangean tori in Hamiltonian systems

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22 October 2008

Abstract

Starting from Poincaré's fundamental problem of dynamics, we consider perturbations of integrable Hamiltonian systems in the neighbourhood of resonant maximal invariant tori with a single resonance. We investigate how such a torus disintegrates when the action variables vary in the resonant surface. For open subsets of this surface the resulting lower dimensional tori are either hyperbolic or elliptic. For a better understanding of the dynamics, both qualitatively and quantitatively, we also investigate the singular tori and the way in which they are being unfolded by the action variables. In fact, if N is the number of degrees of freedom, singularities up to codimension $N-1$ cannot be avoided. In the case of Kolmogorov non-degeneracy the singular tori are parabolic, while under the weaker non-degeneracy condition of Rüssmann the lower dimensional tori may also undergo *e.g.* umbilical bifurcations. We emphasise that this application of Singularity Theory only uses internal (or distinguished) parameters and no external ones.

1 Introduction

Classical perturbation theory largely concerns the continuation of quasi-periodic motions as these occur in integrable Hamiltonian systems for small non-integrable perturbations. The classical perturbation series here diverge on a resonance subset that densely fills the phase space, leading to the notorious small denominators even when avoiding this dense set. This paper deals with the dynamics within such resonance gaps.

We briefly summarise that Kolmogorov–Arnol'd–Moser (KAM) Theory [2] establishes the persistence of quasi-periodic motions that densely fill Lagrangean tori, meaning that the dimension equals the number N of degrees of freedom. If $\omega \in \mathbb{R}^N$ denotes the frequency vector, the resonances alluded to above are given by

$$\langle k, \omega \rangle \;=\; 0 \;, \quad k \in \mathbb{Z}^N \;. \tag{1}$$

KAM Theory imposes strongly non-resonant, Diophantine conditions

$$|\,\langle k, \omega \rangle\,| \;\geq\; \frac{\gamma}{|k|^\tau} \quad \text{for all } k \in \mathbb{Z}^N \backslash \{0\} \tag{2}$$

on the frequency vectors, where $\gamma > 0$, $\tau > N-1$, which guarantee the persistence of many Lagrangean tori in the sense of measure theory. We recall [22] that the persistent N-tori are smoothly parametrised over the nowhere dense union of lines defined by (2), in the sense of Whitney; colloquially we speak of a *Cantor set of lines*.

In this paper we investigate what happens to the Lagrangean tori of the unperturbed system for which the frequency vector is resonant, *i.e.* satisfying (1) for a fixed nonzero $k \in \mathbb{Z}^N$ (and its integer multiples), so which is contained in a gap of the 'Cantor set' defined by (2).

Starting point is the real analytic perturbed N–degree-of-freedom Hamiltonian

$$H_\varepsilon(\varphi, I) \;=\; H_0(I) \;+\; \varepsilon H_1(\varphi, I; \varepsilon) \tag{3}$$

defined on $\mathbb{T}^N \times \mathbb{R}^N$ with perturbation parameter $0 < \varepsilon \ll 1$. We concentrate on a single resonance, whence a normal form approximation can be reduced to one degree of freedom. See below for details. The reduced system is defined on the cylinder $\mathbb{T} \times \mathbb{R}$ with Hamiltonian of the form

$$\mathcal{H}(p, q; \mu) \;, \quad (q, p) \in \mathbb{T} \times \mathbb{R},\; \mu \in \mathbb{R}^{N-1} \;; \tag{4}$$

here $\mu = \mu(I)$ is a (distinguished) parameter varying along the resonance surface

$$\langle k, \omega(I) \rangle \;=\; 0 \;, \quad \text{where} \quad \omega(I) \;:=\; DH_0(I) \;.$$

We will show that (4) is a general family of Hamiltonian functions and our interest is the study of the dynamics they generate. As in all one-degree-of-freedom systems, this dynamics is largely determined by the configuration of level sets. We recall [2] that the closed curves correspond to librational Lagrangean N–tori, while the critical points correspond to invariant $(N-1)$–tori. The latter govern the global geometry of the dynamics and in the $(N-1)$–parameter family (4) of functions this is determined by Singularity Theory [23, 3].

Within such an $(N-1)$–parameter family of functions one may encounter singularities up to co-dimension $N-1$ in a persistent way. Singularity Theory provides us with versal unfoldings of these. These unfoldings also will appear persistently in our general family (4).

We like to note that for families of functions persistence in the sense of structural stability is a generic property. To formulate this property in a precise way one needs a topology on the space of Hamiltonian functions. In the present real analytic setting we use the compact-open topology on holomorphic extensions detailed in [11].

The real-analytic compact-open topology on holomorphic extensions fits with local uniform convergence of the corresponding Hamiltonian functions. We recall from [11] that this compact-open topology has the Baire property, which means that countable intersections of dense-open sets are still dense. Moreover, the compact-open topology is stronger than the Whitney C^k topologies used in [21, 20]. From the latter it immediately follows that any C^k open property also is open in the compact-open sense. The same holds for denseness, as long as we restrict to properties defined in terms of transversality. Therefore a real analytic unfolding that is versal in the C^k sense also is versal with respect to the compact-open topology. In particular the differentiable Singularity Theory also applies for the real analytic case. In fact, Weierstraß originally developed the theory for this case.

We recall that for equilibria to be either hyperbolic or elliptic, the corresponding singularity $\mathcal{H}$ has co-dimension 0, which means that it has open occurrence in the space of families of Hamiltonian functions. However, the singularities of higher co-dimension determine the geometric organization of those of lower co-dimension, therefore in particular the ones corresponding to hyperbolic and elliptic equilibria. This organization reflects both qualitative and quantitative aspects.

Example (A quasi-periodic centre-saddle bifurcation) As an example in the space of $(N-1)$–parameter families (4) we consider

$$\mathcal{H}(p,q;\mu) \;=\; \frac{1}{2}p^2 \;+\; \frac{1}{6}q^3 \;+\; \lambda(\mu)q \tag{5}$$

with $\lambda(0) = 0$ and $D_\mu\lambda(0) \neq 0$. We remark that this family usually is called *fold*, a versal unfolding of the singularity A_1, see [23, 3]. Clearly (5) is a family of one-degree-of-freedom systems for which the equilibrium $(p,q) = (0,0)$ at $\mu = 0$ is parabolic (and hence neither elliptic nor hyperbolic), and for any (real analytic) small perturbation there is a parameter value $\mu = \mu_0 \approx 0$ for which there is a parabolic equilibrium at a certain point $(p,q) = (p_0,q_0) \approx (0,0)$. Thus, there is a full neighbourhood of the 1–parameter family defined by (5) in the above topology such that a parabolic equilibrium occurs nearby. Hence it is impossible for the set of all families of one-degree-of-freedom systems that only have elliptic and/or hyperbolic equilibria to contain a countable intersection of open and dense sets, *i.e.* to be generic.

An interesting aspect is that in a fixed energy level we have the same theory, with one parameter less. Thus in each energy level we have elliptic, hyperbolic and parabolic tori in a center-saddle bifurcation. In both cases a quantitative consequence of this is that for small perturbations the asympotic distance of the two equilibria, as they approach the parabolic equilibrium, is of order $\sqrt{\lambda(\mu)}$ as $\mu \to 0$.

In the reconstruction to the setting of N degrees of freedom, the equilibria correspond to invariant $(N-1)$–tori, where the normal behaviour is inherited, and the closed level curves correspond to librational Lagrangean N-tori. Addition of the non-symmetric higher order terms presents us with a new perturbation problem. This can be solved by KAM

Theory [17]. In this way the entire bifurcation scenario becomes *Cantorised*, for a detailed example also see Section 2.

Observe that for $N = 2$ the lower dimensional tori really are closed orbits. In this case the energy is the only parameter and no bifurcation takes place inside an energy level. For $N \geq 3$ the bifurcating tori have dimension larger than or equal to 2 and the bifurcation can take place within an energy level. $\triangle$

The fold example (5) is a special case of the family of cuspoids that unfold the co-rank 1 singularities A_k, $k \in \mathbb{N}$, see [23, 3] for more details. The corresponding KAM perturbation problem has been solved in [5]. Thus all quasi-periodic bifurcations of normally parabolic tori can be retrieved in resonance gaps.

An aspect that one has to keep in mind in the present case of $(N-1)$–parameter families (4) defined by the normal form of (3) near a single resonance, is that part of the terms in $\mathcal{H}$, say the p–terms, come from H_0 and the remaining terms come from the perturbation H_1. The p^2–term in (5) occurs in the case where the integrable part H_0 satisfies the Kolmogorov condition

$$\det D^2 H_0(I) \quad \neq \quad 0 \ , \tag{6}$$

which is valid on an open and dense set in parameter space, see [2]. For these values of I the above theory of parabolic tori applies, see Section 2 for more details. In particular, one can read off from the degeneracy of the minumum and the maximum of a certain potential V_μ on $\mathbb{T}$ a lower bound for the number of families of invariant $(N-1)$–tori.

Note that the Kolmogorov non-degeneracy condition (6) fails on a co-dimension 1 subset. Here one may still expect a weaker form of non-degeneracy to hold true, expressed by the Rüssmann condition, cf. [8, 25]. In this way also more general singularities can be incorporated, giving rise to all possible applications of Singularity Theory as in [19]. More degenerate singularities may *e.g.* lead to umbilical torus bifurcations, for an example see Section 3; in Section 4 we discuss further possibilities.

We emphasise that we are not to impose genericity conditions on occurring equilibria, but only on the initial Hamiltonian H, see (3). Our genericity assumption on H_0 amounts to Rüssmann non-degeneracy. The remaining genericity assumptions on H are obtained via H_1. Our aim is to obtain versality of the occurring unfolding, thereby excluding too pathological examples. For more details see below.

We discuss some related work. To our knowledge, invariant tori reconstructed from possibly degenerate equilibria have only been addressed for single resonances.

Cheng [13] considers convex H_0, such that in the reduced system (4) the maximum and minimum of the q–dependent part differ. The invariant $(N-1)$–tori corresponding to the maxima are of 'hyperbolic' type, where a degenerate maximum is *not* excluded. In this case it is established that the system (3) has a 'Cantor family' of 'hyperbolic' type invariant $(N-1)$–tori. We stress that here no further genericity conditions are required. Since this also allows the perturbation to be rather pathological, no information is obtained on the fine structure where these tori fail to be truly hyperbolic.

In [14] the minima of the q–dependent part of (4) are treated in a similar way. In this case the degenerate torus is excluded, leading to a 'Cantor family' of elliptic invariant $(N-1)$–tori.

The approach by Gallavotti, Gentile and Giuliani [16] considers the perturbation parameter ε also as a bifurcation parameter. Although they do consider degenerate singularities, the results only concern 'Cantor families' of elliptic and hyperbolic invariant $(N-1)$–tori. Careful considerations yield expansions of the ε–family of $(N-1)$–tori in *e.g.* $\sqrt{\varepsilon}$, and quantitave asymptotic information. The fate of degenerate tori is not explained and *a fortiori* a complete bifurcation scenario is not discussed.

As soon as one makes the assumption that all equilibria of the reduced system are non-degenerate one obtains elliptic and hyperbolic lower dimensional tori also for multiple resonances, see [15, 12, 24, 10] and references therein. We like to emphasise that this assumption is not generic for parametrised systems; one also has to deal with degenerate equilibria and the corresponding bifurcation theory. A starting point for the development of such a persistence result is formed by [7, 4], where multiple resonances are taken into account. For k–fold resonances corank–k–singularities may occur already under the Kolmogorov condition, and under the Rüssmann condition this may further rise to corank $2k$.

2 Kolmogorov Hamiltonians

To explain the kind of results that can be obtained for the perturbation (3) of an integrable Hamiltonian $H_0 = H_0(I)$ on $\mathbb{T}^N \times \mathbb{R}^N$ we first restrict to an open subset $U \subset \mathbb{R}^N$ where the Kolmogorov non-degeneracy condition (6) is valid for every $I \in U$. By classical KAM Theory [22, 2, 8] it then follows that most Lagrangean tori $\mathbb{T}^N \times \{I\}$, $I \in U$ satisfy the Diophantine conditions (2) and persist. The central question of this paper is how Lagrangean tori with resonant frequency vector behave under perturbation.

As announced in the introduction, we concentrate on a single resonance $\langle k, \omega \rangle = 0$, the equation $\langle k, DH_0(I) \rangle = 0$ determining a local hypersurface $\mathbb{Y} \subset U$. For $I \in \mathbb{Y}$ the unperturbed Lagrangean torus $\mathbb{T}^N \times \{I\}$ is foliated into invariant tori of dimension $N-1$. Let us put

$$n \;=\; N-1$$

and on U choose local co-ordinates $\varphi = (x, q)$ with values in $\mathbb{T}^n \times \mathbb{T}$ and $I = (y, p)$ with values in $\mathbb{R}^n \times \mathbb{R}$, where y parametrises the surface $\mathbb{Y}$. In these local co-ordinates the single resonance reads

$$\frac{\partial}{\partial p} H_0(y, 0) \;=\; 0 \quad \text{for all } y \in \mathbb{Y} \;, \tag{7}$$

cf. [13, 14, 24, 10]. The remaining frequencies form the vector $\omega = D_y H_0(y, 0)$ which is non-resonant at a single resonance $(y, p) = (y^*, 0)$. Treating ω as an external parameter, we expect persistence results only for Diophantine ω, where gaps in the resulting 'Cantor set' correspond to multiple resonances. Following [22, 9], we localise to $\hat{y}$ (ε–close to 0) writing $y = y^* + \hat{y}$, thereby shrinking the neighbourhood U if necessary. Moreover we restrict to the lowest order terms

$$H_0(\hat{y}, p; y^*, \omega) \;=\; \langle \omega, \hat{y} \rangle \;+\; \frac{a(y^*)}{2} p^2 \;.$$

From (6) together with (7) we infer

$$a(y^*) \;\neq\; 0 \quad \text{for all } y^* \in \mathbb{Y} \;,$$

whence we find a lower bound of the function $|a|$ by shrinking the co-ordinate domain $U \supset \mathbb{Y}$ a bit if necessary.

We now apply a familiar method to replace the system (3) by a family of one-degree-of-freedom systems, cf. [2, 13, 24, 10]. Starting point is a normalizing transformation that turns the perturbed Hamiltonian into

$$H_\varepsilon(x, \hat{y}, p, q; y^*, \omega) \;=\; H_0(\hat{y}, p; y^*, \omega) \;+\; \varepsilon \bar{H}_1(\hat{y}, p, q; y^*, \omega) \;+\; \mathcal{O}(\varepsilon^2)$$

where $\bar{H}_1$ is the $\mathbb{T}^n$–average along x of H_1 at $\varepsilon = 0$. In the expansion

$$\begin{aligned} \bar{H}_1 \;=\; & \eta(\hat{y}; y^*, \omega) \;+\; \alpha(\hat{y}; y^*, \omega) p \;+\; \beta(\hat{y}; y^*, \omega) q \\ & +\; \frac{A(y^*)}{2} p^2 \;+\; \frac{B(y^*)}{2} q^2 \;+\; C(y^*) pq \;+\; \ldots \end{aligned}$$

we may have $A(y^*) \equiv 0$, but more importantly $|a(y^*) + \varepsilon A(y^*)|$ is still bounded from below on $\mathbb{Y}$. Re-parametrising $\omega \mapsto \omega + \varepsilon D_{\hat{y}} \eta(\hat{y}; y^*, \omega)$ the expansion of H_ε still starts with $\langle \omega, \hat{y} \rangle$. By means of an ε–small shear transformation in p we get rid of terms that are linear in p, and scaling p by $\sqrt{\varepsilon}$ we arrive at

$$H_\varepsilon(x, \hat{y}, p, q; y^*, \omega) \;=\; \langle \omega, \hat{y} \rangle \;+\; \varepsilon \mathcal{H}(p, q; y^*, \omega) \;+\; \mathcal{O}(\varepsilon^2)$$

with

$$\mathcal{H}(p, q; y^*, \omega) \;=\; \frac{\tilde{a}(y^*)}{2} p^2 \;+\; V_{y^*}(q) \;,$$

compare with (4).

Here V_{y^*} can be interpreted as an n–parameter family of 1–dimensional potentials, and critical points q^* of V_{y^*} correspond to invariant n–tori $\mathbb{T}^n \times \{(0, 0, q^*; y^*, \omega)\}$ of the 'intermediate' integrable system with Hamiltonian $\mathcal{H}_\varepsilon = \langle \omega, \hat{y} \rangle + \varepsilon \mathcal{H}$. The $y^* \in \mathbb{R}^n \approx \mathbb{Y}$ around which $\hat{y}$ is localised has the character of a (distinguished) parameter; let us make this explicit by writing

$$\mu \;:=\; y^* \;.$$

While critical points of a single potential are generically non-degenerate, it is generic for the n–parameter family V_μ of potentials to encounter critical points up to co-dimension n. Note that this amounts to a genericity condition on the perturbation H_1 of H_0. More precisely, the μ–values parametrising a potential V_μ with a degenerate critical point of co-dimension k form an $(n-k)$–dimensional submanifold Λ_k in μ–space.

Theorem 2.1 *Let $\mu^* \in \Lambda_k$ and put $d = k+2$. Then all derivatives at q^* of order $j < d$ vanish, so*

$$V_{\mu^*}(q) \quad = \quad \frac{b(\mu^*)}{d!}(q - q^*)^d \; + \; \mathcal{O}\left((q-q^*)^{d+1}\right) \tag{8}$$

with $b(\mu^*) \neq 0$ and $2 \leq d \leq n+2$ near the critical point q^* of V_{μ^*}. When $d = 2$ the critical point q^* is non-degenerate and $(p,q) = (0,q^*)$ is a non-degenerate equilibrium of the one-degree-of-freedom system — a saddle if $\tilde{a}b < 0$ and a centre if both coefficients have the same sign. In case $d \geq 3$ the equilibrium is parabolic and it is furthermore generic for the family V_μ to provide a versal unfolding of the degenerate critical point. Also this genericity condition is a condition on the perturbation H_1.

Let us translate co-ordinates to $q^* = 0$ and concentrate on a degenerate critical point with $d \leq n+1$. Then

$$\mathcal{H}(p,q;\mu,\omega) \quad = \quad \frac{\tilde{a}(\mu)}{2}p^2 \; + \; \frac{\tilde{b}(\mu)}{d!}q^d \; + \; \sum_{j=1}^{d-2}\frac{c_j(\mu)}{j!}q^j \; + \; \mathcal{O}(q^{d+1})$$

and $\mathcal{H}_\varepsilon = \langle \omega, \hat{y} \rangle + \varepsilon \mathcal{H}$ has the form (1.4) of [5] with $\lambda_j = \varepsilon c_j(\mu)$. Thus under generic small perturbation the resonant Lagrangean torus leads to the whole bifurcation scenario detailed in [5]. This amounts to the classical hierarchy of the cuspoids [23, 3] unfolding the singularities A_k, $k \in \mathbb{N}$, that are Cantorised by taking out the dense set of resonances mentioned before and using KAM Theory.

Remark 2.2 Since $\mu = y - \hat{y}$ was introduced by localisation the rôle of the unfolding parameter λ is ultimately played by the action y conjugate to the toral angles, provided that $c : \mu \mapsto c(\mu)$ is at $\mu = 0$ a submersion from $\mathbb{R}^n$ to $\mathbb{R}^{d-2}$. The tori $\mathbb{T}^n \times \{(0,0,q^*)\}$ with $d = n+2$ are generically isolated and may therefore disappear in a resonance gap. In case $n - d + 2 \geq 1$ Diophantine approximation of dependent quantities does yield persistence on Cantor sets, see [8, 5] for more details.

Proof of Theorem 2.1. For a description of the compact-open topology on holomorphic extensions of real analytic systems we refer to the introduction. We recall that the Singularity Theory in the real analytic setting coincides with that in the C^k setting for k large. Therefore the families V_μ are also versal in the real analytic setting.

Genericity of H_1 yields genericity of the family V_μ of potentials since the mapping $H_1 \mapsto \bar{H}_1 \mapsto V_\mu$ is a composition of submersions. Indeed, the latter part $\bar{H}_1 \mapsto V_\mu$ is merely a scaling in p after which the quadratic part $\frac{\tilde{a}}{2}p^2$ is split off by means of the Morse Lemma [20]. The normalization $H_1 \mapsto \bar{H}_1$ consists of a co-ordinate transformation followed by truncation of higher order terms which for sufficiently small ε has the character of a (linear) projection. □

Example (Unfolding a degenerate minimum) We consider the case $d = 4$, for which the one-degree-of freedom family has the form

$$\mathcal{H}(p,q;\mu,\omega) \quad = \quad \frac{1}{2}p^2 \; + \; \frac{1}{24}q^4 \; + \; \lambda_1(\mu)q \; + \; \frac{\lambda_2(\mu)}{2}q^2 \; . \tag{9}$$

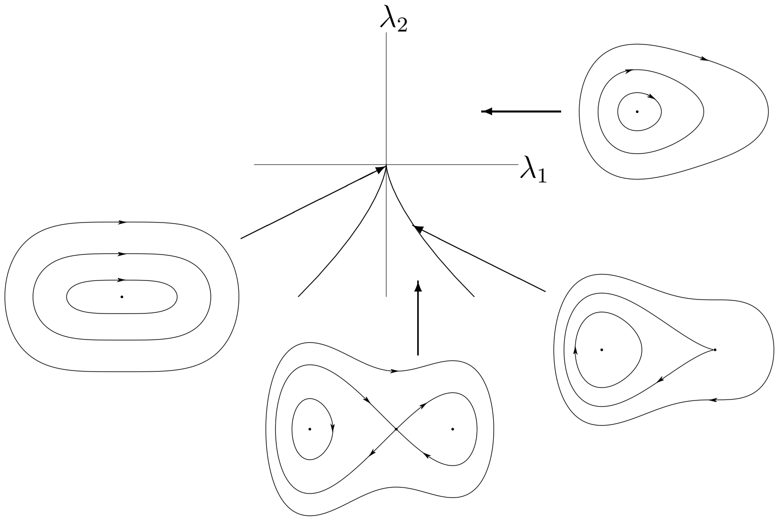

Figure 1: Organisation of the local dynamics near a degenerate minimum of the Hamiltonian. Phase portraits show the reduced one-degree-of-freedom dynamics of (9). For the interpretation for the N–degree-of-freedom system see the text.

Note that this example occurs persistently for $N \geq 4$ degrees of freedom. For definiteness we fix $N = 4$. So we started with four action variables $\mathbb{R}^4 = \{I_1, \ldots, I_4\}$ in which the 3-dimensional resonance hypersurface $\mathbb{Y}$ is defined by $\langle k, DH_0(I)\rangle = 0$. Locally we have the variable p transverse to $\mathbb{Y}$ and $\mu = (\mu_1, \mu_2, \mu_3)$ parametrises $\mathbb{Y}$.

In Figure 1 we show the organisation of the local dynamics. We now describe the meaning of the phase portraits for the 4–degree-of-freedom system. All closed curves correspond to Lagrangean tori and equilibria to 3–dimensional tori. In the case of a center these are elliptic, hyperbolic in the case of a saddle and parabolic in the (two) remaining cases.

We recall that $\mu \mapsto \lambda(\mu)$ is a (local) submersion. Therefore Cantorisation in the (μ_1, μ_2, μ_3)–direction amounts to the following.

1. We begin with the cases corresponding to parabolic tori.

   (a) The central point $\lambda = 0$ corresponds to a line that Cantorises to Cantor dust (i.e. a Cantor set of topological dimension 0) of Hausdorff dimension 1.

   (b) The fold lines emanating from $\lambda = 0$ correspond to planes that Cantorise to Cantor dust as well, now of Hausdorff dimension 2.

2. The hyperbolic tori are parametrised over the open region in between the fold lines, which corresponds to an open 3–dimensional set. Cantorisation leads to a union of lines, smoothly parametrised over the 2–dimensional Cantor dust mentioned above and also ending there. Recall that colloquially we call this a Cantor set of lines.

3. Both open regions parametrise elliptic tori, observing that in between the fold lines each point corresponds to two elliptic tori, and in the other region only to one.

(a) In the latter case Cantorisation leads to Cantor dust.

(b) In the former case Cantorisation leads to two layers of Cantor dust.

As usual, Cantorisation of the librational Lagrangean tori takes place along a Cantor set of lines. Note that this also uses the p variable to obtain the 4–dimensional (μ, p)–set.

As in the example of the quasi-periodic centre-saddle bifurcation, we can give asympotic estimates based on the geometry sketched in Figure 1. For instance, along the symmetry axis $\lambda_1 = 0$ the distance between the two elliptic tori is of order $\sqrt{-\lambda_2(\mu)}$ as $\mu \to 0$. $\triangle$

We now resume our discussion of related work, started in the introduction. Theorem 2.1 allows to recover the results in [13, 14] for generic perturbations εH_1 of H_0. Indeed, the angular variable q takes values in $\mathbb{T}^1$ and on this compact set each potential V_μ, μ fixed, assumes minimum and maximum. It is a genericity condition on H_1 for these to be different from each other, *i.e.* $V_\mu \neq const$ for all μ, and furthermore for V_μ to assume the form (8) with d even and $b(\mu) < 0$ at a maximum q^* while $b(\mu) > 0$ at a minimum; let us denote the latter by q_*. For definiteness we concentrate on $\tilde{a}(\mu) > 0$ (which may always be achieved by reversing time if necessary). Then the maximum q^* corresponds to a 'hyperbolic' type torus $\mathbb{T}^n \times \{(0, 0, q^*)\}$, truly hyperbolic if the maximum is non-degenerate and parabolic otherwise. In both cases we have persistence for ω satisfying the Diophantine conditions (2) with N replaced by n.

The persistence of at least one n–torus from the H_0–resonant $\mathbb{T}^n \times \{(0, 0)\} \times \mathbb{T}^1$ for Diophantine ω had already been established in [13], and without any genericity condition on the perturbation. What our approach adds to this is a precise description how occurring degenerate maxima of the potential, called 'of weaker persistency' in [13], lead to a Cantorised bifurcation scenario of the corresponding n–tori. This latter result cannot be obtained without genericity conditions.

The minima q_* are treated in [14]. In the non-degenerate case these correspond to elliptic n–tori, whence normal-internal resonances have to be avoided as well. Therefore, persistence of a second n–torus is obtained in [14] only on a smaller (though still measure-theoretically large) subset S. For generic perturbations we can now explain the fine structure. In particular the tori coming from degenerate minima q_* do not have to be excluded. In fact, these give the opportunity to enlarge S a bit. For instance, when $d = 4$ in (8) we recover the bifurcation diagram given in Figure 1 and next to at least one centre for all nonzero $\lambda \in \mathbb{R}^2$ we have an additional saddle in between the two fold lines; hence, here only Diophanticity of internal frequencies is needed to obtain persistence of a second family of invariant n–tori in the resonant zone.

Remark 2.3 Without the genericity conditions on the perturbation there may be tori $\mathbb{T}^n \times \{(0, 0, q^*)\}$ with $d > n + 2$. In an attempt to still apply the results of [5] we may introduce extra (external) parameters ν to provide a versal unfolding. The perturbed system then displays again the Cantorised bifurcation scenario and contains the original perturbed system as a subsystem. However, we cannot expect the 'initial' degenerate above torus of the intermediate system to be present in the perturbed system since this torus is moved by the perturbation. We still have the weaker conclusion, though, that the Cantorised family of n–tori contains parabolic tori of at most degeneracy d (as found in the intermediate system).

Note that this approach does not allow us to drop the genericity conditions that we had to impose on the perturbation when recovering the results of [13, 14]. Indeed, the small constant ε in Theorem 2.1 of [5] depends on d and may tend to 0 as $d \to \infty$. Since the results in [13, 14] are valid for non-generic perturbations as well one may speculate that the latter does not happen in this situation.

It is instructive to compare the above results with the corresponding statements in [16]. When d is odd the bifurcation diagram contains empty regions and regions with both elliptic and hyperbolic tori. When d is even all open regions of the bifurcation diagram Cantor-parametrise at least one elliptic torus if $b > 0$, while $b < 0$ yields deformations with hyperbolic tori. See Figure 1. For a similar approach in the latter case (not using Lindstedt series) see [26].

3 An umbilic example

To understand how results similar to those of the previous section can be obtained if the Kolmogorov condition (6) is replaced by Rüssmann's non-degeneracy condition, we now consider the following example in $N = 5$ degrees of freedom. Starting point remains the perturbed Hamiltonian (3) with $I = (y, p)$, and for the unperturbed part we work with

$$H_0(y,p) \;=\; \sum_{i=1}^{4} \mathrm{e}^{i-1} y_i + \frac{1}{2} y_i^2 + \frac{1}{6} y_i^3 \;+\; \frac{1}{6} p^3 \;, \tag{10}$$

where we use that the vector $(1, \mathrm{e}, \mathrm{e}^2, \mathrm{e}^3)$ is Diophantine. Then Rüssmann's non-degeneracy condition

$$< \frac{\partial^{|\ell|}\omega}{\partial I^\ell} \;\Bigg|\; 0 \neq \ell \in \mathbb{N}_0^5 > \;\supseteq\; < \frac{\partial\omega}{\partial y_1}, \frac{\partial\omega}{\partial y_2}, \frac{\partial\omega}{\partial y_3}, \frac{\partial\omega}{\partial y_4}, \frac{\partial^2\omega}{\partial p^2} > \;=\; \mathbb{R}^5$$

is satisfied everywhere. We are interested in the fate of the resonant tori $p = 0$. Again we apply a normalizing transformation that turns the perturbed Hamiltonian into

$$H_\varepsilon(x, \hat{y}, p, q; \mu, \omega) \;=\; H_0(\hat{y}, p; \mu, \omega) \;+\; \varepsilon \bar{H}_1(\hat{y}, p, q; \mu, \omega) \;+\; \mathcal{O}(\varepsilon^2)$$

where $\bar{H}_1$ is the $\mathbb{T}^4$–average along x of H_1 at $\varepsilon = 0$.

A vanishing 1–jet merely amounts to $(p, q) = (0, 0)$ being a relative equilibrium, *i.e.* we again translate co-ordinates to $q^* = 0$. A translation in p allows to remove the p^2–term, and in case the coefficient $a_{02}(0; \mu, \omega)$ of q^2 does not vanish at $\mu = 0$ we scale p by $\sqrt{\varepsilon}$ and q by $\sqrt[4]{\varepsilon}$ to recover the quasi-periodic centre-saddle bifurcation encountered in the previous section. Finally, for nonzero $a_{11}(0; 0, \omega) pq$ we scale p and q both by ε, revealing the relative equilibrium to be hyperbolic. In the expansion

$$\bar{H}_1 \;=\; \sum_{k+l=3} \frac{a_{kl}(\hat{y}; \mu, \omega)}{k!\, l!} p^k q^l \;+\; h.o.t.$$

we therefore start with third order terms. We emphasise that for generic perturbations H_1 this cannot be avoided to happen at 1–parameter subfamilies.

Shrinking $\mathbb{Y}$ a bit, if necessary, the coefficient[1] a_{03} is bounded away from zero. Scaling p by $\varepsilon^{\frac{2}{3}}$ and q by $\varepsilon^{\frac{1}{3}}$ we obtain

$$H_\varepsilon(x, \hat{y}, p, q; \mu, \omega) \;=\; \langle \omega, \hat{y} \rangle \;+\; \varepsilon^2 \mathcal{H}(p, q; \mu, \omega) \;+\; \mathcal{O}(\varepsilon^{\frac{7}{3}})$$

with

$$\mathcal{H}(p, q; \mu, \omega) \;=\; \frac{a}{6} p^3 \;+\; \frac{b}{6} q^3 \;+\; c_1(\mu) q \;+\; c_2(\mu) p \;+\; c_3(\mu) pq$$

where $a \approx 1$, $b = 6a_{03}(0; \mu, \omega)$. The coefficient functions c_1, c_2 and c_3 vanish at $\mu = 0$, this allows to rescale a_{01} by $\varepsilon^{\frac{2}{3}}$ to yield $c_1(\mu)$ and a_{10} by $\varepsilon^{\frac{1}{3}}$ to yield $c_2(\mu)$; a_{11} is not rescaled, we simply put $c_3 = a_{11}(0; \mu, \omega)$. Note that we do not obtain the unfolding of D_4 used in [6], but a form adapted to the critical singularity $\frac{a}{6}p^3 + \frac{b}{6}q^3$, and that this form leads to the hyperbolic umbilic. Still, we conclude that the family of resonant Lagrangean tori $p = 0$ of (10) may lead to umbilical torus bifurcations of invariant 4–tori.

4 Rüssmann Hamiltonians

The example of the previous section looks in one aspect quite degenerate — the resonance $p = 0$ coincides with the hypersurface $p = 0$ where the Kolmogorov condition (6) fails. In general the (transversal) intersection of these should be a co-dimension 2 submanifold, and singularities of the equation $\det D^2 H_0 = 0$ may lead to further complications. For instance, one could consider the example in the previous section with one more degree of freedom and add[2] the terms

$$\mathrm{e}^4 y_5 \;+\; \frac{1}{2} y_5^2 \;+\; \frac{1}{6} y_5^3 \;+\; \frac{1}{2} y_5 p^2$$

to the unperturbed Hamiltonian (10). Then the analysis of the previous section concerns $y_5 = 0$ and the obvious question is how the perturbed system behaves when unfolded by y_5 — scaled by $\varepsilon^{\frac{2}{3}}$. Note that it is generic for H_0 to satisfy some form of Rüssmann non-degeneracy at every point, cf. [25].

Leaving such complications aside for the moment, a 'naive' generalization of the example in the previous section leads to

$$H_0(\hat{y}, p; \mu, \omega) \;=\; \langle \omega, \hat{y} \rangle \;+\; \frac{a(\mu)}{\ell!} p^\ell$$

with $a(\mu) \neq 0$ and $2 \leq \ell \leq n + 2$. For instance, for $\ell = 3$ the potential (8) leads to the quasi-periodic centre-saddle bifurcation (unfolding the singularity A_1) when $d = 2$. For

[1] Here we depart from the general theory of planar singularities that allows to transparently treat the relative equilibria. Indeed, the co-ordinates p and q already have a 'meaning', so we had to sharpen the usual assumption that the (homogeneous) 3–jet does not have multiple roots.

[2] Here we use that the vector $(1, \mathrm{e}, \ldots, \mathrm{e}^4)$ is Diophantine as well.

$d = 3$ this similarly leads to umbilic tori (unfolding the singularity D_4) and to the simple singularities E_6 and E_8 (see [18]) when $d = 4$ and 5, respectively.

The umbilic example of the previous section gives some confidence that it should still be possible to find adapted scalings that turn $\frac{a}{l!}p^l + \varepsilon \bar{H}_1$ for generic perturbation H_1 into versal unfoldings of occuring singularities. Note, however, that the special 'starting point' $\frac{a}{\ell!}p^\ell$ leads to a classification that may slightly differ from the classification of one-degree-of-freedom equilibria by means of planar singularities.

5 Conclusions

Summarising we may state that the for a *single* resonance the Kolmogorov condition (6) restricts the occurring singularities to the family A_k of corank–1–singularities (8), while under the weaker Rüssmann condition also singularities of corank 2 may occur. Theorem 2.1 treats the former situation, while the example in Section 3 concerns the latter case.

The corank–2–singularities D_4, D_5, D_6 and E_6 of low co-dimension are still simple and the complete family D_k, $k \in \mathbb{N}$, unfolded by the umbilics, is simple as well. Next to these also singularities with modal parameters become possible, leading to all quasi-periodic bifurcations of [19]. We note that, for these bifurcations to occur in a resonance gap, a scaling will be needed.

In the case of a k–fold resonance the above normalisation procedure applied to (3) leads to a $(N-k)$–parameter family of Hamiltonian systems defined on $\mathbb{T}^k \times \mathbb{R}^k$. Here, non-degenerate minima correspond to elliptic $(N-k)$–dimensional tori [15, 12, 24, 10]. For these cases there exist many results on quasi-periodic persistence, employing KAM Theory.

In the spirit of the present paper, one should consider degenerate minima as well. Under the Kolmogorov condition (6) this leads to corank–k–singularities. The corresponding quasi-periodic bifurcation theory still has to be developed. Under the Rüssmann condition similarly this gives rise to singularities of corank $2k$. We like to stress that none of these complications can be avoided.

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