THE DISCRETE SPECTRUM OF SELFADJOINT OPERATORS UNDER PERTURBATIONS OF VARIABLE SIGN

Oleg Safronov

Department of Mathematics
Royal Institute of Technology
SE–100 44 Stockholm
Sweden

safronov@math.kth.se

Abstract. Given two selfadjoint operators A and $V = V_+ - V_-$, we study the motion of the eigenvalues of the operator $A(t) = A - tV$ as t increases. Let $\alpha > 0$ and let λ be a regular point for A. We consider the quantity $N(\lambda, A, W_+, W_-, \alpha)$ defined as the difference between the number of the eigenvalues of $A(t)$ that pass the point λ from right to left and the number of the eigenvalues passing λ from left to right as t increases from 0 to α. We study the asymptotic behavior of $N(\lambda, A, W_+, W_-, \alpha)$ as $\alpha \to \infty$. Applications to Schrödinger and Dirac operators are given.

0. Introduction

Let $A = A^*$ be a selfadjoint operator whose spectrum $\sigma(A)$ has gaps. Let $\lambda = \overline{\lambda} \in \rho(A)$ be a fixed "observation point". We take a perturbation V of the form

$$V = W_+^* W_+ - W_-^* W_- \tag{1}$$

and put

$$A(\alpha) = A - \alpha V, \quad \alpha > 0.$$

Let $N(\alpha) = N(\lambda, A, W_+, W_-, \alpha)$ denote the difference between the number of those eigenvalues of $A(t)$ that cross λ moving leftwards as t grows from 0 to α and the similar number related to the eigenvalues moving rightwards. Our concern is with the leading term (in the power expansion) of the asymptotics of $N(\alpha)$ as $\alpha \to \infty$. We find conditions on W_+, W_- reducing the calculation of this asymptotics to the cases of positive $V = W_+^* W_+$ and negative $V = -W_-^* W_-$. In §1 we formulate the problem and describe the main result in detail.

Recall that if $V = W_+^* W_+$, then the eigenvalues of $A(\alpha)$ move leftwards. Thus, the function $N(\alpha)$ is monotonically increasing for any $\lambda = \overline{\lambda} \in \rho(A)$ and coincides with the distribution function of the positive spectrum of a compact selfadjoint operator. A suitable version of the Birman – Schwinger principle can be found in [2]. This forms a basic tool for the investigation of the case $V \geq 0$ (as well as of the case $V \leq 0$), which has been treated in [1], [2], [10] and [13]. Note that a Dirac operator has been considered in the paper [13] , and the papers [1], [2] and [10] deal with Schrödinger operators.

If V is a perturbation of variable sign, the problem becomes much more difficult. Since the motion of eigenvalues of the operator $A(\alpha)$ is no longer monotone, the study of the perturbations (1) requires a "proper" generalization of the function $N(\alpha)$. It has turned out that such a generalization is the difference between the number of eigenvalues having crossed λ in each of the two directions. We remind the reader that closely related problems for perturbations V of variable sign were considered also in the papers [11], [16]. However, another function was examined there, namely, the number $\widetilde{N}(\lambda, \alpha)$ of eigenvalues having reached an interior point λ of a gap. For $\widetilde{N}(\lambda, \alpha)$, some lower asymptotic estimates were found in [11], [16]. Somewhat different questions concerning the fine structure of the motion of the eigenvalues ("trapping and cascading") have been considered in [8]. However the paper [8] deals with one-dimensional problems only. Other literature can also be found in the references (see for example [7] and [9]).

It is interesting to compare the results of this paper with the investigations conducted by the author in [17]-[18]. The papers [17]-[18] are devoted to the case when the asymptotics of $N(\alpha)$ does not depend on λ and the effect of V_- is dominated by the effect of V_+. In the present paper we deal with depending on λ asymptotics to which both parts of the perturbation contribute.

As in the paper [17], in the present article we need certain additional technical means, namely, a special version of asymptotic perturbation theory for operator families of a specific form. The corresponding material is given in §3.

In §6 we apply Theorem 1.1 to the spectral theory of differential operators. The most natural candidates for such applications are Schrödinger operators, but the second example in this subsection deals with the Dirac operator. The starting point of the investigation of Dirac operators are the papers [13] and [3], where the case of positive V was considered.

We mention two notational conventions that are used throughout the work. If in any statement we use the double index "$\pm$", then this statement should be read separately for each of the indices "$+$" and "$-$". Sometimes references to formulae are also given analogous subscripts; for example, $(11)_+$ means that the formula (11) should be read under the index "$+$".

1. Preliminaries and formulation of the main result

1. Below $\mathfrak{H}_j$ $(j = 1,2)$ are separable Hilbert spaces. We denote by $\mathfrak{R} = \mathfrak{R}(\mathfrak{H}_1, \mathfrak{H}_2)$ the space of continuous linear operators and by $\mathfrak{S}_\infty = \mathfrak{S}_\infty(\mathfrak{H}_1, \mathfrak{H}_2)$ the space of compact operators acting from $\mathfrak{H}_1$ to $\mathfrak{H}_2$. If $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}$, we write $\mathfrak{R}(\mathfrak{H})$ and $\mathfrak{S}_\infty(\mathfrak{H})$. The symbols $D(M), \mathrm{Ran} M,\ \mathrm{Ker} M, M^*, \rho(M), \sigma(M)$ denote the domain, the range, the kernel, the adjoint operator, the resolvent set and the spectrum (respectively) of a densely defined linear operator M. Let $T \in \mathfrak{S}_\infty$. We denote by $s_k(T),\ k \in \mathbb{N}$, the singular numbers of an operator T., i.e., the consecutive eigenvalues of the operator $(T^*T)^{1/2}$, and introduce the distribution function

$$n(s,T) = \mathrm{card}\{\ k :\ \ s_k(T) > s\ \}$$

of the $s-$numbers $(s > 0)$. If $T = T^* \in \mathfrak{S}_\infty(\mathfrak{H})$, we put $n_\pm(\cdot, T) = n(\cdot, T_\pm)$, where $2T_\pm = |T| \pm T$. Clearly, $n = n_+ + n_-$.

Some statements equivalent to the inequalities of H. Weyl, Ky Fan, and Horn (see e.g.,[4]) should be mentioned. If $T_j = T_j^* \in \mathfrak{S}_\infty(\mathfrak{H}), j = 1, 2,$

then

$$n_\pm(s_1+s_2, T_1+T_2) \le n_\pm(s_1,T_1) + n_\pm(s_2,T_2), \quad s_1,s_2>0. \tag{2}$$

Similarly, for $T_j \in \mathfrak{S}_\infty(\mathfrak{H}_1,\mathfrak{H}_2), j=1,2$, we have

$$n(s_1+s_2, T_1+T_2) \le n(s_1,T_1)+n(s_2,T_2), \qquad s_1,s_2>0. \tag{3}$$

Also, if $T_1 \in \mathfrak{S}_\infty(\mathfrak{H}_3,\mathfrak{H}_2)$ and $T_2 \in \mathfrak{S}_\infty(\mathfrak{H}_1,\mathfrak{H}_3)$, then

$$n(s_1 s_2, T_1T_2) \le n(s_1,T_1)+n(s_2,T_2), \quad s_1,s_2>0. \tag{4}$$

Let $0<p<\infty$; we consider the class (ideal) $\Sigma_p \subset \mathfrak{S}_\infty$ determined by the condition

$$|T|_p^p := \sup_{s>0} s^p n(s,T) < \infty.$$

The functional $|\cdot|_p$ is a quasinorm on Σ_p. Let Σ_p^0 be the separable closed subspace of Σ_p defined as follows:

$$\Sigma_p^0 := \{\, T\in\Sigma_p : \ n(s,T)=o(s^{-p}),\ s\to 0\}.$$

The set K of finite rank operators is dense in Σ_p^0. We introduce the following functionals on Σ_p:

$$\begin{cases} \Delta_p(T) = \limsup\limits_{s\to 0} s^p n(s,T), \\ \delta_p(T) = \liminf\limits_{s\to 0} s^p n(s,T). \end{cases} \tag{5}$$

If $T=T^*$, we put

$$\begin{cases} \Delta_p^{(\pm)}(T) = \limsup\limits_{s\to 0} s^p n_\pm(s,T), \\ \delta_p^{(\pm)}(T) = \liminf\limits_{s\to 0} s^p n_\pm(s,T). \end{cases} \tag{6}$$

Each of the six functionals (5), (6) is continuous on the space Σ_p (see [4]). Moreover, they are invariant under addition of a summand of class Σ_p^0 to T.

We shall need the following result.

Proposition 1.1. *(a) Let* $T_j \in \Sigma_{q_j}$, $0 < q_j < \infty$, $j = 1, 2$. *Suppose that* $T = T_1 T_2$ *and* $q_1^{-1} + q_2^{-1} = q^{-1}$. *Then* $T \in \Sigma_q$ *and*

$$|T|_q \leq C(q_1, q_2)|T_1|_{q_1}|T_2|_{q_2}. \tag{7}$$

(b) Under the additional condition $T_1 \in \Sigma^0_{q_1}$ *(or* $T_2 \in \Sigma^0_{q_2}$*) we have*

$$T \in \Sigma^0_q.$$

2. Let $a[\cdot, \cdot]$ be a sesquilinear form in a Hilbert space $\mathfrak{H}$. We assume that its domain $d[a]$ is dense in $\mathfrak{H}$ and a is semibounded from below and closed on $d[a]$. The form a induces the self-adjoint operator A in $\mathfrak{H}$. Fix the value of $\gamma \in \mathbb{R}$, such that $a_\gamma := a + \gamma \geq 1$, i.e.

$$a_\gamma[x, x] = a[x, x] + \gamma\|x\|^2 \geq \|x\|^2, \quad x \in d[a],$$

and denote by $H_\gamma[a]$ the (complete) Hilbert space $d[a]$ with the metric form

$$a_\gamma[x, x] = \|(A + \gamma I)^{1/2}x\|^2, \quad x \in d[a].$$

Together with $\mathfrak{H}$, we shall consider some "auxiliary" Hilbert spaces $\mathfrak{G}_\pm$. In what follows, the inner products and the norms in various spaces are denoted by the symbols $(\cdot, \cdot)$ and $\|\cdot\|$ without any subscripts.

Let $W_\pm : \mathfrak{H} \to \mathfrak{G}_\pm$ be two closable linear operators, satisfying $D(W_\pm) \supset d[a]$ and

$$G_\pm := W_\pm(A + \gamma I)^{-1/2} \in \Sigma_{2p}(\mathfrak{H}, \mathfrak{G}_\pm) \tag{8}$$

for some $p \in (0, \infty)$. Put

$$v_\pm[x, y] = (W_\pm x, W_\pm y). \tag{9}$$

Then $v_\pm$ are compact on $d[a]$. This means that the $v_\pm$ are continuous on $H_\gamma[a]$ and the corresponding operators $Q_\pm$ (determined by the relations

$a_\gamma[Q_\pm x, y] = v_\pm[x, y]$ for $x, y \in d[a]$) are compact on $H_\gamma[a]$. Therefore the forms

$$a_\pm(\alpha) = a \mp \alpha v_\pm$$

are lower semibounded and closed on $d[a]$. Under the above assumptions, the difference between the resolvents of the operators A and $A_\pm(\alpha)$ is compact. Hence, the spectrum of $A_\pm(\alpha)$ is discrete in the gaps of the spectrum $\sigma(A)$ of A.

3. Let an interval $\Lambda = (\lambda_-, \lambda_+)$ be a gap in $\sigma(A)$. It is easy to check (see [2], §1), that the eigenvalues of $A_+(\alpha)$ (of $A_-(\alpha)$) move inside Λ monotonically from the right to the left (from the left to the right) as α grows.

We denote by $N_\pm(\lambda, A, W_\pm, \alpha)$ the number of eigenvalues of $A_\pm(t)$ that cross a point $\lambda \in \Lambda$ as t grows from 0 to α, and introduce the quantities

$$\Delta_p^{(\pm)}(\lambda; A, W_\pm) := \limsup_{\alpha \to \infty} \alpha^{-p} N_\pm(\lambda, A, W_\pm, \alpha), \tag{10}$$

$$\delta_p^{(\pm)}(\lambda; A, W_\pm) := \liminf_{\alpha \to \infty} \alpha^{-p} N_\pm(\lambda, A, W_\pm, \alpha) \tag{11}$$

(these quantities are finite by (8)). Below we shall always suppose that

$$\Delta_p^{(\pm)}(\lambda; A, W_\pm) = \delta_p^{(\pm)}(\lambda; A, W_\pm) =: J_p^{(\pm)}. \tag{12}$$

Further explanations concerning the material of this subsection can be found in §2.

4. Let a form a and an operator A be as above, and let $v_\pm$ be the forms defined in (9); we assume that condition (8) is fulfilled. The form

$$v = v_+ - v_- \tag{13}$$

is compact on $d[a]$. We introduce a family of lower semibounded closed forms $a(\alpha)$ on $d[a]$:

$$a(\alpha) = a - \alpha v, \quad \alpha > 0. \tag{14}$$

Below, $A(\alpha)$ denotes the self-adjoint operator on $\mathfrak{H}$, which corresponds to the form (14). Since the difference of the resolvents of A and $A(\alpha)$ is compact, the spectrum of $A(\alpha)$ in the gaps of $\sigma(A)$ is discrete. Let the interval $\Lambda = (\lambda_-, \lambda_+)$ be a gap in the spectrum $\sigma(A)$. We fix an "observation point" λ,

$$\lambda_- < \lambda < \lambda_+.$$

Assume that λ is an eigenvalue of multiplicity k of the operator $A(\alpha_0)$ for some $\alpha_0 > 0$. Then one can choose a numbering of the eigenvalues $\lambda_j(\alpha)$, $j = 1, \dots, k$, of $A(\alpha)$ such that they are real-analytic functions of α near α_0 and

$$\lambda_j(\alpha_0) = \lambda, \quad j = 1, \dots, k.$$

We suppose that the multiplicity of recurrence of some value in the set $\{\lambda_j(\alpha)\}_{j=1}^k$ coincides with the multiplicity of the respective eigenvalue.

Let us choose a suitable neighborhood of the point α_0. Since none of the functions λ_j is constant, the zeros of the derivatives $d\lambda_j/d\alpha$ are isolated. Hence, we can choose such a neighborhood of α_0 that

$$d\lambda_j(\alpha)/d\alpha \neq 0 \;\text{ for } \alpha \neq \alpha_0, \quad j \in \overline{1,k}.$$

Assume that among these λ_j exactly k_+ functions decrease and k_- functions increase in the chosen neighborhood (non-monotone functions are not counted). Then we say, that k_+ eigenvalues pass the point λ from the right to the left, and k_- eigenvalues pass the point λ from the left to the right, as the coupling constant α increases near α_0.

By $N(\lambda, A, W_+, W_-, \alpha)$ we denote the difference between the number of eigenvalues of $A(t)$, which pass the point $\lambda \in \Lambda$ from the right to the left, and the number of eigenvalues of $A(t)$, which pass the point $\lambda \in \Lambda$ from the left to the right, as t increases from 0 to α, excluding α. In other words, we sum up the differences $k_+ - k_-$ over all $t \in (0, \alpha)$ for which $\lambda \in \sigma(A(t))$. It is easily seen that such t's are isolated; so, there are finitely many of them in the interval $(0, \alpha)$. The sum obtained is $N(\lambda, A, W_+, W_-, \alpha)$, and this quantity is continuous from the left with respect to α for every fixed $\lambda \in \Lambda$.

If $v > 0$ ($v < 0$), then the eigenvalues of $A(\alpha)$ are monotonically decreasing (monotonically increasing) as α grows. In this case, $|N(\lambda, A, W_+, W_-, \alpha)|$ coincides with the number of the eigenvalues of $A(t)$ that cross λ as t grows from 0 to α.

The condition (8) guarantees that the quantities

$$\Delta_p(\lambda; A, W_+, W_-) = \limsup_{\alpha \to \infty} \alpha^{-p} N(\lambda, A, W_+, W_-, \alpha), \tag{15}$$

$$\delta_p(\lambda; A, W_+, W_-) = \liminf_{\alpha \to \infty} \alpha^{-p} N(\lambda, A, W_+, W_-, \alpha) \tag{16}$$

take finite values.

5. Let $W_\pm$ be the same as in Subsection 2 of §1, and let condition (8) be fulfilled. For $\lambda \in \rho(A)$ we consider the operators

$$X_\lambda(A) := G_+(A + \gamma I)(A - \lambda I)^{-1} G_-^*,$$

$$X_\lambda^\pm(A) := G_\pm(A + \gamma I)(A - \lambda I)^{-1} G_\pm^*$$

(the operators $G_\pm$ are defined in (8)). By (8), these operators are of class Σ_p. If $\lambda = \overline{\lambda}$, the operators $X_\lambda^\pm(A)$ are selfadjoint in $\mathfrak{G}_\pm$. If Λ is a gap in $\sigma(A)$, then $X_\lambda(A)$, $X_\lambda^\pm(A)$ admit the following equivalent definition:

$$X_\lambda(A) g = W_+(A - \lambda I)^{-1} W_-^* g, \quad g \in \mathcal{D}(W_-^*), \tag{17}$$

$$X_\lambda^\pm(A) g = W_\pm(A - \lambda I)^{-1} W_\pm^* g, \quad g \in \mathcal{D}(W_\pm^*). \tag{18}$$

Now we are ready to formulate our main result on the quantities (15), (16). Recall that the quantities $J_p^{(\pm)}$ are defined in (12).

Theorem 1.1. *Let* $\lambda = \overline{\lambda} \in \rho(A)$ *and let the form* v *in* (14) *be defined in* (13), *where* $v_\pm$ *are the forms given by* (9). *Assume that relations* (8) *and* (12) *are fulfilled. Finally, let*

$$X_\lambda(A) \in \Sigma_p^0. \tag{19}$$

Then the identity

$$\Delta_p(\lambda; A, W_+, W_-) = \delta_p(\lambda; A, W_+, W_-) = J_p^{(+)} - J_p^{(-)} \tag{20}$$

holds true.

The relevant auxiliary material is presented in §§2-4; the proof is completed in §5.

2. The Birman-Schwinger principle

1. Let $A_\pm(\alpha)$ be the same as in §1. It is well known that the description of the discrete spectrum of $A_\pm(\alpha)$ can be reduced to the study of the spectrum of the compact operator $X_\lambda^\pm(A)$. We borrow from [2] a suitable version of this reduction.

Proposition 2.1. *Let* $\lambda = \overline{\lambda} \in \rho(A)$. *The following two statements are equivalent:*

1)λ *is an eigenvalue of multiplicity* k *for the operator* $A_\pm(\alpha)$*;*

2)the point $\pm\alpha^{-1}$ *is an eigenvalue of multiplicity* k *for the operator* $X_\lambda^\pm(A)$.

The next assertion is a direct consequence of Proposition 2.1.

Proposition 2.2. *If* $\lambda = \overline{\lambda} \in \rho(A)$, *then*

$$N_{\pm}(\lambda, A, W_{\pm}, \alpha) = n_{\pm}(\alpha^{-1}, X_{\lambda}^{\pm}(A)), \quad \alpha > 0. \tag{21}$$

Assume that condition (8) is fulfilled. Then $X_{\lambda}^{\pm}(A) \in \Sigma_p$ if $\lambda = \overline{\lambda} \in \rho(A)$. Now (21) ensures that the quantities (10), (11) are finite, because

$$\Delta_p^{(\pm)}(\lambda; A, W_{\pm}) = \Delta_p^{(\pm)}(X_{\lambda}^{\pm}(A)),$$

$$\delta_p^{(\pm)}(\lambda; A, W_{\pm}) = \delta_p^{(\pm)}(X_{\lambda}^{\pm}(A)). \tag{22}$$

2. Let $A, A_{\pm}(\alpha), A(\alpha)$ be the same as in §1. We fix $\lambda = \overline{\lambda} \in \rho(A)$ and introduce the set

$$\mathcal{Y}_{\pm} = \{\ \alpha > 0 : \lambda \in \rho(A_{\pm}(\alpha))\}.$$

Along with the operators $X_{\lambda}^{\pm}(A)$, we can consider the operators $X_{\lambda}^{\pm}(A_{\mp}(\alpha)), \alpha \in \mathcal{Y}_{\mp}$ (with similar definition). These operators are bounded and selfadjoint on $\mathfrak{G}_{\pm}$; from (8) it follows that $X_{\lambda}^{\pm}(A_{\mp}(\alpha)) \in \Sigma_p$.

We describe a way of reducing the study of the discrete spectrum of $A(\alpha)$ to that of the spectrum of the compact operators discussed in this paragraph. This reduction is based on the following assertion (see [17]).

Proposition 2.3. **a)** *Let* $\lambda \in \rho(A(\alpha)) \cap \rho(A_{-}(\alpha)) \cap \Lambda,\ \alpha > 0$. *Then*

$$\begin{gathered} N(\lambda, A, W_{+}, W_{-}, \alpha) = \\ n_{+}(s, X_{\lambda}^{+}(A_{-}(\alpha))) - n_{-}(s, X_{\lambda}^{-}(A)), \quad s\alpha = 1. \end{gathered} \tag{23}$$

b) *Let* $\lambda \in \rho(A(\alpha)) \cap \rho(A_{+}(\alpha)) \cap \rho(A), \alpha > 0$. *Then*

$$N(\lambda, A, W_{+}, W_{-}, \alpha) = n_{+}(s, X_{\lambda}^{+}(A)) - n_{-}(s, X_{\lambda}^{-}(A_{+}(\alpha))), \quad s\alpha = 1.$$

3. Operator-valued functions (mappings) of classes $\mathcal{S}_p$ and $\mathcal{S}_p^0$

In this section we construct some generalizations of the classes Σ_p and Σ_p^0 needed in what follows. Our generalizations are related to the replacement of "individual" operators by certain mappings of the form

$$\mathcal{T} : \mathbb{R}_+ \to \mathfrak{S}_\infty(\mathfrak{H}_1, \mathfrak{H}_2). \tag{24}$$

1. For $0 < p < \infty$, we introduce the class $\mathcal{S}_p(\mathfrak{H}_1, \mathfrak{H}_2)$ of mappings (24) for which the quantity

$$\sup_{s>0} s^p n(s, \varepsilon \mathcal{T}(s^{-1})) \tag{25}$$

is finite for every $\varepsilon > 0$. It is clear that $\mathcal{S}_p$ is a linear set (linear space). We are not going to dwell on a possibility of supplying the space $\mathcal{S}_p$ with a quasinorm. Keeping the previous notation, we introduce some natural analogs of the functionals (5),(6) on the space $\mathcal{S}_p$. For $\mathcal{T} \in \mathcal{S}_p(\mathfrak{H}_1, \mathfrak{H}_2)$ we put

$$\begin{cases} \Delta_p(\mathcal{T}) = \limsup\limits_{s\to 0} s^p n(s, \mathcal{T}(s^{-1})), \\ \delta_p(\mathcal{T}) = \liminf\limits_{s\to 0} s^p n(s, \mathcal{T}(s^{-1})). \end{cases} \tag{26}$$

If $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}$ and

$$\mathcal{T}(\alpha) = \mathcal{T}(\alpha)^*, \quad \alpha > 0, \tag{27}$$

then we put

$$\begin{cases} \Delta_p^{(\pm)}(\mathcal{T}) = \Delta_p(\mathcal{T}_\pm), \\ \delta_p^{(\pm)}(\mathcal{T}) = \delta_p(\mathcal{T}_\pm). \end{cases} \tag{28}$$

where $2\mathcal{T}_\pm(\cdot) = |\mathcal{T}(\cdot)| \pm \mathcal{T}(\cdot)$. The subspace of $\mathcal{S}_p$ determined by the condition

$$\Delta_p(\varepsilon\mathcal{T}) = 0 \ \text{ for any } \varepsilon > 0$$

will be denoted by $\mathcal{S}_p^0$.

2. Now we discuss the simplest properties of the functionals (26), (28). In the following statement we consider products of two mappings of the form (24).

Proposition 3.1. *Let* $\mathcal{T}_1 \in \mathcal{S}_p(\mathfrak{H}_3, \mathfrak{H}_2)$, $\mathcal{T}_2 \in \mathcal{S}_p(\mathfrak{H}_1, \mathfrak{H}_3)$, $\mathcal{T}(\alpha) = \alpha\mathcal{T}_1(\alpha)\mathcal{T}_2(\alpha)$. *Then* $\mathcal{T} \in \mathcal{S}_p(\mathfrak{H}_1, \mathfrak{H}_2)$ *and for every* $\varepsilon > 0$

$$\Delta_p(\mathcal{T}) \leq \Delta_p(\varepsilon^{-1}\mathcal{T}_1) + \Delta_p(\varepsilon\mathcal{T}_2). \tag{29}$$

We finish this subsection with considering the behavior of the quantities (26), (28) under additive perturbations of class $\mathcal{S}_p^0$. Full analogy with the case of the functionals (5), (6) cannot be achieved here, because the functionals (26), (28) are not homogeneous relative to multiplication of $\mathcal{T}$ by constants; we are forced to compensate this shortcoming by imposing an additional requirement on $\mathcal{T}$.

Proposition 3.2. *Assume that the mappings* $\mathcal{T} \in \mathcal{S}_p(\mathfrak{H})$ *and* $\mathcal{T}_0 \in \mathcal{S}_p^0(\mathfrak{H})$ *satisfy condition* (27). *Then*

a) *if* $\lim_{t\to 1} \Delta_p^{(\pm)}(t\mathcal{T}) = \Delta_p^{(\pm)}(\mathcal{T})$, *then* $\Delta_p^{(\pm)}(\mathcal{T} + \mathcal{T}_0) = \Delta_p^{(\pm)}(\mathcal{T})$;

b) *if* $\lim_{t\to 1} \delta_p^{(\pm)}(t\mathcal{T}) = \delta_p^{(\pm)}(\mathcal{T})$, *then* $\delta_p^{(\pm)}(\mathcal{T} + \mathcal{T}_0) = \delta_p^{(\pm)}(\mathcal{T})$.

Proposition 3.3. *Let* $\mathcal{T} \in \mathcal{S}_p(\mathfrak{H}_1, \mathfrak{H}_2)$, $\mathcal{T}_0 \in \mathcal{S}_p^0(\mathfrak{H}_1, \mathfrak{H}_2)$.

a) *If* $\lim_{t\to 1} \Delta_p(t\mathcal{T}) = \Delta_p(\mathcal{T})$, *then* $\Delta_p(\mathcal{T} + \mathcal{T}_0) = \Delta_p(\mathcal{T})$.

b) *If* $\lim_{t\to 1} \delta_p(t\mathcal{T}) = \delta_p(\mathcal{T})$, *then* $\delta_p(\mathcal{T} + \mathcal{T}_0) = \delta_p(\mathcal{T})$.

The proof of Propositions 3.1-3.3 can be found in [17].

3. We present some sufficient conditions for a mapping of the form (24) to be of class $\mathcal{S}_p^0$.

Let $T_1 \in \mathfrak{S}_\infty$, and let $\mathcal{T}_2$ be a mapping of the form (24). We formulate some conditions ensuring that the mapping

$$\mathcal{T}(\alpha) = \alpha T_1 \mathcal{T}_2(\alpha), \quad \alpha > 0, \tag{30}$$

belongs to the class $\mathcal{S}_p^0$.

Proposition 3.4. *Let $\mathcal{T}$ be the mapping (30) where $T_1 \in \Sigma_p^0(\mathfrak{H}_3, \mathfrak{H}_2)$ and $\mathcal{T}_2 \in \mathcal{S}_p(\mathfrak{H}_1, \mathfrak{H}_3)$. Assume that*

$$\lim_{\varepsilon \to 0} \Delta_p(\varepsilon \mathcal{T}_2) = 0. \tag{31}$$

Then $\mathcal{T} \in \mathcal{S}_p^0(\mathfrak{H}_1, \mathfrak{H}_2)$.

Proof. Proposition 3.1 readily implies that $\mathcal{T} \in \mathcal{S}_p$. Next, we make the substitution $\mathcal{T}_1 \mapsto \eta^{1/2} T_1$, $\mathcal{T}_2 \mapsto \eta^{1/2} \mathcal{T}_2$ in (29) to obtain

$$\Delta_p(\eta \mathcal{T}) \le \Delta_p(\varepsilon^{-1} \eta^{1/2} T_1) + \Delta_p(\varepsilon \eta^{1/2} \mathcal{T}_2), \quad \varepsilon, \eta > 0.$$

This leads to the estimate $\Delta_p(\eta \mathcal{T}) \le \Delta_p(\varepsilon \eta^{1/2} \mathcal{T}_2)$; letting $\varepsilon \to 0$, from (31) we see that $\Delta_p(\eta \mathcal{T}) = 0$ for every $\eta > 0$. $\square$

The next result is deduced from Proposition 3.4.

Proposition 3.5. *Let $T \in \Sigma_p^0(\mathfrak{H}_3, \mathfrak{H}_2)$, $\mathcal{T} \in \mathcal{S}_p(\mathfrak{H}_1, \mathfrak{H}_3)$ and*

$$\lim_{\varepsilon \to 0} \Delta_p(\varepsilon \mathcal{T}) = 0. \tag{32}$$

Then the mapping

$$\mathbb{R}_+ \ni \alpha \mapsto \alpha^2 T \mathcal{T}(\alpha) T^* \in \mathfrak{S}_\infty$$

belongs to the class $\mathcal{S}_p^0$.

Proof. The singular numbers of T and T^* coincide. Therefore, the conditions of type $T \in \Sigma_p^0$, $\mathcal{T} \in \mathcal{S}_p^0$, etc., are invariant under conjugation.

Moreover, passing to the adjoint operators in (30), we see that the factors T_1 and $\mathcal{T}_2$ in Propositions 3.4 may be interchanged.

In Proposition 3.4, we make the substitution $T_1 \mapsto T$, $\mathcal{T}_2 \mapsto \mathcal{T}$. As a result we obtain that the mapping $\mathcal{F}_1(\alpha) = \alpha T \mathcal{T}(\alpha)$ belongs to $\mathcal{S}_p^0$. For the mapping $\mathcal{F}_2(\alpha) = \alpha \mathcal{F}_1(\alpha) T^*$ the relation $\mathcal{F}_2 \in \mathcal{S}_p^0$ follows from Proposition 3.4 if we interchange the factors. $\square$

4. So far, we have dealt with the mappings (24) on $\mathbb{R}_+$. In what follows we shall need some mappings of the form

$$\mathcal{T} : \mathcal{Y} \longrightarrow \mathfrak{S}_\infty(\mathfrak{H}_1, \mathfrak{H}_2), \tag{33}$$

where $\mathcal{Y}$ is a fixed set dense in $\mathbb{R}_+$. All the said above remains valid for such mappings. When referring, we shall assume that $\mathbb{R}_+$ is changed for $\mathcal{Y}$ in Proposition 3.1-3.7. Below, the role of $\mathcal{Y}$ will be played by the half-axis $\mathbb{R}_+$ from which some sequence $\alpha_k \to \infty$ is deleted. The dependence of the classes $\mathcal{S}_p(\mathfrak{H}_1, \mathfrak{H}_2)$, $\mathcal{S}_p^0(\mathfrak{H}_1, \mathfrak{H}_2)$ on $\mathcal{Y}$ will not be indicated explicitly.

4. Some special operator-valued functions

Let $A, A_-(\alpha)$ be the same as in §§1,2. We fix $\lambda = \overline{\lambda} \in \rho(A)$. In this section we consider the mappings (33); $\mathcal{Y}$ will be a fixed set satisfying the conditions $\overline{\mathcal{Y}} = \mathbb{R}_+$ and

$$\mathcal{Y} \subset \{\ \alpha > 0 : \lambda \in \rho(A_-(\alpha))\ \}.$$

The set $\mathcal{Y}$ will be further specified in §5.

1. Our nearest goal is to investigate the mapping

$$\mathcal{T}_0 : \mathcal{Y} \ni \alpha \mapsto X_\lambda^+(A_-(\alpha)) - X_\lambda^+(A) \in \Sigma_p. \tag{34}$$

Theorem 4.1. *Assume that the conditions* (8), (19) *are fulfilled. If* $\mathcal{T}_0$ *is defined as in* (34), *then*

$$\mathcal{T}_0 \in \mathcal{S}_p^0(\mathfrak{G}_+). \tag{35}$$

The proof of this theorem will be presented in the next subsection; here we establish two preliminary statements.

Proposition 4.1. *Let* $\alpha \in \mathcal{Y}$ *and* $s > \alpha^{-1}$. *Then*

$$\begin{aligned} &n(s, X_\lambda^-(A_-(\alpha))) \\ &\quad = N_-(\lambda, A, W_-, \alpha + s^{-1}) - N_-(\lambda, A, W_-, \alpha - s^{-1} + 0). \end{aligned} \tag{36}$$

Proof. We fix a point $\alpha \in \mathcal{Y}$ and view $A_-(\alpha)$ as an unperturbed operator; a perturbation is introduced via the forms $\mp\beta v_-, \beta > 0$. The parameter β plays the part of a coupling constant. Then, the perturbed operator is $A_-(\alpha \mp \beta)$. Proposition 2.2 implies that

$$\dim \operatorname{Ker}(X_\lambda^-(A_-(\alpha)) \mp \beta^{-1} I) = \dim \operatorname{Ker}(A_-(\alpha \mp \beta) - \lambda I).$$

Therefore,

$$\begin{aligned} &n(s, X_\lambda^-(A_-(\alpha))) \\ &\qquad\qquad = n_+(s, X_\lambda^-(A_-(\alpha))) + n_-(s, X_\lambda^-(A_-(\alpha))) = \\ &= \sum_{0<\beta<s^{-1}} \dim \operatorname{Ker}(A_-(\alpha - \beta) - \lambda I) + \sum_{0<\beta<s^{-1}} \dim \operatorname{Ker}(A_-(\alpha + \beta) - \lambda I) \\ &\qquad = \sum_{\alpha - s^{-1} < t < \alpha + s^{-1}} \dim \operatorname{Ker}(A_-(t) - \lambda I) \\ &\qquad = N_-(\lambda, A, W_-, \alpha + s^{-1}) - N_-(\lambda, A, W_-, \alpha - s^{-1} + 0). \end{aligned}$$

□

The next proposition deals with the mapping

$$\Pi(\alpha) = X_\lambda^-(A_-(\alpha)), \ \alpha \in \mathcal{Y}, \tag{37}$$

and is a consequence of $(12)_-$ and (36).

Proposition 4.2. *Let* Π *be defined by* (37) *and* $0<\varepsilon<1$. *Then*

$$\Delta_p(\varepsilon\Pi) = ((1+\varepsilon)^p - (1-\varepsilon)^p)J_p^{(+)}.$$

In particular

$$\lim_{\varepsilon\to 0}\Delta_p(\varepsilon\Pi) = 0. \tag{38}$$

2. Proof of Theorem 4.1. We shall rely on the following assertion which is proved in [17].

Proposition 4.3. *Let* $\alpha\in\mathcal{Y}$. *Then*

$$\begin{aligned} X_\lambda^+(A_-(\alpha)) =& X_\lambda^+(A) - \alpha X_\lambda(A)X_\lambda(A)^* \\ &+ \alpha^2 X_\lambda(A)X_\lambda^-(A_-(\alpha))X_\lambda(A)^*. \end{aligned} \tag{39}$$

From (19) we see that $R := X_\lambda(A)X_\lambda(A)^* \in \Sigma^0_{p/2},$. Therefore, $n(\alpha^{-1},\alpha\varepsilon R) = o(\alpha^p)$ as $\alpha\to\infty$ ($\varepsilon>0$).

It remains to show that the mapping

$$\mathcal{Y}\ni\alpha\mapsto\alpha^2 X_\lambda(A)X_\lambda^-(A_-(\alpha))X_\lambda(A)^*$$

is of class $\mathcal{S}_p^0(\mathfrak{G}_+)$. Using (19) and (38) we see that this is a consequence of Proposition 3.5. □

4. Under the assumptions of Theorem 4.1 consider the mapping

$$\mathcal{L}_\lambda : \mathcal{Y}\ni\alpha\mapsto X_\lambda^+(A_-(\alpha))\in\Sigma_p. \tag{40}$$

The following result is of importance for our consideration.

Theorem 4.2. *Assume that conditions* (8), (12) *and* (19) *are fulfilled. If* $\mathcal{L}_\lambda$ *is defined as in* (40), *then*

$$\Delta_p^{(+)}(\mathcal{L}_\lambda) = \delta_p^{(+)}(\mathcal{L}_\lambda) = J_p^{(+)}. \tag{41}$$

Proof. We employ Proposition 3.2 with $\mathcal{T}(\alpha) = X_\lambda(A)$ and $\mathcal{T}_0$ from (34). The condition $\mathcal{T}_0 \in \mathcal{S}_p^0$ follows from Theorem 4.1. Also, we need to verify that

$$\lim_{t\to 1} \Delta_p^{(+)}(tX_\lambda(A)) = \Delta_p^{(+)}(X_\lambda(A)), \tag{42}$$

$$\lim_{t\to 1} \delta_p^{(+)}(tX_\lambda(A)) = \delta_p^{(+)}(X_\lambda(A)). \tag{43}$$

But these relations follow from the continuity of the functionals $\Delta_p^{(+)}, \delta_p^{(+)}$ on Σ_p. □

5. The proof of the main theorem (Theorem 1.1)

1. Proposition 2.3 implies that the two-sided estimate

$$-n_-(s, X_\lambda^-(A)) \le N(\lambda, A, W_+, W_-, \alpha) \le n_+(s, X_\lambda^+(A)),$$
$$\alpha > 0,\ s\alpha = 1,\ \lambda \in \rho(A(\alpha)) \cap \rho(A_-(\alpha)) \cap \rho(A_+(\alpha)) \cap \Lambda,$$

is valid for all $\alpha > 0$ except for some sequence $\alpha_k \to \infty$. Since N is continuous from the left with respect to α, the following estimate is valid for all $\alpha > 0$:

$$\begin{aligned} -n_-(s+0, X_\lambda^-(A)) &\le N(\lambda, A, W_+, W_-, \alpha) \\ &\le n_+(s+0, X_\lambda^+(A)), \qquad s\alpha = 1. \end{aligned} \tag{44}$$

We have $X_\lambda^\pm(A) \in \Sigma_p$ by (8); therefore, (44) ensures that the quantities (15), (16) are finite.

2. Proof of Theorem 1.1. We put

$$\mathcal{Y} = \{\alpha > 0 : \lambda \in \rho(A(\alpha)) \cap \rho(A_-(\alpha))\}.$$

The set $\mathcal{Y}$ is obtained from $\mathbb{R}_+$ by deleting two sequences tending to infinity. Since $N(\alpha) = N(\lambda, A, W_+, W_-, \alpha)$ is a left continuous function of α, for any $\alpha_0 > 0$ we have

$$\sup_{\alpha > \alpha_0} \alpha^{-p} N(\alpha) = \sup_{\alpha > \alpha_0, \alpha \in \mathcal{Y}} \alpha^{-p} N(\alpha),$$

$$\inf_{\alpha>\alpha_0} \alpha^{-p} N(\alpha) = \inf_{\alpha>\alpha_0, \alpha\in\mathcal{Y}} \alpha^{-p} N(\alpha).$$

Consequently, for the calculation of the functionals (15), (16) it suffices to consider $\alpha \in \mathcal{Y}, \alpha \to \infty$. On the other hand, from (23) we deduce that

$$\Delta_p(\lambda; A, W_+, W_-) \le \Delta_p^{(+)}(\mathcal{L}_\lambda) - \delta_p^{(-)}(X_\lambda^-(A)),$$
$$\delta_p(\lambda; A, W_+, W_-) \ge \delta_p^{(+)}(\mathcal{L}_\lambda) - \Delta_p^{(-)}(X_\lambda^-(A)).$$

Now the required relation (20) follows immediately from formulae $(12)_-$ and (41). □

6. Applications to differential operators

In this section we present some applications of Theorem 1.1 to the study of differential operators.

1. Below we write $\int = \int_{\mathbb{R}^d}$. We denote $D_j = -i\frac{\partial}{\partial x_j}$, $D = -i\nabla = (D_1, \dots, D_d)$, $B_1 = \{x \in \mathbb{R}^d : |x| < 1\}$ and $\omega_d = \mathrm{vol} B_1$. By $H^s(\mathbb{R}^d)$ we denote the Sobolev classes of order $s \in \mathbb{N}$.

For a measurable function $u : \mathbb{R}^d \to \mathbb{C}$ and $t > 0$, we set

$$\mu_u(t) = \mathrm{meas}\{x \in \mathbb{R}^d : \ |u(x)| > t\}.$$

We say that u is in the class $L_{p,\infty}(\mathbb{R}^d)$, $0 < p < \infty$, if the following quasinorm is finite

$$\|u\|_{p,\infty} := \sup_{t>0} t^p \mu_u(t) < \infty.$$

The subspace $\{u \in L_{p,\infty}(\mathbb{R}^d) : \ \mu_u(t) = o(t^{-p}),\ t \to 0,\ t \to \infty\}$ is denoted by $L^0_{p,\infty}(\mathbb{R}^d)$. Finally, χ_r, $r \in \mathbb{R}$, is the operator of multiplication by the function

$$\chi_r(x) := (1 + |x|^2)^{-r/2}, \quad x \in \mathbb{R}^d. \tag{45}$$

2. Let $d \geq 2$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a real function. Assume that this function is (Lebesgue) measurable and that

$$f \in L_\infty(\mathbb{R}^d); \tag{46}$$

$$f(x+n) = f(x), \quad x \in \mathbb{R}^d, \quad n \in \mathbb{Z}^d. \tag{47}$$

In our first example we set $\mathfrak{H} = \mathfrak{G}_\pm = L_2(\mathbb{R}^d)$, $d[a] = H^1(\mathbb{R}^d)$, and

$$a[u,u] = \int (|\nabla u|^2 + f(x)|u|^2)dx, \quad u \in d[a]. \tag{48}$$

Let A be the selfadjoint operator in $\mathfrak{H}$ generated by the form a.

Then A is a Schrödinger operator, i.e., $A = -\triangle + f(x)$. Without loss of generality we shall assume that $f \geq 0$ and therefore $A > 0$. Let V be a measurable real-valued function, and

$$v[u,u] = \int V(x)|u|^2 dx. \tag{49}$$

Also, let $W_\pm$ be the operators of multiplication by the functions $V_\pm^{1/2}$, where $2V_\pm = |V| \pm V$. Then the forms v_+ and v_- defined as in (9) satisfy (13). The following condition is imposed on V so as to ensure (8):

$$V(x) = \Psi(\theta)\chi_s(x), \quad \theta = x/|x|,$$

where $0 < s < 2$ and

$$\Psi \in L_p(\mathbb{S}^{d-1}), \quad p = d/s. \tag{50}$$

Denote by $\|\cdot\|_p$ the norm in $L_p(\mathbb{S}^{d-1})$. Then

$$|W_\pm(A+I)^{-1/2}|_{2p} \leq C\|\Psi_\pm\|_p^{1/2}. \tag{51}$$

Notice that for $f = 0$ this inequality simply follows from the Cwikel [6] estimate:

Proposition 6.1. *(a) Let Φ be the operator of Fourier transformation, $W \in L_{q,\infty}(\mathbb{R}^d)$ and $b \in L_q(\mathbb{R}^d)$, $q > 2$. Then the operator $T = W\Phi^* b$ belongs to the class $\Sigma_q(\mathfrak{H})$ and the following estimate holds*

$$|T|_q \leq C(q,d)\|W\|_{q,\infty}\|b\|_{L_q}. \tag{52}$$

(b) Under the additional condition

$$W \in L^0_{q,\infty}(\mathbb{R}^d)$$

we have

$$T \in \Sigma^0_q(\mathfrak{H}).$$

Putting $W(x) = W_\pm(x),\ b(\xi) = (|\xi|^2+1)^{-1/2},\ q = 2d/s$, we obtain (51) for $f = 0$. In the general situation (51) follows from the relation

$$(-\Delta + I)^{1/2}(A+I)^{-1/2} \in \mathfrak{R}(\mathfrak{H}).$$

Proposition 6.2. *Let a, v be defined as in* (48), (49), *and let* (50) *be fulfilled. Then*

$$X_\lambda(A) \in \Sigma^0_p. \tag{53}$$

Proof. Let us introduce the following functions

$$\mathcal{F}_\pm = \Psi_\pm^{1/2}, \qquad \Psi_\pm = (|\Psi| \pm \Psi)/2.$$

Substituting $q_1 = q_2 = 2p$, $T_1 = W_+|A-\lambda I|^{-1/2}\mathrm{sgn}(A-\lambda I)$, $T_2 = W_-|A-\lambda I|^{-1/2}$ in (7) and employing (51) with $W = W_+$ or $W = W_-$, we obtain

$$|X_\lambda(A)|_p \leq C\|\mathcal{F}_+\|_{2p}\|\mathcal{F}_-\|_{2p}, \tag{54}$$

where $\|\cdot\|_{2p}$ denotes the norm in $L_{2p}(\mathbb{S}^{d-1})$.

Let $\widetilde{\mathcal{F}}_\pm \in C^\infty(\mathbb{S}^{d-1})$, $\widetilde{\mathcal{F}}_\pm \geq 0$, and let $\widetilde{X}_\lambda(A)$ denotes the operator $X_\lambda(A)$ with the replacement of $\mathcal{F}_+$, $\mathcal{F}_-$ by $\widetilde{\mathcal{F}}_+$, $\widetilde{\mathcal{F}}_-$. Put also

$$\widetilde{W}_\pm(x) = \widetilde{\mathcal{F}}_\pm(\theta)\chi_s(x)^{1/2}.$$

Then

$$|X_\lambda(A) - \widetilde{X}_\lambda(A)|_p \leq |W_+(A-\lambda I)^{-1}(W_- - \widetilde{W}_-)^*|_p$$
$$+ |(W_+ - \widetilde{W}_+)(A-\lambda I)^{-1}\widetilde{W}_-^*|_p.$$

On the right we have the quasinorms of some operators of type $X_\lambda(A)$, but with W_+ and W_- changed. Since estimates of the form (54) are applicable to such operators, we obtain

$$|X_\lambda(A) - \widetilde{X}_\lambda(A)|_p$$
$$\leq c\big(\|\mathcal{F}_+\|_{2p}\|\mathcal{F}_- - \widetilde{\mathcal{F}}_-\|_{2p} + \|\mathcal{F}_+ - \widetilde{\mathcal{F}}_+\|_{2p}\|\widetilde{\mathcal{F}}_-\|_{2p}\big). \tag{55}$$

The functions $\mathcal{F}_\pm$ can be approximated in $L_{2p}(\mathbb{S}^{d-1})$ (as closely as we wish) by functions $\widetilde{\mathcal{F}}_\pm$, and we may additionally require that $\widetilde{\mathcal{F}}_+\widetilde{\mathcal{F}}_- = 0$. By (55), this shows that the operator $X_\lambda(A)$ admits approximation in the class Σ_p by operators of type $\widetilde{X}_\lambda(A)$. Thus, it suffices to establish (53) in the case where $\mathcal{F}_+$, $\mathcal{F}_- \in C^\infty(\mathbb{S}^{d-1})$ and

$$\mathcal{F}_+\mathcal{F}_- = 0. \tag{56}$$

In what follows it is important that $\mathcal{D}(A) = H^2(\mathbb{R}^d)$. Let $\mathcal{F}_0$ be the operator of multiplication by the function $\mathcal{F}_0(x) = \mathcal{F}_-(\theta)\rho(x)$, where ρ is a fixed real function such that $1-\rho \in C_0^\infty(\mathbb{R}^d)$, $0 \notin \operatorname{supp}\rho$. In order to find the commutator of the operators $(A-\lambda I)^{-1}$ and $\mathcal{F}_0$, we note that $[B^{-1}, \mathcal{F}_0] = -B^{-1}[B, \mathcal{F}_0]B^{-1}$, where $B = A - \lambda I$. It is easily seen that $[B, \mathcal{F}_0]$ is a first order differential operator with smooth coefficients decaying as $|x|^{-1}$ when $|x| \to \infty$. Therefore,

$$Z := \chi_{-1}[B, \mathcal{F}_0](A+I)^{-1/2} \in \mathfrak{R},$$

where χ_{-1}, is the operator of multiplication by the function (45) with $s=-1$. Later we shall prove the inclusion

$$\Pi := W_+(A-\lambda I)^{-1}(W_- - \mathcal{F}_0\chi_s) \in \Sigma_p^0. \tag{57}$$

Therefore we need only to show that $W_+(A-\lambda I)^{-1}\mathcal{F}_0\chi_s \in \Sigma_p^0$. The relation (56) yields

$$\begin{aligned} W_+(A-\lambda I)^{-1}\mathcal{F}_0\chi_s &= -W_+B^{-1}\chi_1(\chi_{-1}[B,\mathcal{F}_0])B^{-1}\chi_s \\ &= -W_+B^{-1}\chi_1 Z(A+I)^{1/2}B^{-1}\chi_s. \end{aligned} \tag{58}$$

By (51) with $W=\chi_s$ we have $(A+I)^{1/2}B^{-1}\chi_s = (A+I)B^{-1}(\chi_s(A+I)^{-1/2})^* \in \Sigma_{2p}$. Moreover, since $W_+B^{-1}\chi_1 = W_+(A+I)^{-1/2}\big(B^{-1}(A+I)^{-1}\big)(A+I)^{-1/2}\chi_1$, the relation (51) with $W=W_+$ and the inclusion $(A+I)^{-1/2}\chi_1 \in \mathfrak{S}_\infty$ shows that $W_+B^{-1}\chi_1 \in \Sigma_{2p}^0$. Thus, by (58), $W_+(A-\lambda I)^{-1}\mathcal{F}_0\chi_s \in \Sigma_p^0$.

It remains to prove (57). In order to do that we represent this operator in the following form

$$\Pi = W_+(A+I)^{-1/2}\Omega((W_- - \mathcal{F}_0\chi_s)(A+I)^{-1/2})^*$$

where $\Omega=(A+I)(A-\lambda I)^{-1}$ is bounded. Since $(W_- - \mathcal{F}_0\chi_s) \in L^0_{p,\infty}(\mathbb{R}^d)$ the relation (57) follows from the second part of Proposition 6.1. □

In order to apply Theorem 1.1, we introduce the notion of the integrated density of states. Let A_n be the operator $-\triangle + f(x)$ in $L_2(Q_n)$, $Q_n = (-n,n)^d$, $n\in\mathbb{N}$, with Dirichlet boundary conditions on ∂Q_n. Denote by $N(\lambda, A_n)$ the number of eigenvalues of A_n lying to the left of $\lambda\in\mathbb{R}$. Then the following limit

$$\tau(\lambda) := \lim_{n\to\infty}(2n)^{-d}N(\lambda, A_n)$$

exists and it is called integrated *density of states* for the operator A.

Theorem 6.1. *Let a, v be defined as in (48), (49), and let (50) be fulfilled. Then for $\lambda = \overline{\lambda} \in \rho(A)$ and $p = d/s$ we have*

$$\Delta_p(\lambda; A, W_+, W_-) = \delta_p(\lambda; A, W_+, W_-) =$$

$$\Theta_+(\lambda) \int\limits_{\mathbb{S}^{d-1}} \Psi_+^p(\theta) d\theta - \Theta_-(\lambda) \int\limits_{\mathbb{S}^{d-1}} \Psi_-^p(\theta) d\theta, \tag{59}$$

where

$$\pm\Theta_\pm(\lambda) = \int (\tau(\lambda \pm |x|^{-s}) - \tau(\lambda)) dx.$$

Proof. If $V(x) = \pm\Psi_\pm(\theta)\chi_s(x)$, $\theta = x/|x|$, and $\Psi_\pm$ are continuous, the asymptotic relation (59) is proved in [1]. It is extended to $V(x) = \pm\Psi_\pm(\theta)\chi_s(x)$ with $\Psi_\pm \in L_p(\mathbb{S}^{d-1})$ due to the estimate (51). The full scale assertion of the theorem is a consequence of Theorem 1.1. □

3. In Theorem 6.1 A is semibounded. This restriction may be omitted. In the next example we deal with the Dirac operator perturbed by a decreasing electric potential. Let $\mathbf{g} = (g_1, g_2, g_3)$ and g_0 be (4×4)-Dirac matrices; $\mathbf{1}$ denotes the unit matrix. The Dirac matrices satisfy the relations

$$g_j g_k + g_k g_j = \delta_{jk}\mathbf{1}, \qquad j, k = 0, 1, 2, 3. \tag{60}$$

Let us consider the unperturbed Dirac operator in $\mathfrak{H} = L_2(\mathbb{R}^3; \mathbb{C}^4)$

$$A = \mathbf{g} \cdot D + g_0,$$
$$\mathbf{g} \cdot D = -i \sum_{j=1}^{3} g_j \frac{\partial}{\partial x_j},$$

and perturb the operator by a real potential

$$A(\alpha) = A - \alpha V, \quad \alpha > 0, \tag{61}$$

$$V \in L_3(\mathbb{R}^3), \quad \overline{V} = V. \tag{62}$$

The operator (61) needs to be correctly defined. Under the condition (62) it is impossible to introduce the operator as the difference of two operators, but it can be understood in a sense of the sum of the sesquilinear forms. This definition could be used not only for semibounded operators but for general ones, too. The corresponding scheme for non-semibounded operators is given in [21].

The spectrum of the operator A is absolutely continuous and covers the complement of the interval $\Lambda = (-1, 1)$. The essential spectrum of the operator $A(\alpha)$ coincides with the spectrum of A. Besides, the operator $A(\alpha)$ has discrete spectrum in the gap Λ.

Throughout this subsection we use the notation

$$W_{\pm} = (V_{\pm})^{1/2}.$$

Theorem 6.2. *Let A be the Dirac operator and $\lambda \in \Lambda$. Under the condition* (62) *the following asymptotics holds*

$$N(\lambda, A, W_-, W_+, \alpha) \sim \frac{1}{3\pi^2}\alpha^3\Big(\int V_+^3 dx - \int V_-^3 dx\Big), \tag{63}$$

$$\alpha \to \infty.$$

The proof of (63) is similar to the proof of Theorem 6.1. Instead of the condition (8) here we need the inclusion

$$W_{\pm}|A|^{-1/2} \in \Sigma_6(\mathfrak{H}),$$

which follows from Proposition 6.1. Indeed, since $(-\triangle + I)^{1/4}|A|^{-1/2}$ is a bounded operator, it is sufficient to establish that

$$\mathcal{G} := W_{\pm}(-\triangle + I)^{-1/4} \in \Sigma_6(\mathfrak{H}).$$

Since the functions W and b in Proposition 6.1 can be interchanged, by (52) with $W(\xi) = (|\xi|^2+1)^{-1/4}$, $b(x) = W_\pm$ and $q = 6$ we have

$$|\mathcal{G}|_6 \le C_0 \|W_\pm\|_{L_6},$$

where $\|\cdot\|_{L_6}$ denotes the norm in $L_6(\mathbb{R}^3)$. Hence,

$$|W_\pm |A|^{-1/2}|_6 \le C\|W_\pm\|_{L_6}. \tag{64}$$

Besides establishing (64), as in the previous example, we have to investigate the operator

$$X_\lambda(A) = W_+(A-\lambda I)^{-1}W_-.$$

Proposition 6.3. *Let* (62) *be fulfilled. Then*

$$X_\lambda(A) \in \Sigma_3^0. \tag{65}$$

Proof. Substituting $q_1 = q_2 = 6$, $T_1 = W_+|A-\lambda I|^{-1/2}\operatorname{sgn}(A-\lambda I)$, $T_2 = (W_-|A-\lambda I|^{-1/2})^*$ in (7) and employing (64), we obtain

$$|X_\lambda(A)|_3 \le C\|W_+\|_{L_6}\|W_-\|_{L_6}. \tag{66}$$

Let $\widetilde{W}_\pm \in C_0^\infty(\mathbb{R}^3)$, $\widetilde{W}_\pm \ge 0$ and $\widetilde{W}_+\widetilde{W}_- = 0$. Let $\widetilde{X}_\lambda(A)$ denotes the operator $X_\lambda(A)$ with the replacement of W_+, W_- by $\widetilde{W}_+$, $\widetilde{W}_-$. The relation (66) shows that the operator $X_\lambda(A)$ admits approximation in the class Σ_3 by operators of type $\widetilde{X}_\lambda(A)$. Thus, it suffices to establish (65) in the case where W_+, $W_- \in C_0^\infty(\mathbb{R}^3)$ and

$$W_+W_- = 0. \tag{67}$$

In order to find the commutator of the operators $(A-\lambda I)^{-1}$ and W_-, we note that $[B^{-1}, W_-] = -B^{-1}[B, W_-]B^{-1}$, where $B = A - \lambda I$. It is easily seen that $Z := [B, W_-]$ is an operator of multiplication by a bounded compactly supported matrix-function. Moreover,

$$Z = \chi_- Z,$$

where χ_- is the operator of multiplication by the characteristic function of suppW_-. The relation (67) yields

$$\begin{aligned} W_+(A-\lambda I)^{-1}W_- &= -W_+B^{-1}[B,W_-]B^{-1}\chi_- \\ &= -(W_+B^{-1}\chi_-)(ZB^{-1}\chi_-). \end{aligned} \tag{68}$$

By (51) with $W_\pm$ substituted by Z or χ_-, we have

$$ZB^{-1}\chi_- \in \Sigma_3.$$

Moreover, since $W_+B^{-1}\chi_- \in \Sigma_3$, by (68), we have $X_\lambda(A) \in \Sigma_{3/2} \subset \Sigma_3^0$. $\square$

For the case $V \geq 0$ the asymptotics (63) (under the condition (62)) was established in [3]. Moreover the case $V \leq 0$ can be treated in the similar way (see Proposition 2.3 in [3]). The full scale assertion of Theorem 6.2 follows from the suitable for the Dirac operator modification of Theorem 1.1 .

7. Acknowledgments

It is a pleasure for the author to express his deep gratitude to Professor M. Sh. Birman (who suggested the problem), A. A. Laptev and M. Klein for their attention to the work. The author is grateful for financial support given by SFB 288 and for the hospitality of the Department of Mathematics of the University of Potsdam.

References

[1] Alama, S.; Deift, P.A.; Hempel, R. Eigenvalue brunches of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$. Comm. Math. Phys. **1989**, *121*, 291-321.

[2] Birman, M.Sh. Discrete spectrum in the gaps of a continuous one for perturbations with large coupling constants. Adv. Soviet Math. **1991**, *7*, 57-73.

[3] Birman, M.Sh.; Laptev, A.A. Discrete spectrum of the perturbed Dirac operator. Ark. Mat. **1994**, *32*, 13-32.
[4] Birman, M.S.; Solomjak, M.Z. *Spectral theory of self-adjoint operators in Hilbert space. Mathematics and Its Applications*; D. Reidel Publishing Company, Dordrecht, 1987.
[5] Birman, M.Sh.; Solomyak, M.Z. Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations. Adv. Soviet Math. **1991**, *7*, 1-55.
[6] Cwikel, M. Weak type estimates for singular values and the number of bound states of Schrödinger operators. Ann. Math. **1977**, *106*, 93-100.
[7] Deift, P.A.; Hempel, R. On the existence of eigenvalues of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$. Comm. Math. Phys. **1986**, *103*, 461-490.
[8] Gesztesy, F.; Gurarie, D.; Holden, H.; Klaus, M.; Sadun, L.; Simon, B.; Vogl, P. Trapping and Cascading of eigenvalues in the large coupling limit. Comm. Math. Phys. **1988**, *118*, 597-634.
[9] Gesztesy, F.; Simon, B. On a Theorem of Deift and Hempel. Comm. Math. Phys. **1988**, *116*, 503-505.
[10] Hempel, R. On the asymptotic distribution of the eigenvalue branches of a Schrödinger operator $H - \lambda W$ in a spectral gap of H. J. Reine Angew. Math. **1989**, *399*, 38-59.
[11] Hempel, R. Eigenvalues in gaps and decoupling by Neumann boundary conditions. J. Math. Anal. Appl. **1992** , *169* (1), 229-259.
[12] Hempel, R. On the asymptotic distribution of eigenvalues in gaps. Quasiclassical methods (Minneapolis, MN, 1995). IMA Vol. Math. Appl. **1997**, *95*, 115-124.
[13] Klaus, M. On the point spectrum of Dirac operators. Helv. Phys. Acta **1980**, *53*, 453-462.
[14] Klaus, M. Some applications of the Birman-Schwinger principle. Helv. Phys. Acta **1982**, *55*, 49-68.
[15] Konno, R.; Kuroda, S.T. On the finiteness of perturbed eigenvalues. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **1966**, *13*, 55-63.
[16] Levendorskiĭ, S.Z. Lower bound for the number of eigenvalue branches for the Schrödinger operator $H - \lambda W$ in a gap of H: the case of indefinite W. Comm. Partial Differential Equations **1995**, *20*, 827-854.
[17] Safronov, O.L. Discrete spectrum in a gap of the continuous spectrum for variable sign perturbations with large coupling constant. St. Petersburg Math. Journal **1997**, *8* (2), 307-331.
[18] Safronov, O.L. The discrete spectrum in the gaps of the continuous one for nonsigndefinite perturbations with a large coupling constant. Comm. Math. Phys. **1998**, *193*(1), 233-243.
[19] Safronov, O.L. The discrete spectrum in the spectral gaps of semibounded operators with non-signdefined perturbations. KTH-preprint, Sweden, TRITA-MAT-98-MA-24, ISSN 1401-2278.
[20] Simon, B. Brownian motion, L^p properties of Schrödinger operators and the localization of binding. J. Funct. Anal. **1980**,*35*, 215-229.
[21] Yafaev, D. *Mathematical scattering theory*; Transl. Math. Monographs, Amer. Math. Soc., Providence, R.I., 1992; Vol. 105.