Functional Integrals and their Applications

Notes for a course for the "Troisième Cycle de la Physique en Suisse Romande"
given in Lausanne, Switzerland, during the summer of 1992

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*This work is partially supported by NSF grant DMS 9102564
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Abstract

These lectures are concerned with the analysis and applications of functional integrals defined by small perturbations of Gaussian measures. The central topic is the renormalization-group. Following Wilson and Polchinski, an effective potential is studied as a function of an ultra-violet cutoff. By changing the cutoff in a continuous manner, one obtains a differential equation for the effective potential. It is shown that by converting this equation to an integral equation and generating an iterative solution one obtains the Mayer expansion of classical statistical mechanics. Some results on the convergence of such an expansion are deduced, with applications to Coulomb and Yukawa gases. However, this method of solving for the effective potential turns out to be of limited value due to "large-field problems", which we explain. To achieve better results we abandon the effective action and represent the partition function as a polymer gas. The polymer gas representation has enough in common with the effective action that we are able to exactly repeat the previous method of iterating an integral equation, concluding with the flow of the activities of the polymers given by "cluster expansions". This is expounded in considerable detail, using as illustration the example of dipole gases. In addition, there are two introductory sections—independent of the other lectures—in which, as a motivation, problems in the theory of Coulomb gases and polymer physics are shown to be related to functional integrals of the type studied here. In particular, a heuristic discussion is presented on how quantum effects can destroy Debye screening.

Contents

1 Introduction

These lectures are concerned with the analysis and applications of functional integrals which look like

$$\int \prod_x d\phi(x)\, e^{-S(\phi)}; \tag{1.1}$$

where "$\prod_x d\phi(x)$" denotes a very mysterious "measure"—often denoted $\mathcal{D}\phi$ in the literature—and

$$S(\phi) = \int dx \left[\frac{1}{2}(\nabla\phi)^2 + v(\phi, \nabla\phi, \ldots)\right]. \tag{1.2}$$

In the physics literature, these formal expressions are mostly used to generate perturbation series, semiclassical approximations and more recently Monte-Carlo algorithms. Since Wilson's invention of the renormalization group (RG) [60] the perturbation theory of these integrals has developed into a deep subject which at least for small perturbations v is by now well understood and we will have nothing to add to this topic.

Instead, we would like to concentrate on *analytic* aspects. We shall try to explain how one can control the errors beyond perturbation theory so that the Wilson RG becomes the basis for a mathematically complete discussion of this type of functional integral. In Wilson's language, we will be concerned with the "irrelevant" terms. The question is: are these irrelevant terms *really* irrelevant, or are they so numerous that their overall contribution becomes comparable to that of the "relevant" terms? Here are some problems that reflect this question: Is there Debye screening in Quantum Coulomb systems? Does the expansion of the pressure in powers of activity (the Mayer expansion) converge for a gas of dipoles or for the Kosterlitz-Thouless phase of the classical two dimensional Coulomb gas? What is the end-to-end distance of a polymer on a four dimensional lattice?

A program like this was, after Wilson, first taken up by Gallavotti et al. [7] in their construction of super-renormalizable field theories such as ϕ^4_3. Since then a number of problems, which as an optimistic youth I would have considered to be a hundred years in the future, have been successfully attacked [19, 3, 31, 40]. However in our zeal to go ever further into new territory we have sometimes neglected to explain what we are doing in language that can easily be understood by outsiders. I consider these lectures to be a time to *begin(!)* this task. The plan of campaign is as follows.

To display the surprisingly rich range of applications of these functional integrals, in Sections 2 and 3 I review how problems in statistical mechanics such as the Coulomb gas and polymer physics can be transformed into these integrals. One of the problems that I discuss here (heuristically) is to prove that quantum effects destroy screening in the quantum Coulomb gas. These introductory sections are purely for motivation. The main topics are independent of these sections.

In Section 4 I begin the main topic—the renormalization group (RG). A standard approach is to introduce an ultra-violet cutoff at a length scale $l(t)$ into the Gaussian part $d\mu(\phi)$ of (1.1), so that it becomes $\int d\mu_t(\phi)$, and then to define an evolution of the effective potential V_t by insisting that $\int d\mu_t(\phi)e^{V_t}$ remain unchanged when the cutoff $l(t)$ is changed by varying the parameter t. In Section 5 I describe how this leads to a differential equation for V_t and attempt to solve it by converting it to an integral equation and generating a series by iteration. The resulting solution is closely related to the Mayer expansion of classical statistical mechanics and in Section 6 I pause to develop this idea. This leads into some pretty results on the Yukawa and Coulomb gases in two dimensions but is of limited use in the control of the RG because this is an expansion in powers of the initial potential $V_{t=0}$. Any method based on the convergence of such an expansion is unable to distinguish between $V_{t=0}$ and $-V_{t=0}$, yet the change from e.g., ϕ^4 to $-\phi^4$ is a change from a stable to an unstable interaction. This is an instance of a "large field problem".

To get around this large field problem, in Section 7 I partly abandon the representation e^{V_t} in favor of

$$e^{V_t} \longrightarrow \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{X_1,\ldots,X_N} \prod_j K(X_j,\phi) e^{-\int_{-U} dx\, v_t(\phi,\nabla\phi)}.$$

In these notes we only have time for those systems where the v_t can be taken to be zero[1]. The function v_t contains parts of the original effective action V_t which are dominant and play an essential role in maintaining stability—for the ϕ_4^4 theory it would have the form $\frac{\lambda_t}{4!} : \phi^4 : + \frac{\mu_t^2}{2} : \phi^2 : + \frac{z_t}{2} : (\nabla\phi)^2 :$—whilst the functionals $K_t(X,\phi)$ contain corrections which are higher order in λ. The argument X signifies that K_t depends on the field $\phi(x)$ only for $x \in X$. Each set X is called a polymer. The set $-U$ is the complement of the union of the polymers $X_1,\ldots,X_N$. Under the action of the renormalisation group the polymer activities $K_t(X)$ evolve by equations that can be studied in very close analogy to the ones satisfied by the effective action V_t. One can write beautiful series solutions (called cluster expansions) for these activities but the main objective in Section 7 is to devise a method of measuring the size (a norm) of the functionals $K(X,\phi)$. This norm captures the information necessary to ensure that

[1] I will discuss the the case ϕ_4^4 in which $v_t \neq 0$ in an expanded version of these notes to be published elsewhere

under the RG evolution, whilst λ_t remains small, these polymers will remain a dilute gas, so that the factor $e^{-\int_{-U} dx\, v_t(\phi,\nabla\phi)}$ is the dominant part.

There is some clumsiness in the construction of this norm. Perhaps someone else will be able to do better. The less information one needs to hold within this norm the more efficient the scheme will be. The norm keeps track of variational derivatives of all orders, in fact it enforces a kind of analyticity in ϕ, which probably is not necessary.

In Section 8 I discuss in considerable detail the case where $v(\phi)$ actually depends only on $\nabla\phi$. The dipole gas or $e^{\int dx\, (\nabla\phi)^4(x)}$ are examples of such a functional integral. With some further work, this leads to a proof that the pressure for the dipole gas has a convergent expansion in powers of activity, a result first obtained by Gawedzki and Kupiainen, who were the first to really understand the dipole gas. The methods of this section have also been used in the course of work on Coulomb and Yukawa gases in two dimensions by Dimock and Hurd [43]. They have extended these methods to calculate the correlations as opposed to just the effective action. They also have proved that the two dimensional Coulomb gas in a finite volume without ultra-violet cutoff has pressure analytic in the activity for all inverse temperatures up to the Kosterlitz Thouless transition at $\beta = 8\pi$ provided at each of the thresholds $\beta = 8\pi(1-\frac{1}{2n})$, $n = 1, 2, \cdots$ an infinite vacuum energy renormalization is performed.

I thank my colaborators Paul Federbush, Tom Kennedy, H.T. Yau, John Imbrie, Pronob Mitter and Georg Keller for much of the work presented here, Roberto Fernández for generously agreeing to help in preparing these notes and Andreas Pordt for corrections. Of course I will have only myself to thank for any errors. I wish to thank Gallavotti and Benfatto for their hospitality in Rome whilst some of this work was being prepared. Ideas close to the ones presented here have been separately advanced by Mack and Pordt [45, 46, 54] and indeed it should be realized that the work presented here is a cleaned up version of ideas that are implicit in all the work on phase cell expansions [25, 24]. Finally I am very grateful to Philippe Martin and Prof. L. Schellenberg and the Troisième Cycle de la Physique en Suisse Romande, for the opportunity to give these lectures.

2 The Sine-Gordon Transformation

This identity rewrites the partition function for a system with two-body forces in the Grand Canonical Ensemble as a superposition of partition functions of ideal gases with (imaginary) external fields. The superposition is over these external fields.

Suppose our particles are described by a coordinate $\xi = (x, \epsilon)$, where x is the position and ϵ an internal coordinate such as charge, or orientation in the case of dipoles. We consider particles in a finite container, and with a finite (or compact) space of internal coordinates, so ξ ranges over a compact set that we shall denote Λ. The partition function for particles interacting with a two-body potential u and external field f is

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \int \tilde{z}^N \, d^N\xi \; e^{-\beta U(\xi_1,\ldots,\xi_N)} \tag{2.1}$$

where

$$U = \frac{1}{2} \sum_{i,j=1}^{N} u(\xi_i, \xi_j) + \sum_{i=1}^{N} f(\xi_i). \tag{2.2}$$

Here $\int d\xi$ represents the integral both over coordinates and internal space variables, and it ranges over the compact set Λ. In particular, the partition function for an ideal gas in an external field ϕ is

$$\begin{aligned} Z_{\text{ideal}}(\phi) &= \sum \frac{1}{N!} \int \tilde{z}^N \, d^N\xi \; e^{\beta \sum_1^N \phi(\xi_j)} \\ &= \exp\left[\int \tilde{z} d\xi \; e^{\beta\phi(\xi)} \right] . \end{aligned} \tag{2.3}$$

Let us first consider the case of particles on a *lattice*. For simplicity we consider first $f = 0$, the reader can easily furnish the small changes needed to deal with a general f. We write the interaction energy U as a sum of two-body energies $u(\xi_i, \xi_j)$ times the "particle densities" at ξ_i and ξ_j:

$$U = \frac{1}{2} \sum_{\xi,\xi'} \rho(\xi) u(\xi, \xi') \rho(\xi'), \tag{2.4}$$

with

$$\rho(\xi) = \sum_{i=1}^{N} \delta(\xi - \xi_i); \tag{2.5}$$

where the δ refers to the Kronecker delta. Then $\exp(-\beta U) = \exp[(-\beta/2)\sum \rho u \rho]$ is a Gaussian function of the vector ρ. Provided u is positive definite we can use the identity (Fourier transform of a Gaussian is a Gaussian)

$$\exp\left[-\frac{\beta}{2}(\rho, u\rho)\right] = \text{Const.} \int \prod_{\xi} d\phi(\xi)\ \exp\left[-\frac{1}{2}\beta(\phi, u^{-1}\phi) - i\beta(\phi,\rho)\right]. \tag{2.6}$$

Here $(\rho, u\rho) = \sum_{\xi,\xi'\in\Lambda} \rho(\xi) u(\xi,\xi')\rho(\xi')$ and u^{-1} is the inverse of the matrix with rows and columns labelled by the particle coordinates in Λ and entries $u(\xi,\xi')$. In a more compact way, this formula can be written as

$$e^{-\frac{\beta}{2}(\rho,u\rho)} = \int d\mu(\phi)\, e^{-i\beta(\phi,\rho)}, \tag{2.7}$$

where $d\mu$ is a zero-mean Gaussian probability measure completely defined by its covariance (a matrix)

$$\int d\mu(\phi)\, \phi(\xi)\phi(\xi') = \beta^{-1} u(\xi,\xi'). \tag{2.8}$$

If we substitute (2.4) and (2.7) into (2.1), we obtain

$$Z = \int d\mu(\phi) \sum_{N=0}^{\infty} \frac{1}{N!} \tilde{z}^N \sum_{\xi_1,\ldots,\xi_N \in\Lambda} e^{i\sum_j \phi(\xi_j)\rho(\xi_j)} \tag{2.9}$$

and, using (2.5),

$$\begin{aligned} Z &= \int d\mu(\phi) \sum_{N=0}^{\infty} \frac{1}{N!} \tilde{z}^N \sum_{\xi_1,\ldots,\xi_N\in\Lambda} e^{i\sum_j \phi(\xi_j)} \\ &= \int d\mu(\phi)\, Z_{\text{ideal}}(i\phi). \end{aligned} \tag{2.10}$$

This last formula is the *Sine-Gordon transformation.* We notice that in order to use (2.6), the diagonal terms $u(\xi,\xi)$ ("self-energy") *must* be included. This will need further attention when aplying the Sine-Gordon transformation to forces singular at short distances—for example Coulomb or Yukawa.

Siegert [57] and Kac [44] discovered the Sine-Gordon transformation and used it for continuum systems. A cautious statement of the result is:

Proposition 2.1 (The Sine-Gordon Transformation) *Let $u(\xi,\xi')$ be positive semi-definite and continuously differentiable in the position coordinates, then there is a Gaussian measure $d\mu(\phi)$ on the space of continuous fields $\phi \in \mathcal{C}(\Lambda)$, which is characterized* *by (1) $\int d\mu(\phi)\phi(\xi) = 0$ for all $\xi \in \Lambda$; and (2) covariance $\beta^{-1}u$, i.e., $\int d\mu(\phi)\,\phi(\xi)\phi(\xi') = \beta^{-1}u(\xi,\xi')$; such that*

$$Z = \int d\mu(\phi)\, Z_{\rm ideal}(i\phi + f). \tag{2.11}$$

The reader might wonder what is meant by a *Gaussian* measure in this functional context. This is defined by requiring that

$$\int d\mu(\phi)\, e^{i\int d\rho(\xi)\phi(\xi)} = e^{-\frac{1}{2\beta}\int\int d\rho(\xi)u(\xi,\xi')d\rho(\xi')}, \tag{2.12}$$

where $d\rho$ is any finite Borel measure on Λ. This is a natural generalization of (2.7) (the Fourier transform of a Gaussian is a Gaussian).

Proof. The proof is basically a transcription of the above discussion. Starting from the formula above, the proof goes exactly as above, except that (2.5) turns into $d\rho(\xi) = \beta\sum_1^N \delta(\xi - \xi_j)d\xi$, with δ the Dirac delta. The remaining (difficult) question is the existence of a Gaussian measure on a space of continuous functions. This is discussed, for instance, in [59, 33]. In these references the measure is constructed on a Hilbert space of functions. Our hypotheses on u guarantee that the Hilbert space is a space of *continuous* functions. □

2.1 Coulomb Systems

For Coulomb forces between charges $\epsilon = \pm 1$, one would like to choose $\xi = (x, \epsilon)$ and $u(\xi,\xi') = \epsilon\epsilon' G(x - x')$ where G satisfies $-\Delta G(x,y) = \delta(x - y)$, that is, with G being the kernel of a Green function $(-\Delta)^{-1}(x,y)$. In more than one dimension this kernel is singular at $x = y$ so there are problems with the Sine-Gordon transformation which, as we have described it, requires all the integrals (2.8) to be finite, and in particular, as emphasized above, *finite diagonal terms*, i.e.

$$\int d\mu(\phi)\, \phi^2(\xi) = u(\xi,\xi) < \infty. \tag{2.13}$$

Fourier-transforming the coordinates:

$$u((x,\epsilon),(y,\epsilon')) = \text{Const.} \int \hat{u}(\epsilon,\epsilon',k)e^{ik(x-y)}\, dk\ ,$$

we see that condition (2.13) is equivalent to requiring the integrability of $\hat{u}(k)$. This is not satisfied for the Coulomb ($\hat{u} \propto 1/k^2$) or Yukawa ($\hat{u} \propto 1/(1 + k^2)$) forces in dimensions $d > 1$, because of the slow decay at high wave numbers k (for the Coulomb potential there is also trouble on the infrared end for $d \leq 2$).

The physical interpretation of these problems is the possible existence of an instability with respect to collapse. Repulsive short-range forces or quantum mechanics are necessary to prevent such a collapse in three or more dimensions. A standard way to model such forces is to suppose that the point charges are spread out into little spherical charge-distributions on a length scale ℓ. Then u is the Fourier transform of

$$\hat{u}(\epsilon, \epsilon', k) = \epsilon\epsilon' \frac{F(\ell^2 k^2)}{k^2}, \tag{2.14}$$

where $\sqrt{F}$ is the Fourier transform of the charge distribution, and F is assumed to be a sufficiently smooth function that takes the value 1 at the origin and decays rapidly as $k \to \infty$. The length ℓ corresponds to an ultraviolet cutoff; roughly speaking, for distances greater than ℓ the original potential differs little from the one defined by (2.14). In the case that the particles are in a periodic box u is obtained from $\hat{u}$ by a Fourier series in which the $k = 0$ term is omitted.

When applying the Sine-Gordon transformation to systems with interactions of the form (2.14) it is useful to recognise that

$$\phi(x, \epsilon) = \epsilon\phi(x)\ . \tag{2.15}$$

where on the left hand side $\phi(x, \epsilon)$ is distributed according to a Gaussian measure which has a covariance of the form $\frac{1}{\beta}\epsilon\epsilon' u(x, x')$ and on the right hand side $\phi(x)$ is distributed according to a Gaussian measure which has covariance $\frac{1}{\beta}u(x, x')$. This is because both sides are Gaussian variables with the same covariance and therefore are equal as random variables. In particular, if $\tilde{z}d\xi$ gives equal weight to $\epsilon = \pm 1$ (charge symmetry), then the Sine-Gordon transformation rewrites the partition function as:

$$Z = \int d\mu(\phi) \exp\left[2 \int \cos\beta[\phi(x) + f(x)]\tilde{z}dx\right]. \tag{2.16}$$

where f is the external field.

A particular case of interest here, is that of a gas of *dipoles*. In this case $\xi = (x, \hat{p})$ where $\hat{p}$ is a unit vector in $\mathbf{R}^d$, the direction of the dipole. $d\xi = dx\, dS(\hat{p})$, where dS is the Lebesgue measure on the surface of the unit sphere. In k-space the covariance of $d\mu(\phi)$ is

$$\hat{u}(k, \hat{p}, \hat{p}') = (\hat{p} \cdot k) \frac{F(\ell^2 k^2)}{k^2} (\hat{p}' \cdot k). \tag{2.17}$$

By the argument we used for (2.15) $\phi(\xi) \to (\hat{p}\cdot\nabla)\phi(x)$ so

$$Z = \int d\mu(\phi) \exp\left[2\int \tilde{z} dx\, dS(\hat{p}) \cos\beta[\hat{p}\cdot\nabla\phi(x) + f(x,\hat{p})]\right], \tag{2.18}$$

where $d\mu$ is the same as in (2.16).

2.2 Energy and Normal Ordering

It is more standard to exclude self energies in defining the interaction (2.2). These are the terms $i = j$ in U. To achieve this we put the self energies into the activity by defining

$$z = e^{-\frac{1}{2}\beta u(\xi,\xi)}\tilde{z}, \tag{2.19}$$

and take them out of the interaction. Now one writes the Sine-Gordon transformation in the form

$$Z = \int d\mu(\phi) \exp\left(\int z d\xi \, : e^{i\beta\phi(\xi)} :\right), \tag{2.20}$$

where the *normal ordered* $: e^{i\phi} :$ is defined by

$$: e^{i\beta\phi(\xi)} := e^{\frac{1}{2}\beta u(\xi,\xi)} e^{i\beta\phi(\xi)}. \tag{2.21}$$

It will turn out that that z is the "right" activity from the physical point of view. For example, if we remove the ultra-violet cutoff by taking ℓ to zero in the $d = 2$ Yukawa gas than the limit is a gas at zero density if $\tilde{z}$ is held fixed but has density approximately equal to z if z is held fixed and small. We will be discussing this further in Section 5

2.3 The Quantum Statistical Coulomb System

This section is contains some <u>heuristic</u> arguments drawn from a paper I am writing with Georg Keller [15]. We thank Erhard Seiler for his help in many conversations on these topics. We should also mention that the original observation that there might not be screening in Quantum Coulomb plasma was due to Paul Federbush.

If we include momenta and kinetic energy then the partition function for a (charge-symmetric) *classical* Coulomb gas is

$$Z = \sum \frac{1}{N!} \int d^N p \tilde{z}^N d^N \xi \, e^{-\beta H} \tag{2.22}$$

with

$$H = \sum \frac{p_i^2}{2m} + \frac{1}{2}\int \rho u \rho \tag{2.23}$$

where

$$\rho(x) = \sum \epsilon_i \delta(x - x_i) \tag{2.24}$$

so that

$$\int \rho u \rho = \sum \epsilon_i \epsilon_j u(x_i - x_j)\ . \tag{2.25}$$

$u(x-y) =$ "$1/r$". The quotes around the $1/r$ remind us that, as discussed in Example 1, it is necesary to cutoff the singularity of the Coulomb potential at short distances in order to have a stable interaction. This cutoff will be characterised by a length l. The natural choice for this length is the size of the typical one-particle wavefunction in a corresponding quantum ideal gas

$$l = \sqrt{\frac{\beta \hbar^2}{m}}\ ; \tag{2.26}$$

since it is the Pauli exclusion principle and quantum mechanics that give rise to a stable system which we are approximating classically.

The other lengths which naturally arise are β and the Debye length

$$l_D = \frac{1}{\sqrt{2z\beta}}\ , \tag{2.27}$$

where z includes also the contribution from the momentum integral:

$$z = \tilde{z}\int dp e^{-\frac{\beta}{2m}p^2} e^{-\beta\cdot(\mathrm{self-energy})/2}\ . \tag{2.28}$$

In general, for reasonable cutoff prescriptions $\mathrm{self-energy} = \mathrm{O}(1/\ell)$.

For this system the following theorem has been proved [12, 13, 38, 39].

Theorem 2.2 *For*

$$zl^3 \ll e^{-\frac{\beta}{2l}}, \quad zl_D^3 \gg 1, \tag{2.29}$$

all charge-charge correlations decay exponentially, i.e., there are constants C_1 and $L > 0$ such that

$$|\langle \rho(x)\rho(y)\rangle| \leq C_1 e^{\frac{-|x-y|}{L}} \tag{2.30}$$

and higher truncated charge correlations decay exponentially with the length of the shortest tree joining the positions of the observables.

Also $L \simeq l_D$ when zl^3 and zl_D^3 are as in the theorem. As z is roughly the density of particles, the interpretation of the hypothesis (2.29) is that there are many particles within a Debye sphere (rightmost condition), which constitute an approximately ideal gas (leftmost condition). An exponential decay of the kind proved in this theorem is generally referred to as "Debye screening". The reason for this screening can be understood by the following simplified argument. By the Sine-Gordon representation, the partition function for the interacting gas is a superposition of partition functions for ideal gases of particles in external fields:

$$Z = \int d\mu(\phi)\, Z_{\mathrm{ideal}}(i\phi) \ , \tag{2.31}$$

with

$$Z_{\mathrm{ideal}}(i\phi) = e^{\int dp\tilde{z}d\xi\, e^{-\beta h(i\phi)}} \tag{2.32}$$

and putting the mass, $m = 1$,

$$h(i\phi) = \frac{p^2}{2} + i\epsilon\phi \ . \tag{2.33}$$

As there is charge symmetry (c.f. (2.16))

$$Z = \int d\mu(\phi)\, \exp\left[2\int z : \cos\beta\phi(x) : dx\right], \tag{2.34}$$

where z is as in (2.28). In the regime (2.29) only small values of $\beta\phi(x)$ contribute, so approximating $\cos() \simeq 1 - \frac{1}{2}()^2$ we obtain

$$Z \simeq \int d\mu(\phi)\, \exp -\frac{\beta}{2l_D^2}\int dx\, (: \phi^2 : +\mathrm{Const.}) \ , \tag{2.35}$$

where "Const." is independent of ϕ. Hence the partition function is approximately determined by a new Gaussian measure [2]

$$d\mu'(\phi) = \frac{1}{\mathrm{Norm.}} d\mu(\phi)\, \exp -\frac{\beta}{2l_D^2}\int dx\ : \phi^2 : , \tag{2.36}$$

of covariance proportional to $(-\Delta + l_D)^{-1}$, and hence exhibiting exponential damping (it is the same covariance matrix as in the Yukawa interaction). Within this approximation

[2]$: \phi^N :$ is defined by $: e^{i\beta\phi} := \sum \frac{\beta^N}{N!} : \phi^N :$

this covariance predicts exponential decay for the charge-charge correlations (Problems 2.4.1-2.4.2 below). So, once we do a substantial amount of work to show that this approximation is valid, (2.30) follows.

For the *quantum-mechanical* Coulomb gas, there is good evidence that the (exponential) screening is lost. Rather it seems that after a number of Debye lengths, there is a cross-over from exponential to power-law decay. This possibility was first noted in [13], p428. Since then several papers [10, 1, 2] have left us in no doubt that this actually occurs, but there is as yet no proof. In particular the detailed calculations of Alastuey and Martin [1, 2] show within perturbation theory (the Wigner-Kirkwood expansion) that screening is destroyed by effects due to diagrams with power-law decay at order $\hbar^4$ and higher. They show, for example, that for NaCl ions in water at room temperature there will be screening out to about 60 Debye lengths at which point there is a cross-over to a power law tail. This is a tiny effect but they note that it may be much larger in electron gases. According to their analysis the typical power-law is r^{-6} but the power can be higher depending on the correlation and the system. This violation of screening has nothing to do with statistics. It is a mechanism similar to that of the Van der Waals forces, but it occurs [2] even for one component plasmas in which there are no atoms or molecules. Possible implications for the binding energy of Noble Metals have been discussed by Maggs and Ascroft in [47] who have independently noted the failure of screening.

Let us present a simple model to explain why quantum fluctuations can destroy the screening in the sense of the theorem above. For simplicity we discuss the case of Boltzmann statistics, but there are similar representations for Fermion and Bose statistics. This simplification is reasonable since we are discussing a regime in which the gas is very far from degenerate; by the hypotheses (2.29) of the theorem $l \ll$ interparticle distance.

We first must find the analogous of the Sine-Gordon representation for the quantum case. The starting expression is

$$Z = \sum \frac{\tilde{z}^N}{N!} \operatorname{tr}\left(e^{-\beta H_N}\right) . \tag{2.37}$$

To deal with the failure of commutativity—$[p, x] \neq 0$—we proceed as in the standard derivations of the Feynman-Kac path-integral formula: we decompose $H_N = K_N + U_N$, where K_N is the kinetic term, and use Trotter's formula:

$$\operatorname{tr} e^{-\beta H_N} \quad = \quad \lim_{m \to \infty} \operatorname{tr}\left(e^{-\frac{\beta}{m} K_N} e^{-\frac{\beta}{m} U_N}\right)^m$$

$$
\begin{aligned}
&= \lim_{m\to\infty} \int d\underline{X}_0\, d\underline{X}_1\, \ldots, d\underline{X}_m\, \delta(\underline{X}_m - \underline{X}_0) \\
&\quad \times \prod_{k=1}^{m} \exp\left[(-1/2)|\underline{X}_{k-1} - \underline{X}_k|^2 \left(\frac{\beta}{m}\right)^{-1} + U_N(\underline{X}_k)\frac{\beta}{m}\right] \qquad (2.38)
\end{aligned}
$$

Now, as in the Sine-Gordon transformation for classical systems, we write *each* factor $e^{-\frac{\beta}{m}U_N(\underline{X}_i)}$ as a Fourier transform of a Gaussian, using the identity (2.6). This generates a family of external fields $\phi_k(x)$, each one integrated with a Gaussian measure $d\mu_k$. All these Gaussian measures are in fact identical, with zero mean and covariance $\frac{m}{\beta}u$.

$$
\begin{aligned}
Z &= \sum_N \frac{\tilde{z}^N}{N!} \lim_{m\to\infty} \int d\underline{X}_0\, d\underline{X}_1\, \ldots, d\underline{X}_m\, \delta(\underline{X}_m - \underline{X}_0) \\
&\quad \times \prod_{k=1}^{m} \exp\left[(-1/2)|\underline{X}_{k-1} - \underline{X}_k|^2 \left(\frac{\beta}{m}\right)^{-1}\right] \\
&\quad \times \int d\mu_k(\phi_k) \exp\left[-i(\phi_k, \rho_k)\frac{\beta}{m}\right] . \qquad (2.39)
\end{aligned}
$$

In the limit $m \to \infty$, the product over k with increment β/m becomes (formally!) an integral in the exponent. That is k is replaced by a continuum "time" τ; the sequence of fields $\phi_k(x)$ becomes a time-dependent external field $\phi(\tau, x)$; and the sequence of Gaussian measures $d\mu_k$ assemble a "space-time" Gaussian measure $d\mu_{u\otimes I}$ whose covariance is $u(x-y)\delta(\tau - \sigma)$. Permuting this measure with the sum over N we obtain again the Sine-Gordon transformation (2.31)

$$
Z = \int d\mu_{u\otimes I}\, Z_{\text{ideal}}(i\phi)\,, \qquad (2.40)
$$

but with

$$
Z_{\text{ideal}} = \exp \sum_{\epsilon} \int dW^\beta(X) e^{-i\epsilon \int_0^\beta d\tau\, \phi(\tau, X(\tau))}\,. \qquad (2.41)
$$

Here dW^β is the Wiener measure—formally proportional to $\exp\left(-\int_0^\beta d\tau\, \frac{1}{2}(dx/d\tau)^2\right)$—supported over continuous closed paths $X(\tau), \tau \in [0, \beta]$. The combination (2.40)–(2.41) is a representation for the quantum partition function which appears in [28]. It is also derived and used in [2]. The expression of Z_{ideal} in terms of the Wiener measure is due to Ginibre [34].

Notice that there is a Goldstone mode:

$$
\phi(\tau, x) \to \phi(\tau, x) + f(\tau)
$$

where f is any function such that $\int_0^\beta d\tau f(\tau) = 0$. This will be the origin of the long range forces. The intuition is that the Feynman-Kac formula represents the quantum gas as a classical gas of closed charge loops with instantaneous Coulomb interactions. Each loop represents the quantum uncertainty around a classical position. This leads to a time-dependent dipole force superimposed on the Coulomb force for the classical system. A dipole can polarise other dipoles leading to induced dipole-dipole or multipole-multipole forces which are power laws. The standard textbook discussions do not see this effect because they make a static approximation which loses these time dependences. The mechanism is very similar to that of the Van der Waals forces, except that it takes place without any need for neutral objects such as atoms or molecules.

Unfortunately the delta function in the covariance $u(x-y)\delta(\tau-\sigma)$ means that the time-dependent external fields ϕ are as singular as white noise in their t-dependence and so (2.40) is formal. We are going to fail to face these questions and instead concentrate on a simple model obtained by making the following simplifications:

1. We replace the continuum of possible particle positions by a simple cubic lattice of spacing l with periodic boundary conditions. If this is done then the natural object to stand in for the Wiener measure dW^β is the measure on all simple random walks that form loops $X(\tau)$ parametrised by $\tau \in [0, \beta]$.

2. We approximate the measure dW^β by a new measure supported only on the following two types of walk:

   (a) The paths that stay at some site x for all $\tau \in [0, \beta]$.

   (b) The paths that jump between two neighboring sites, that is walks characterized by two nearest neighbour lattice sites x, y, such that

$$X(\tau) = \begin{cases} x & \text{if } \tau \in [0, \frac{\beta}{2}) \\ y & \text{if } \tau \in [\frac{\beta}{2}, \beta) \end{cases} .$$

We assume that dW^β assigns probability $1-q$ to the paths of type (a), where $q \ll 1$; while paths of type (b) occur with remaining probability, q. This parameter q does not depend on the parameters $\beta, z, \ldots$ because we have set the lattice spacing equal to the natural length scale l. Although the mass m was set equal to one, our analysis depends on m only through l so this normalization is no longer in effect. The two-body Coulomb interaction energy is replaced by u_{lattice}, the inverse of the lattice Laplacian with periodic boundary conditions and the zero-mode projected out. We call this model the *lattice Semi-Quantum Model.*

The consequence of these changes is that the dependence of $Z_{\rm ideal}(i\phi)$ on $\phi(\tau,x)$ is only through

$$\phi_1(x) = \frac{1}{\beta}\int_0^\beta d\tau\,\phi(\tau,x)$$

and

$$\phi_2(x) = \frac{1}{\beta}\int_0^\beta d\tau\,\phi(\tau,x)\theta(\tau)\ ,$$

where

$$\theta(\tau) = \begin{cases} 1 & \text{if } \tau\in[0,\frac{\beta}{2}) \\ -1 & \text{if } \tau\in[\frac{\beta}{2},\beta) \end{cases} \quad .$$

These fields are independently distributed according to the periodic massless Gaussian measure $d\mu$ with covariance $\beta^{-1}u_{\rm lattice}$ on fields whose spatial average vanishes.

After a brief calculation the partition function becomes

$$\begin{aligned} Z &= \int d\mu(\phi_1)\int d\mu(\phi_2)\exp\left[2z(1-q)\sum_x l^3 : \cos(\beta\phi_1(x)) :\right] \\ &\quad \exp\left[2z\frac{q}{6}\sum_{xy} l^3 : \cos\Big(\frac{\beta}{2}[\phi_1(x)+\phi_1(y)]\Big) :\, : \cos\Big(\frac{\beta}{2}[\phi_2(x)-\phi_2(y)]\Big) :\right] \end{aligned} \tag{2.42}$$

If q is set to zero we revert to the classical Coulomb gas. If ϕ_1 is set to zero then by reversing the Sine-Gordon transformation we obtain

$$Z \to \sum \frac{(zq/3)^N}{N!}\sum_{x_1y_1,\ldots,x_Ny_N} l^{3N} e^{-\beta U(x_1y_1,\ldots,x_Ny_N)}$$

where $U(x_1y_1,\ldots,x_Ny_N)$ is the Coulomb energy, without self energies, of N lattice dipoles composed of positive and negative charges at $(x_1,y_1),\ldots,(x_N,y_N)$; positive charges at x, negative charges at y coordinates.

We conjecture that this model will have the following unusual behavior: for z, β, l as in the screening theorem above there is a $L\approx l_D$ such that as $r\equiv\frac{x-y}{L}\to\infty$

$$\langle\phi_1(x);\phi_1(y)\rangle \sim \frac{c_1}{\beta L}\frac{e^{-r}}{r} \tag{2.43}$$

$$\langle\phi_1^2(x);\phi_1^2(y)\rangle \sim \left[\frac{c_1}{\beta L}\frac{e^{-r}}{r}\right]^2 + \frac{c_2}{L^4}\left[\frac{l}{L}\right]^4 r^{-6} \tag{2.44}$$

$\langle A\,; B\rangle \equiv \langle AB\rangle - \langle A\rangle\langle B\rangle$. We have put in the exponentially decaying term in the last equation to emphasise that since l is much smaller than L there is a intermediate region where exponential decay is dominant. More generally two-point correlations of odd functions of ϕ_1 will decay exponentially and two-point correlations of even functions will decay by a power law. Static charge densities[3] ρ_{stat} are odd functions so we expect $\langle\rho_{\text{stat}}\rho_{\text{stat}}\rangle$ to have exponential decay in this model but $\langle\rho^2_{\text{stat}};\rho^2_{\text{stat}}\rangle$ will decay by a power law. This behavior has come about because our approximations have inadvertently created a model which is even separately in ϕ_1 and ϕ_2 so every Feynman diagram for $\langle\phi_1(x)\,;\phi_2(y)\rangle$ has a path of massive ϕ_1 propagators connecting the two points, but this need not be the case with $\langle\phi_1^2(x)\,;\phi_2^2(y)\rangle$. The ϕ_1 propagators are massive because of screening from the $\cos\phi_1 \approx 1 - \frac{1}{2}\phi_1^2$.

To obtain (2.43) and (2.44) we rewite the action in (2.42) in terms of dimensionless variables $x = l_D x'$ and $\phi_i(x) = \frac{1}{\sqrt{\beta l_D}}\phi_i'(x')$ with the approximations $\sum_x l^3 \longrightarrow \int dx$ and $\cos() \sim 1 - \frac{1}{2}()^2$. Omitting constants that are independent of the parameters z, β and l the action becomes

$$\frac{1}{2}\int(\nabla\phi_1')^2 + \frac{1}{2}\int(\nabla\phi_2')^2 + \frac{1}{2}\int(\phi_1')^2 + \frac{\beta}{l_D}(\frac{l}{l_D})^2\int :\phi_1'^{\,2}::(\nabla\phi_2')^2:\,, \tag{2.45}$$

and perturbation theory for this action leads to (2.43) and (2.44).

2.4 Problems

2.4.1 Find out what the charge-density correlations become in Sine-Gordon language:

$$\langle\rho(\xi)\,\rho(\xi')\rangle \;=\; ? \qquad\qquad \xi \neq \xi'$$

Hint: Differentiate with respect to an external field.

2.4.2 Let $U(x_1\,y_1,\ldots,x_N\,y_N)$ be the Coulomb energy including seff-energies for a system of charges of $+\epsilon$ at the sites $x_1,\ldots,x_N$ and $-\epsilon$ at the sites $y_1,\ldots,y_N$, where $x_i\,y_i$ are nearest neighbors and are in a finite subset Λ of a lattice. Consider Dirichlet boundary conditions. (That is, the two-body interaction u is the solution of $-\Delta u(x,y) = \delta(x-y)$, where Δ is the finite-difference laplacian, with

[3]These are observables obtained by differentiating the (logarithm of) the partition function (2.37) with respect to a time independent external field $f(x)$ inserted into the Hamiltonian by $H_N \longrightarrow H_N + \int dx\,\rho(x)f(x)$. This corresponds to $\phi_1 \longrightarrow \phi_1 + f$ in the Sine-Gordon integrand. They measure the response to an infinitesmal external field.

boundary conditions $u(x,y)=0$ if x or y is not in Λ.) Show that

$$\sum_{N\geq 0}\frac{z^N}{N!}\sum_{x_1\,y_1}\cdots\sum_{x_N\,y_N}\exp\left[-U(x_1\,y_1,\ldots,x_N\,y_N)\right]$$
$$=\int d\mu(\phi)\,\exp\left[2z\sum_{x\,y}\cos[\phi(x)-\phi(y)]\right]\ ,$$

where the sums run over *all* nearest-neighbor pairs x_i, y_i, x, y, even if one point is outside the lattice, but $\phi(x)=0$ if $x\notin\Lambda$. The measure $d\mu$ is defined by

$$d\mu(\phi)=\frac{1}{\text{Norm.}}\,d^{\Lambda}\phi\ \exp\left[-(1/2)\sum_{x\,y\in\Lambda}(\phi(x)-\phi(y))^2\right]\ .$$

3 Polymers, Levy Processes, Functional Integrals

In this section we review the transformation of problems in polymer physics into functional integrals of the type (1.2), but sometimes including Fermionic fields. These topics are discussed in great detail in the book [27].

A polymer $\omega = (x_0, x_1, \ldots, x_N)$ is a sequence of monomers described by the coordinate x belonging to a finite set Λ. The position x of a monomer in Λ unites information about the species of the polymer and its position in a discrete approximation to space. The typical Λ will be a finite subset of $\mathbf{Z}^d$ with d the dimension of space. In this case all monomers would be of the same species. To a first approximation the activity of a polymer ω would be expected to be a product of Gibbs factors, one for each monomer and one for each link, thus

$$z(\omega) = \prod_{i=0}^{N} \frac{1}{A(x_i, x_i)} \prod_{j=0}^{N-1} (-A(x_j, x_{j+1})), \tag{3.1}$$

where we have encoded the link activities as off-diagonal elements of a matrix $A(x, y)$ and monomer activities as inverses of the diagonal entries. The matrix A has the properties

$$\begin{aligned} A(x, y) &\le 0 \quad \forall x \neq y \in \Lambda; \\ A(x, x) &\ge 0 \qquad \forall x \in \Lambda, \end{aligned} \tag{3.2}$$

and in order that $\sum z(\omega)$ be finite if one end of ω is fixed,

$$\begin{gathered} \sum_y A(x, y) \ge 0, \qquad \forall x; \\ A + A^t \text{ is a positive definite matrix.} \end{gathered} \tag{3.3}$$

We use $\omega : u \to v$ to denote the event that there is a polymer ω with $x_0 = u$ and $x_N = v$ for some N. Then for a non-interacting and non-self-interacting polymer system,

$$P_u(\omega : u \to v) = \frac{\sum_{\omega : u \to v} z(\omega)}{\sum_y \sum_{\omega : u \to y} z(\omega)} \tag{3.4}$$

where P_u is the probability conditioned on there being a polymer which starts at u.

The connection between this non-interacting system and integrals of type (1.1) is a triviality: define a Gaussian measure

$$d\mu(\phi) = \prod_{x \in \Lambda} (d\bar{\phi}(x) \wedge d\phi(x))\, e^{-(\phi, A\bar{\phi})}/\text{Normalization} \tag{3.5}$$

for the *complex* field $\phi = \{\phi(x) \in \mathbf{C} : x \in \Lambda\}$, with $(\phi, A\bar{\phi}) = \sum \phi(x)A(x,y)\bar{\phi}(y)$. Then

$$P_u(\omega : u \to v) = \frac{\int d\mu \bar{\phi}(u)\phi(v)}{\sum_y \int d\mu \bar{\phi}(u)\phi(y)}\,. \tag{3.6}$$

The probability that given a polymer starting at a and another polymer starting at b, one of them ends at c and the other at d is likewise

$$\frac{\int d\mu \bar{\phi}(a)\bar{\phi}(b)\phi(c)\phi(d)}{\sum_{x,y} \int d\mu \bar{\phi}(a)\bar{\phi}(b)\phi(x)\phi(y)}\,; \tag{3.7}$$

and so on for higher order correlations. To prove these statements we have only to note that the right hand side of (3.6) can be expressed in terms of the propagator $(A^{-1})(u,v)$ and $\sum z(\omega)$ is a neat way of writing the expansion $A^{-1} \equiv (D-O)^{-1} = D^{-1} + D^{-1}OD^{-1} + D^{-1}OD^{-1}OD^{-1} + \cdots$, where D is the digaonal part of A and O is the off-diagonal part of A.

In summary: we have seen that the correlations of Gaussian integrals can be represented in terms of probabilities of points being connected by polymers provided the inverse propagator obeys conditions (3.2), (3.3), i.e., is *ferromagnetic*. Let us note that if in (1.1) $\int(\nabla\phi)^2$ were replaced by the finite difference approximation for a lattice $\mathbf{Z}^d$ with $\phi = 0$ outside some set $\Lambda \subset \mathbf{Z}^d$ then, with $v = 0$, (1.1) would describe nearest-neighbor polymers inside Λ. The real field in (1.1) as opposed to the complex field in our discussion has to do with whether the polymers are oriented or not.

Interactions. Suppose all the monomers which comprise all the polymers are interacting by two-body potentials $u(x,y)$. This would introduce a Gibbs factor

$$G(\underline{n}) = \exp\left\{-\frac{\beta}{2}\sum_{x,y} n(x,\underline{\omega})u(x,y)n(y,\underline{\omega})\right\}, \tag{3.8}$$

$n(x,\underline{\omega})$ is the total number of times the collection of polymers $\underline{\omega} = \{\omega_1, \omega_2, \ldots, \omega_M\}$ contains a monomer of x. For a polymer ω, $n(x,\omega)$ is the number of times x occures in the list $\omega = (x_0, x_1, \ldots, x_N)$, and if there are many polymers $\{\omega_1, \omega_2, \ldots, \omega_M\}$, then $n(x,\underline{\omega}) = \sum n(x,\omega_i)$.

Question. Does $\phi^2(x) \equiv \phi(x)\bar{\phi}(x)$ correspond to $n(\omega, x)$ in the sense that correlations of $d\mu(\phi)G(\underline{\phi}^2)/\int d\mu G(\underline{\phi}^2)$ describe polymers interacting with the Gibbs factor (3.8)? We have set $\underline{\phi}^2 \equiv (\phi(x)\bar{\phi}(x))_{x\in\Lambda}$ The answer to this equations is actually *NO*, but following Symanzik [58], we will see that if polymers are replaced by Levy processes which are functions $X(s)$ of a continuous parameter s, and if the quantities $n(x,\omega)$ are replaced by *occupation times*

$$\tau^t(x) = \int_0^t ds\, \delta_x(X(s)), \tag{3.9}$$

then the rule

$$\phi^2(x) \equiv \phi(x)\bar{\phi}(x) \longleftrightarrow \tau(x) \tag{3.10}$$

becomes valid so that the measure $d\mu(\phi)G(\phi^2)$ does indeed describe a (Grand Canonical Ensemble) of objects $\underline{X} = \{X_1(s_1), \ldots, X_M(s_M)\}$ with interaction $G(\sum \tau_j^t)$ as opposed to $G(\underline{n})$. It is possible to descend to polymers by integrating over all $X(s)$ which are different parameterizations of the same polymer ω. Then $G(\sum \tau_j^t)$ becomes another (more complicated) function $\tilde{G}(\underline{n})$.

3.1 Levy Processes

A Levy process $X(t)$ is a random function of a continuous time $t \geq 0$ which describes the *history* of a drunkard who, given that he is at a site x, waits for a random time t distributed according to

$$dp_x(t) = A(x,x)e^{-A(x,x)t}dt, \tag{3.11}$$

and then, with probability $-A(x,y)/A(x,x)$ jumps to site y, and with the remaining probability $1 + \sum_{y \neq x} A(x,y)/A(x,x)$ jumps to a state called *the graveyard* —and is never again seen amongst the states of the living, which are the sites in Λ.

This process is associated with e^{-tA} as we can see formally by decomposing A into its diagonal part (D) and its off-diagonal part $(-O)$ and writing

$$\begin{aligned} e^{-tA} = e^{-t[D-O]} &= \lim_{N\to\infty} \left[e^{-\frac{t}{N}D} e^{\frac{t}{N}O}\right]^N \\ &= \lim_{N\to\infty} \left[e^{-\frac{t}{N}D}\left(1 + \frac{t}{N}O\right)\right]^N . \end{aligned} \tag{3.12}$$

Expand the right hand side as a matrix product and observe that the resulting sequences of matrix indices can be interpreted as histories of the type described in the paragraph above sampled at intervals of time of length $\frac{t}{N}$.

A more accurate way of putting this is that to the exponential e^{-tA} of any matrix satisfying the conditions (3.2) is associated a probability measure $dP_u(X)$ on the space of all histories $\{X(t) : X(0) = u\}$ with the properties:

$$(e^{-tA})(u,v) = \int dP_u(X)\, \delta_v(X(t)), \tag{3.13}$$

where $\delta_x(y) = 1$ if $x = y$, 0 otherwise, and more generally, the *Feynman-Kac* formula

$$\left(e^{-t(A+V)}\right)(u,v) = \int dP_u\, e^{\int_0^t V(X(s))ds}\delta_v(X(t)). \tag{3.14}$$

On the left-hand side, $V(x)$ is acting as a diagonal matrix $V(x)\delta_x(y)$.

If we present dP_u this way, then the question arises: What does this have to do with our first description of $X(t)$ as a dilatory drunkard and what does it have to do with the polymer? To answer this, define, in terms of dp_x given in (3.11),

$$\begin{aligned} dp_x^n(t) &= (dp_x * dp_x * \cdots * dp_x)(t) \\ &\equiv \int dp_x(t_1)\cdots dp_x(t_n)\delta\left(t - \sum_{i=1}^n t_i\right) dt. \end{aligned} \tag{3.15}$$

If a drunkard is conditioned to visit n times the point x, then this is how his total time spent at x will be distributed. When $n = 0$ define $dp^0(t) \equiv \delta(t)dt$. Define, for a polymer ω, the probability measure

$$d^\omega p(\underline{t}) = \prod_{x\in\Lambda} dp_x^{n(x,\omega)}(t(x)) \tag{3.16}$$

on the collection $\underline{t} = \{t(x) : x \in \Lambda\}$. Then one can pass from the measure $dP_u(X)$ to a sum over polymers by

Proposition 3.1

$$\int_0^\infty dt \int dP_u(X)F(\underline{\tau}^t)\delta_v(x(t)) = \sum_{\omega:u\to v} z(\omega)\int d^\omega p(\underline{t})F(\underline{t}), \tag{3.17}$$

where $\underline{\tau}^t = \{\tau^t(x) : x \in \Lambda\}$ *is the collection of random variables*

$$\tau^t(x) \equiv \int_0^t \delta_x(X(s))ds, \tag{3.18}$$

and F *is any bounded smooth function. We see that* $\tau^t(x)$ *is the total time that* $X(s)$, $0 \le s \le t$, *spends at* x.

Proof of Proposition 3.1. Verify it for the special case

$$F(\underline{t}) = e^{-i\sum_x V(x)t(x)} \tag{3.19}$$

by evaluating the right hand side explicitly and recognizing that it is the resolvent expansion for $(A+iV)^{-1}(u,v)$ when $A+iV$ is split into its diagonal part D and off-diagonal part 0 so that $(A+iV)^{-1} = (D+O)^{-1} = D^{-1} + D^{-1}OD^{-1} + \cdots$. For the left hand side of (3.17) we notice that $\sum_x V(x)\tau^t(x) = \int_0^t ds\, V(X(s))$, so it too can be evaluated by the Feynman-Kac formula (3.14). It is

$$\int_0^\infty dt\, e^{-t(A+iV)} = (A+iV)^{-1}. \tag{3.20}$$

Now the general case follows by approximating F with linear superposition of functions of the form (3.19), e.g.,

$$F(\underline{t}) = \int d\underline{V}\, \hat{F}(\underline{V}) \exp\left(-i \sum V(x)t(x)\right). \tag{3.21}$$

□

Now we return to our discussion of interactions and make the $\phi^2 \longleftrightarrow \tau$ assertion more precise with

Theorem 3.2 *[14] For any bounded smooth function $G(\underline{t})$*

$$\begin{aligned} \int d\mu\, G(\underline{\phi}^2)\bar{\phi}(u)\phi(v) &= \int_0^\infty dt \int dP_u\, \delta_v(X(t)) \\ &\quad \times \int d\mu\, G(\underline{\phi}^2 + \underline{\tau}^t) \end{aligned} \tag{3.22}$$

where $\underline{\phi}^2 \equiv (\phi(x)\bar{\phi}(x))_{x\in\Lambda}$.

Proof. Verify both sides for the special case

$$G(\underline{t}) = e^{-i\sum_x V(x)t(x)}$$

using the same reasoning as in the proof of Proposition 3.1 to evaluate the right hand side and using the fact that $\frac{d\mu\, G}{\int d\mu\, G}$ is a Gaussian measure whose covariance is $(A+V)^{-1}$ to evaluate the left hand side. Then pass to more general functions G using superposition as in (3.11). □

At this point one can use the Proposition 3.1 to rewrite the right hand side of (3.22) as a sum over polymers, however the formulas with the Levy processes are simpler [21, 22].

Symanzik showed [58] that ϕ^2 can be interpreted as the sum of the occupation times for a Grand Canonical Ensemble of Lévy processes which are closed loops, e.g.,

$$\int d\mu G(\underline{\phi}^2 + \underline{\tau}) = \lim_{\epsilon \to 0} (1/\mathcal{N}_\epsilon) \sum \frac{1}{M!} \int dP_\epsilon(X_1) \cdots \int dP_\epsilon(X_M)\, G\left(\sum_{j=1}^{M} \underline{\tau}_j + \underline{\tau}\right), \quad (3.23)$$

where $\mathcal{N}_\epsilon$ is the normalization factor—obtained by replacing $G \to 1$ in the right-hand side—and

$$\int dP_\epsilon(X) \equiv \int_\epsilon^\infty \frac{dt}{t} \sum_x \int dP_x(X) \delta_x(X(t)). \quad (3.24)$$

3.2 Further Developments

De Gennes [20] considered the limit where the number of components (two in our case) of ϕ tends to zero. This has the effect of driving the activity of all closed loops in Symanzik's formula to zero so that quantities such as the end-to-end distance of a single self-interacting polymer can be studied by Theorem 3.2.

We prefer another method introduced by Parisi-Sourlas [51] and separately McKane [49] to achieve the same result. They introduce anti-commuting variables $\psi(x), \bar{\psi}(x)$ as well as $\phi(x)$. Following the usual rules of Berezin calculus there is a partner

$$d\mu(\psi) = \prod_{x \in \Lambda} (d\bar{\psi}(x) d\psi(x)) \cdot e^{-(\psi, A\bar{\psi})} / \text{Normalization} \quad (3.25)$$

to $d\mu(\phi)$, and we can obtain the Parisi-Sourlas-McKane formula

Theorem 3.3 *For any* smooth *bounded* $G(\underline{t})$*,*

$$\int d\mu(\phi) \int d\mu(\psi)\, G(\underline{\phi}^2 + \underline{\psi}^2) \bar{\phi}(u) \phi(v) = \int_0^\infty dt \int dP_u \delta_v(X(t)) G(\underline{\tau}^t). \quad (3.26)$$

where $\underline{\phi}^2 + \underline{\psi}^2 \equiv (\phi(x)\bar{\phi}(x) + \psi(x)\bar{\psi}(x))_{x \in \Lambda}$.

Proof. We can exactly repeat the proof of Theorem 3.2. The Fermions play no role in the algebra. We obtain the same right hand side except that $\int d\mu G(\underline{\phi}^2 + \underline{\tau})$ is replaced by

$$\int d\mu(\phi) \int d\mu(\psi)\, G(\underline{\phi}^2 + \underline{\psi}^2 + \underline{\tau}) \quad (3.27)$$

which is equal to $G(\underline{\tau})$! This claim follows by checking the special case $G(\underline{t}) = \exp(-i \sum V(x) t(x))$. In this case the integral (3.27) factors into two Gaussian integrals, one over ϕ giving a determinant, and the other over ψ giving an inverse determinant which exactly cancel. As usual, more general functions G are then obtained by superposition as in (3.21). $\square$

4 Flow Equations

4.1 Perturbations of Gaussian Measures

The objective of the rest of these lectures will be the accurate evaluation of functional integrals of *small* perturbations of Gaussian measures, of the form

$$\int d\mu(\phi)\; Z(\Lambda, \phi). \tag{4.1}$$

Here $d\mu$ is a Gaussian measure on the space $\mathcal{C}(\Lambda)$ of continuous functions on a compact set $\Lambda \subset \mathrm{R}^d \times \Omega$, where Ω is a space for *internal* indices such as charge or direction of a dipole. The measure $d\mu$ has mean zero and is specified uniquely by its covariance[4] $C(\xi, \xi')$,

$$\int d\mu_C(\phi)\; \phi(\xi)\phi(\xi') = C(\xi, \xi') \tag{4.2}$$

The integrand $Z(\phi) \equiv Z(\Lambda, \phi)$ is a functional defined on $\mathcal{C}(\Lambda)$, which is a Banach space under the supremum norm. The reader could keep the following applications of the integrals (4.1) in mind:

(i) If we write $Z(\Lambda, \phi) = e^{-V(\phi)}$ and $V(\phi)$ is an integral of a local function of ϕ, then (4.1) can be thought as a Feynman integral for a quantum field theory with ultraviolet and infrared cutoff.

(ii) By the Sine-Gordon representation, the grand-canonical partition functions of statistical mechanics systems with two-body interactions can be written in the form (4.1). In this case Z corresponds to the (grand-canonical) partiton function of an ideal gas.

(iii) In the above cases, the log of (4.1) corresponds to a free energy. However, if we introduce an extra field ψ, the corresponding log

$$G(\Psi) = \log \int d\mu(\phi)\; Z(\Lambda, \phi + \psi) \tag{4.3}$$

is the generating functional for the amputated Greens functions of the theory. The understanding of integrals of the form (4.1) allows us to take continuum limits or discover the long distance properties of a statistical-mechanical theory.

[4]Not all functions $C(\xi, \xi')$ can arise as covariances of Gaussian measures: sufficient conditions are discussed in [59]. In particular it is sufficient if $C(\xi, \xi')$ is a positive-semidefinite continuously differentiable function on $\Lambda \times \Lambda$.

But in the sequel, we shall consider general expressions of the form (4.1), requiring only that $Z(\phi)$ not grow more rapidly as $|\phi| \to \infty$ than what is required to be integrable with respect to a Gaussian measure and that $Z(\phi)$ has variational derivatives (same as Fréchet derivatives) of all orders and that these be measures. Let us recall that the variational derivatives of a functional are defined via usual derivatives with respect to linear increments of the arguments. For instance, derivatives of the form $\delta^N Z/\delta\phi^N$ are functionals with the defining property that for any functions $f_1, \ldots, f_N$ in $\mathcal{C}(\Lambda)$,

$$\frac{\partial}{\partial\lambda_1} \cdots \frac{\partial}{\partial\lambda_N} Z\left(\phi + \sum_{j=1}^{N} \lambda_j f_j\right)\Bigg|_{\underline{\lambda}=0} =$$
$$\int d\xi_1 \cdots \int d\xi_N \, \frac{\delta^N Z(\phi)}{\delta\phi(\xi_1)\cdots\delta\phi(\xi_N)} f(\xi_1)\cdots f(\xi_N). \tag{4.4}$$

In many cases, for each field-configuration ϕ these functionals are measures on Λ^N, for example:

$$\frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)} \int dz \, \phi^2(z) = 2\delta(x-y), \tag{4.5}$$

but they could be more singular objects, like derivatives of delta functions. We shall require that $(\delta^N Z/\delta\phi^N)(\phi)$ exists for each N and ϕ *as a signed Borel measure* on $\Lambda \times \cdots \times \Lambda$. Those cases producing derivatives of deltas, for instance functionals of $\nabla\phi$, will be handled by introducing additional fields plus additional constraints, for instance the requirements that these extra fields coincide with the gradients.

Definition 4.1 *A functional $A(\phi)$ defined on $\mathcal{C}(\Lambda)$ is smooth if it posesses derivatives of all orders which exist as Borel measures as described above. It is of limited growth if these derivatives are all bounded according to*

$$\| \int d^N\xi \, f(\xi_1, \ldots, \xi_N) \frac{\delta^N A(\phi)}{\delta\phi(\xi_1)\cdots\delta\phi(\xi_N)} \| \leq C_{N,\epsilon} e^{\epsilon \int_\Lambda d\xi \, \phi^2(\xi)} \tag{4.6}$$

for any $N, f, \epsilon > 0$ with f continuous and $|f| \leq 1$.

4.2 Wilson Renormalization

Wilson has taught us that an effective way to study integrals of the form (4.1) is to successively eliminate the "degrees of freedom" present in the Gaussian measure $d\mu$ by performing the infinite number of integrations on the integrand Z in small bunches.

In this way one obtains a sequence of smoother and smoother Gaussian measures but applied to integrands which have poorer locality properties. If one manages to control the loss of locality of the integrand then at the end one is left with an integral over a trivial delta-function measure and the problem is solved. To develop this program we decompose the covariance of ϕ in the form

$$C(\xi,\xi') = C'(\xi,\xi') + B(\xi,\xi').$$

Then

$$\begin{aligned}\int d\mu_C(\phi) Z(\phi) &= \int d\mu_{C'}(\phi') \int d\mu_B(\zeta) Z(\phi'+\zeta) \\ &\equiv \int d\mu_{C'}(\phi')\,(\mu_B * Z)(\phi')\,.\end{aligned} \tag{4.7}$$

To prove this it suffices, by using the defining property of Gaussian measures (2.12), to prove it in the special case where $Z(\phi) = \exp i(\phi, f)$, i.e., $\int d\mu_{C'}(\phi') \int d\mu_B(\zeta) \exp[i(\phi' + \zeta, f)] = \exp[(-1/2)(f, Cf)]$. This in turn is a consequence of (2.12).

Formula (4.7) is the formalization of Wilson's idea: the RHS involves a "smoother" Gaussian measure—$d\mu_{C'}$—and a less local integrand—the convolution of $d\mu_B$ and Z. Heuristically, B is chosen so to include all fluctuations up to a certain space scale characterized by a "length" $\ell(t)$ so that the random field asociated to the measure $d\mu_{C'}$ is approximately constant on cubes of side of order $\ell(t)$. In many RG approaches, this removal of fluctuations is done in discrete steps, for instance increasing $\ell(t)$ by a fixed factor in each iteration. We have found, following [53], that it is more convenient to proceed in a *continuum* of steps, that is, integrating out fluctuations over an "infinitesimal" interval so to obtain a "flow" of Gaussian measures with associated integrands described by *differential equations*. Let us present an heuristic discussion of this procedure. The "infinitesmal" integration of degrees of freedom corresponds to letting the covariance B be "infinitesmal" in the sense that

$$\sup_\xi B(\xi,\xi') \ll 1\ ; \quad \sup_\xi \int |B(\xi,\xi')|\, d\xi' \ll 1\ . \tag{4.8}$$

In this case, expanding $Z(\phi' + \zeta)$ up to fourth order in ζ we obtain:

$$\begin{aligned}(\mu_B * Z)(\phi') &= \int d\mu_B(\xi) \Big\{ Z(\phi') + \int d\xi\, \frac{\delta Z(\phi')}{\delta\phi'(\xi)} \zeta(\xi) d\xi \\ &\quad + \frac{1}{2} \int d\xi \int d\xi'\, \frac{\delta^2 Z(\phi')}{\delta\phi'(\xi)\delta\phi'(\xi')} \zeta(\xi)\zeta(\xi') \;+\; O(\zeta^4) \Big\} \\ &= Z(\phi') \;+\; \frac{1}{2} \int d\xi \int d\xi'\, \frac{\delta^2 Z(\phi')}{\delta\phi'(\xi)\delta\phi'(\xi')} B(\xi,\xi') \;+\; O(B^2)\,.\end{aligned} \tag{4.9}$$

(As μ_B has zero mean, the expectation of odd-degree monomials in the field ζ is zero.) Therefore, to leading order in B, formula (4.9) implies that

$$\int d\mu_C(\phi)\, Z(\phi) \;=\; \int d\mu_{C'}(\phi')\, Z'(\phi') \tag{4.10}$$

with

$$Z'(\phi') - Z(\phi) \;=\; \frac{1}{2}\int d\xi \int d\xi' \, \frac{\delta^2 Z(\phi')}{\delta\phi'(\xi)\delta\phi'(\xi')} B(\xi,\xi') \,. \tag{4.11}$$

In the limit where C' becomes "infinitesimally" close to C this finite-diference equation becomes a differential equation, and the neglect of the term $O(B^2)$ is fully justified. We can then iterate (4.10) until we reach a zero-covariance Gaussian measure, that is, a δ-function.

The formalization of this idea corresponds to constructing a family of covariances $C_{\infty t}$ parametrized by a continuous parameter t, such that $C_{\infty t} \to 0$ as $t \to \infty$. To reproduce the argument sketched above, we need some additional conditions related to (4.8), which ensures a sufficiently smooth variation of the covariances with t. We shall call such a family a *deformation*. In this and what follows, a double subindex "ts" where $t \geq s$ in a covariance or measure is intended to emphasize the interpretation that the corresponding random field has fluctuations only on the scales "between s and t".

Definition 4.2 *A* deformation *of C is a family $C_{\infty t}$ of covariances such that $C_{\infty 0} = C$; $\lim_{t\to\infty} C_{\infty t}(\xi,\xi) = 0$ uniformly in ξ; and $\dot{C}_t \equiv dC_{\infty t}/dt$ exists and satisfies: (i) it is continuously differentiable in ξ, ξ'; (ii) it is negative-definite; and (iii) it is integrable in the sense that $\sup_\xi \int |\dot{C}_t(\xi,\xi')d\xi'|$ is finite.*

Remark: The range $[0,\infty)$ for the parameter t is, of course, arbitrary. In some cases we shall use instead $t \in [0,1]$ or some other range. We put a single index on $\dot{C}_t$ because differentiation with respect to t makes this a single scale object.

Now we can formalize the steps (4.9)–(4.11). Formula (4.11) becomes a partial (functional) differential equation for a family of functionals Z_{t0}. As Gaussian functions are fundamental solutions of the heat equation, it is no surprise to find out that Z_{t0} is the solution of a suitably formulated heat equation. The result is the following lemma. A comment about notation: in what follows, a functional will be labelled with subindexes "$t0$" to help us remember that it was constructed by integration over the scales between 0 and t. Thus Z_{t0} is always to be integrated with respect to a measure $d\mu_{\infty t}$ which integrates over all scales larger than t.

Lemma 4.3 *Let $d\mu_{\infty t}$ be the Gaussian measures whose covariances are $C_{\infty t}$ in a deformation of C, then for any $t \geq 0$,*

$$\begin{aligned}\int d\mu Z &= \int d\mu_{\infty t} Z_{t0} & (4.12)\\ &= \lim_{t\to\infty} Z_{t0}(0) & (4.13)\end{aligned}$$

if Z_{t0} obeys the functional heat equation

$$\begin{aligned}\frac{\partial Z_{t0}}{\partial t} &= \frac{1}{2}\Delta_t Z_{t0} & (4.14)\\ Z_{00} &= Z\ .\end{aligned}$$

Here

$$\Delta_t Z_{t0} = \int d\xi \int d\xi' \, \frac{\delta^2 Z_{t0}}{\delta\phi(\xi)\delta\phi(\xi')}\,(-\dot{C}_t(\xi,\xi')). \tag{4.15}$$

Also, a solution for the heat equation is

$$Z_{t0}(\phi) = \int d\mu_{t0}(\zeta) Z(\phi+\zeta), \tag{4.16}$$

where $d\mu_{t0}$ has covariance $C_{\infty 0} - C_{\infty t}$.

Proof. It is a transcription of the previous steps (4.8)–(4.11). Write $C_{\infty t} = C_{\infty t'} + C_{t't}$. As in (4.7), for *any* smooth functional Z_{t0}

$$\int d\mu_{\infty t}(\phi)\, Z_{t0}(\phi) = \int d\mu_{\infty t'}(\phi') \int d\mu_{t't}(\zeta) Z_{t0}(\phi'+\zeta)\ . \tag{4.17}$$

A Taylor expansion of $Z_{t0}(\phi'+\zeta)$ yields the analogue of (4.9):

$$(\mu_{t't} * Z_{t0})(\phi') \;=\; Z_{t0}(\phi') \;+\; \frac{1}{2}\int d\xi \int d\xi' \, \frac{\delta^2 Z_{t0}(\phi')}{\delta\phi'(\xi)\delta\phi'(\xi')} C_{t't}(\xi,\xi') \;+\; O\Big((C_{t't})^2\Big)\ . \tag{4.18}$$

Expand the first term in this identity in powers of $(t-t')$ and use the hypotheses on $\dot{C}_t$ and Z:

$$\mu_{t't} * Z_{t0} \;=\; Z_{t'0} + \left[\frac{\partial Z_{t0}}{\delta t}\right]_{t=t'} (t-t') - \frac{1}{2}\left[\Delta_t Z_{t0}\right]_{t=t'} (t-t') + o\Big(t-t'\Big)\ . \tag{4.19}$$

(We have used the facts that $\frac{\partial C_{t't}}{\partial t} = \frac{\partial}{\partial t}(C_{\infty t'} - C_{\infty t}) = -\dot{C}_t$; and that $C_{tt} = 0$.) Therefore, from (4.17) and (4.19)

$$\begin{aligned}
\frac{d}{dt}\int d\mu_{\infty t} Z_{t0} &= \lim_{t'\to t} \frac{\int d\mu_{\infty t} Z_{t0} - \int d\mu_{\infty t'} Z_{t'0}}{t'-t} \\
&= -\int d\mu_{\infty t}\left[\frac{\partial Z_{t0}}{\partial t} - \frac{1}{2}\Delta_t Z_{t0}\right] ;
\end{aligned}$$

which shows that (4.14) is sufficient for $\int d\mu_{\infty t} Z_{t0}$ to be constant with t. This proves (4.12).

The identity (4.13) follows from the fact that $d\mu_{\infty t}(\phi) \to \delta(\phi)$ weakly as $t \to \infty$, by the condition $C_{\infty t}(\xi,\xi) \to 0$ in the definition of a deformation of C.

The proof of (4.16) is left as an exercise for the reader (Problem 4.3.1). □

Remark: The standard Wilson procedure corresponds to the identity (4.13). We have, however, stated the lemma so to highlight also the intermediate step (4.12). This, in a sense, is a slight generalization of the usual prescription that provides an extra flexibility: it shows how the ideas can be used even in those cases in which for some reason we are unable to iterate the procedure all the way to $t = \infty$.

4.3 Problems

4.3.1 Prove that (4.16) solves (4.14). Notice that this means that the solution of the *functional* heat equation is the convolution of a Gaussian measure with the initial condition. This is exactly analogous to the familiar PDE case. [Hint: Proceed as in the previous proof to show that $\frac{\partial}{\partial t}\mu_{t0} * Z = (1/2)\Delta_t(\mu_{t0} * Z)$.]

4.3.2 More generally, show that a functional A_t solves the differential equation

$$\begin{aligned}
\frac{\partial A_t}{\partial t} &= \Delta_t A_t + f_t(A_t) \\
A_0 &= A ,
\end{aligned}$$

where, for each t, f_t is a reasonable function (e.g. involving powers), if A_t solves the integral equation

$$A_t = \mu_{t0} * A + \int_0^t \mu_{ts} * f_s(A_s)\, ds .$$

This is the functional-equation analogous of the well-known process of replacing a non-linear PDE by the integral equation obtained by convolving the non-linearity with the fundamental solution of the linear part. This integral equation is then used to generate a diagramatic expansion.

4.3.3 *A form of Wick's theorem.* Using Exercise 4.3.1 prove that if V is a polynomial in ϕ then

$$\begin{aligned}\mu_C * V &= \exp[\frac{1}{2}\Delta_C]V \\ &\equiv \sum \frac{1}{N!}\left(\frac{1}{2}\Delta_C\right)^N V \end{aligned} \tag{4.20}$$

where

$$\Delta_C \equiv \int_\Lambda d\xi \int_\Lambda d\xi' \frac{\delta}{\delta\phi_i(\xi)} C(\xi,\xi') \frac{\delta}{\delta\phi_j(\xi')} . \tag{4.21}$$

Prove that the Wick powers defined by

$$:\phi^n:(x) \equiv e^{-\frac{1}{2}\Delta_C}\phi^n(x) \tag{4.22}$$

are orthogonal, $\int d\mu_C :\phi^n:(x) :\phi^m:(y) = 0$ if $n \neq m$, and that this way of defining $:\phi^n:$ agrees with another standard definition: $: e^{i\phi(x)} :\equiv e^{\frac{1}{2}C(x,x)}e^{i\phi(x)} = \sum \frac{i^N :\phi^N:}{N!}$, c.f. Exercise 5.5.2 below.

5 The Effective Action

In physical applications, one expects that the functional Z will have *locality properties*. By this we mean that it approximately factorizes when evaluated on a function ϕ which is the sum of two functions with distant supports. One way to see the presence of locality properties is to define the *effective action* V by $Z = e^V$. In terms of V, locality is the property that V is (approximately) of the form $\int d\xi\, f(\phi(\xi), \nabla\phi(\xi), \ldots)$. When applying Lemma 4.3 we are interested in the extent to which locality properties of Z_{t0} are preserved under evolution by the heat equation (4.14). This can be done by defining V_{t0} by $Z_{t0} = \exp(V_{t0})$ and studying the resulting equation for it.

5.1 Functional Equations

The heat equation (4.14) for Z_{t0} implies that V_{t0} solves

$$\frac{\partial V_{t0}}{\partial t} = \frac{1}{2}\left[\Delta_t V_{t0} + (\nabla V_{t0}\,,\, \nabla V_{t0})_t\right], \tag{5.1}$$

where ∇V denotes the measure $\frac{\partial V}{\partial\phi(\xi)}d\xi$ and, given a suitable family of covariances $C_{\infty t}$ (Definition 4.2), $(\ ,\ )_t$ is the bilinear form ("dot product") defined by

$$(\mu_1\,,\, \mu_2)_t \equiv \int d\mu_1(\xi_1) \int d\mu_2(d\xi_2)\, [-\dot{C}_t(\xi_1, \xi_2)]$$

for any two measures μ_1, μ_2 in Λ. Explicitly

$$(\nabla V_{t0}\,,\, \nabla V_{t0})_t \;=\; \int d\xi_1 \int d\xi_2\, \frac{\delta V_{t0}}{\delta\phi(\xi_1)} \left[-\dot{C}_t(\xi_1, \xi_2)\right] \frac{\delta V_{t0}}{\delta\phi(\xi_2)} \tag{5.2}$$

("dot product" between two "gradients"). The equation (5.1) was already introduced by Wilson [60], and its importance was emphasized by Polchinsky [53].

A solution to (5.1) can sometimes be found by rewriting it as an integral equation (Exercise 4.3.2)

$$V_{t0} = \mu_{t0} * V_{00} + \frac{1}{2}\int_0^t ds\, \mu_{ts} * (\nabla V_{s0}\,,\, \nabla V_{s0})_s \tag{5.3}$$

and generating a series by iteration. We observe that the first term in the right-hand-side corresponds to the solution of the linear part. The notation is

$$(\mu_{ts} * F)(\phi) \equiv \int d\mu_{ts}(\zeta) F(\phi + \zeta)$$

$$d\mu_{ts} \text{ is Gaussian with covariance } C_{ts} \equiv C_{\infty s} - C_{\infty t} \tag{5.4}$$

(see comment before Definition 4.2).

From (4.13) we see that to compute the free energy (or the generating functionals) we only need the value of $V_{\infty 0}$:

$$\log \int d\mu \, Z = \lim_{t\to\infty} V_{t0}(0) \ , \tag{5.5}$$

but, as commented above, we wish to be slighty more general and consider V_{t0} for all values of t.

We turn to the iterative solution of the integral flow (5.3), which will be an expansion in terms of tree diagrams. We point out that this integral flow generates essentially the same expansion as Gallavotti *et al.* have been using [8, 29], but they like to work integrating out the degrees of freedom in *finite steps* instead of *infinitesimal steps*. They also have tree diagrams but their trees are not to be confused with ours: their trees label the hierarchy of scales in their expansion whereas out trees are conventional Feynman diagrams or sometimes Mayer diagrams.

5.2 Formal Series for the Classical Action

To familiarize ourselves with the process of generating a series solution from (5.3), let us first see what happens if all the convolutions with respect to μ are left out; that is, if we replace (5.3) by

$$V_{t0}^{\mathrm{cl}} = V_{00} + \frac{1}{2}\int_0^t ds \, (\nabla V_{s0}^{\mathrm{cl}} \, , \, \nabla V_{s0}^{\mathrm{cl}})_s \ . \tag{5.6}$$

We notice that this is the integral equation associated to the functional PDE

$$\begin{aligned} \frac{\partial V_{t0}^{\mathrm{cl}}}{\partial t} &= \frac{1}{2}(\nabla V_{t0}^{\mathrm{cl}} \, , \, \nabla V_{t0}^{\mathrm{cl}})_t \\ V_{00}^{\mathrm{cl}} &= V_{00} \end{aligned} \tag{5.7}$$

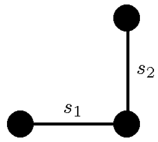
which is the Hamilton-Jacobi equation obtained by leaving out the Laplacian in (5.1). So our warm-up study of (5.6) yields in fact an expression for what, in the quantum-field interpretation (application (i) in Section 4.1), corresponds to the classical action.

Equation (5.6) lends itself to a natural process of iteration: to zeroth order $V_{t0}^{\mathrm{cl}} \approx V_{00}$; inserting this approximation in the RHS ($V_{s0}^{\mathrm{cl}} \approx V_{00}$) we get a first-order approximation for V_{t0}, which can be substituted in the RHS to generate a second-order approximation, and so on. The first few terms of the series generated in this way are:

$$
\begin{aligned}
V^{\rm cl}_{t0} &= V_{00} + \frac{1}{2}\int_0^t ds\, (\nabla V_{00}\,,\, \nabla V_{00})_s + \left(\frac{1}{2}\right)^2 \int_0^t ds_1 \int_0^{s_1} ds_2 \\
&\quad \times \Big[\Big(\nabla V_{00}\,,\, (\nabla^2 V_{00}\,,\, \nabla V_{00})_{s_2}\Big)_{s_1} + \Big(\nabla V_{00}\,,\, (\nabla V_{00}\,,\, \nabla^2 V_{00})_{s_2}\Big)_{s_1} \\
&\qquad + \Big((\nabla^2 V_{00}\,,\, \nabla V_{00})_{s_2}\,,\, \nabla V_{00}\Big)_{s_1} + \Big((\nabla V_{00}\,,\, \nabla^2 V_{00})_{s_2}\,,\, \nabla V_{00}\Big)_{s_1}\Big] \qquad (5.8) \\
&= g_0(t) + g_1(t) + g_2(t) + \cdots \,.
\end{aligned}
$$

Here $\nabla^k V$ denotes the measure $\frac{\delta^k V}{\delta\phi(\xi_1)\cdots\delta\phi(\xi_k)}\, d\xi_1 \cdots d\xi_k$. We notice that in the iteration of (5.6) each additional level of brackets $(\;,\;)_C$ comes together with a "∇" operator; hence, a term of the form $\nabla^k V_{00}$ appears only inside k nested levels of such brackets. The bookeeping of this type of series is usually done with the aid of diagrams: Terms are associated to graphs whose edges represent covariances $-\dot{C}_s$ ("propagators") and where a vertex with k impinging lines corresponds to a measure ∇^k. For instance

$$\Big(\nabla V_{00}\,,\, (\nabla^2 V_{00}\,,\, \nabla V_{00})_{s_2}\Big)_{s_1} \equiv$$

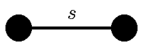


(5.9)

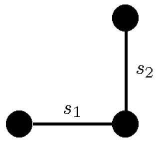
$$= \int d\xi_1\, d\xi_2\; d\xi_3\, d\xi_4\, \frac{\delta V_{00}}{\delta\phi(\xi_1)} \left[-\dot{C}_s(\xi_1,\xi_2)\right] \frac{\delta^2 V_{00}}{\delta\phi(\xi_2)\delta\phi(\xi_3)} \left[-\dot{C}_s(\xi_3,\xi_4)\right] \frac{\delta V_{00}}{\delta\phi(\xi_4)}\,.$$

In this way, (5.8) can be depicted as

$$V^{\rm cl} \;=\; \bullet \;+\; \frac{1}{2}\int_0^t ds$$



$$+\; \frac{1}{4}\int_0^t ds_1 \int_0^{s_1} ds_2 \Big[$$



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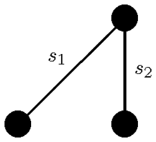

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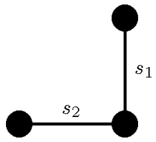

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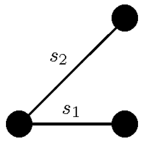
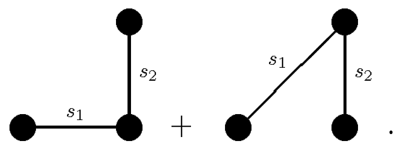


$$\Big] + \cdots \,. \qquad (5.10)$$

Let us analyze a bit more the double-integral term on the right, which we denote as $g_2(t)$. Of course, the four diagrams give the same contribution, but it is convenient to keep the distinction and do some juggling to obtain an expression with better mnemonics. The first two diagrams represent the term

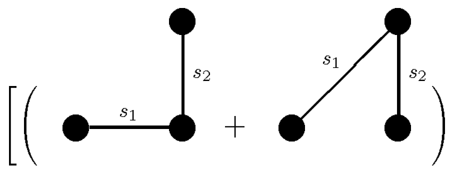
$$\Big(\nabla V_{00}\,,\,\nabla(\nabla V_{00}\,,\,\nabla V_{00})_{s_2}\Big)_{s_1} \;\equiv$$

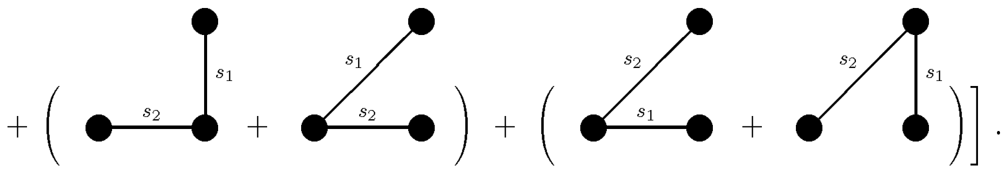


(5.11)

This corresponds to a partition of the three vertices ① ② ③ into two sets: a vertex representing the left factor V_{00} in the dot product (5.11) and a set of two vertices representing the right factor $(\nabla V_{00}\,,\,\nabla V_{00})_{s_2}$. We obtain a more attractive formula by *symmetrization* over the labels of the V factors. That is, one writes instead

$$\Big(\nabla V_{00}\,,\,\nabla(\nabla V_{00}\,,\,\nabla V_{00})_{s_2}\Big)_{s_1} \;=\; \frac{1}{3}$$

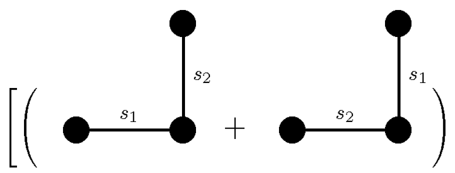


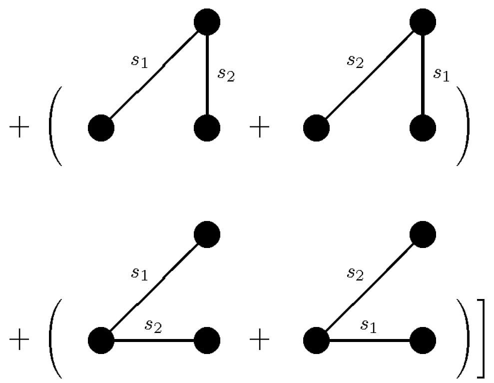


(5.12)

The same RHS is obtained from the analogous symmetrization of the permuted dot product $\Big((\nabla^2 V_{00}\,,\,\nabla V_{00})_{s_2}\,,\,\nabla V_{00}\Big)_{s_1}$. Adding both contributions one obtains

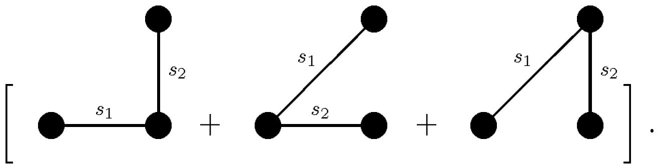
$$g_2(t) \;=\; \frac{1}{3!}\int_0^t ds_1 \int_0^{s_1} ds_2$$





(5.13)

$$= \frac{1}{3!}\int_0^t ds_1 \int_0^t ds_2$$



(5.14)

In fact, as we shall prove, the same type of symmetrization can be applied at each order N of the iteration. We obtain a sum over all trees T of N vertices. At each vertex sits a copy $\phi_i(\xi)$ of the field $\phi(\xi)$ and to each line $(ij) \in T$ there corresponds a parameter s_{ij} and an operator

$$\Delta_{s_{ij}}^{(ij)} \equiv \int_\Lambda d\xi \int_\Lambda d\xi' \frac{\delta}{\delta\phi_i(\xi)} \left[-\dot{C}_{s_{ij}}(\xi,\xi')\right] \frac{\delta}{\delta\phi_j(\xi')} \,. \tag{5.15}$$

Therefore the whole tree represents the operator

$$\Delta_{\underline{s}}^T \equiv \prod_{(ij)\in T} \Delta_{s_{ij}}^{(ij)} \tag{5.16}$$

acting on the N-field functional $\prod_{i=1}^N V_{00}(\phi_i)$. The parameters s_{ij} must be integrated from 0 to t, a fact we shall represent with the notation $\int_0^t d^T s$. The final result is

Proposition 5.1 *Let V_{00} be a smooth functional. Then in the sense of formal power series in powers of V_{00}, the unique solution to (5.6), or (5.7), is*

$$V_{t0}^{\mathrm{cl}} = \sum_N \frac{1}{N!} \sum_{T \text{ on } N} \int_0^t d^T s\, \Delta_{\underline{s}}^T \prod_{i=1}^N V_{00}(\phi_i)\Big|_{\phi_i=\phi} \,, \tag{5.17}$$

where T is summed over all connected tree graphs on N vertices labelled by $i, j = 1, \ldots, N$ and there is one integral over a parameter s_{ij} for each line ij in the tree graph T.

Remarks: This is the statement of the known fact that tree graphs give the classical action i.e. the solution to the Hamilton-Jacobi equation. If the functional V_{00} is of the form $\int d\xi\, \phi^n(\xi)$, the vertex functions are represented by a set of n "legs" associated to the dummy index ξ that is integrated over. With such representation, the successive terms of (5.8) correspond to the ϕ^n*-tree* Feynman diagrams (Exercise 5.5.2).

An elegant way to prove (5.17) is to check that it satisfies the differential equation (5.7). This is sketched in Exercise 5.5.4 for the fully quantum case. Here is a more constructive proof.

Proof. Write

$$g_N(t) \;=\; \frac{1}{N!} \sum_{T \text{ on } N} I_t^T \;.$$

We shall proof by induction on N that it takes the form (5.17). From the previous discussion we know that g_1 (in fact also g_2 and g_3) has the form given in (5.17). Assume it is so for all g_M with $M \leq N-1$. From the integral equation (5.6) we have:

$$g_N(t) \;=\; \sum_{K=1}^{N-1} \frac{1}{2} \int_0^t ds \left(\nabla g_K(s) \,,\, \nabla g_{N-K}(s) \right)_s \tag{5.18}$$

$$= \;\frac{1}{2} \sum_{K=1}^{N-1} \frac{1}{K!(N-K)!} \sum_{\substack{T_1 \text{ on } K \\ T_2 \text{ on } N\backslash K}} \int_0^t ds \left(\nabla I_s^{T_1} \,,\, \nabla I_s^{T_2} \right)_s . \tag{5.19}$$

At this point we symmetrize, as above, among all possible ways of dividing the vertices $N \equiv \{1, \ldots, N\}$ into two sets. For a given integer K, there are $\binom{N}{K}$ ways to choosing a set $X \subset \{1, \ldots, N\}$ of cardinality $|X| = K$; hence

$$g_N(t) \;=\; \frac{1}{2} \frac{1}{N!} \sum_{X \subset N} \sum_{\substack{T_1 \text{ on } X \\ T_2 \text{ on } N\backslash X}} \int_0^t ds \left(\nabla I_s^{T_1} \,,\, \nabla I_s^{T_2} \right)_s . \tag{5.20}$$

The two trees T_1 and T_2 and the connecting bond (parametrized by s) coming from the dot product, assemble a tree T on N. However, the same tree T appears exactly *twice* in the sum (5.20), namely in the partitions $(X, N \setminus X)$ and $(N \setminus X, X)$. Therefore, in passing to a sum over trees $T \subset N$ we must cancel the factor $1/2$. The result is

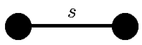
$$g_N(t) \;=\; \frac{1}{N!} \sum_{T \text{ on } N} \sum_{b \in T} I_t^T(b)$$

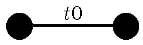
where

$$I_t^T(b) \;=\; \int_0^t ds_b \int_0^{s_b} d^{T \setminus b} s \, \Delta_{\underline{s}}^T \prod_{i=1}^N V_{00}(\phi_i)\bigg|_{\phi_i = \phi} .$$

The sum over all $b \in T$ yields full integrals $\int_0^t ds_{ij}$ for all $(ij) \neq b$ because $[0,t]^N = \cup_{i=1}^N \{s_1, \ldots, s_N | s_j \leq s_i \text{ for } j = 1, \ldots, N\}$ (same step as the transition between (5.13) and (5.14) above). $\square$

We have stated Proposition 5.1 in a form suitable for comparisons with the fully quantum case (Proposition 5.2 below). However, for the present classical case the integrals over the parameters s_{ij} can trivially be performed as the integrand in (5.17) factors into independent integrals $\int_0^t ds_{ij}\,[-\dot{C}_{s_{ij}}] = C_{t0}$. Therefore, adopting the convention that unlabeled lines correspond to covariances C_{t0}:

 $\equiv \int_0^t ds$ 

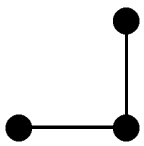
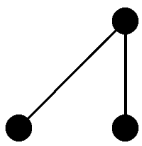
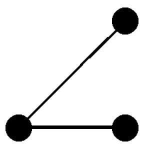
$=$ 

, (5.21)

we can write $V_{t0}^{\rm cl}$ as a sum over unlabelled tree-diagrams:

$$V_{t0}^{\rm cl} \;=\; \sum_N \frac{1}{N!} \sum_{T \text{ on } N} \Delta_{t0}^T \prod_{i=1}^N V_{00}(\phi_i)\bigg|_{\phi_i=\phi} ,$$

$=$  $+ \frac{1}{2}$ 

$+ \frac{1}{3!}\Big[$  $+$  $+$  $\Big] + \cdots$. (5.22)

5.3 Formal Series for the Quantum Action

Let us turn now to the series for the fully quantum action defined by (5.3). The additional ingredient is the convolution by Gaussian measures. To understand its role, consider first the term $\mu_{t0} * V_{00}$ (solution to the linear part of (5.1)). Assuming for simplicity that V_{00} is a polynomial in ϕ we can evaluate $\mu_{t0} * V_{00}$ with the following handy formulas. See Exercise 4.3.3.

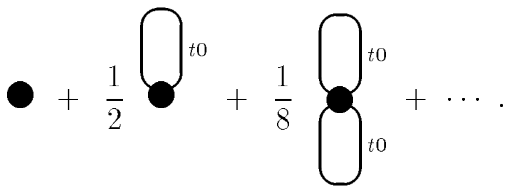
$$\begin{aligned} \mu_{t0} * V_{00} &= \exp[\frac{1}{2}\Delta_{t0}]\, V_{00} \\ &\equiv \sum \frac{1}{N!}\left(\frac{1}{2}\Delta_{t0}\right)^N V_{00} \end{aligned} \tag{5.23}$$

where

$$\Delta_{t0} \equiv \int_\Lambda d\xi \int_\Lambda d\xi' \frac{\delta}{\delta\phi_i(\xi)} C_{t0}(\xi,\xi') \frac{\delta}{\delta\phi_j(\xi')} . \tag{5.24}$$

Following the above convention of using lines for 2-point correlations ("propagators") we can diagramatically depict

$$\mu_{t0} * V_{00} \quad = \tag{5.25}$$



The vertices represent functional derivatives of V_{00}. The number of derivatives is equal to the number of ends of lines at the vertex. The next term in the iteration of (5.3) involves the integral over s of

$$\mu_{ts} * (\nabla(\mu_{s0} * V_{00}) \,,\, \nabla(\mu_{s0} * V_{00}))_s \;=\; [\mu_{ts} * (\mu_{s0} \otimes \mu_{s0})] * (\nabla V_{00} \,,\, \nabla V_{00})_s \,. \tag{5.26}$$

As the convolution of Gaussians is again a Gaussian whose covariance matrix is the sum of the covariance matrices and the tensor product of Gaussians is a Gaussian whose covariance matrix is the direct sum of the covariance matrices, we see that the measure

$$\mu_{ts_{\max}} \equiv \mu_{ts} * (\mu_{s0} \otimes \mu_{s0}) \;. \tag{5.27}$$

is in fact a new Gaussian measure (the reasons for the subscript $s_{\max}$ will be seen more clearly below). This measure integrates over variables $(\phi_i(\xi))_{i=1,2}$ where the index i is there because there are two factors of V_{00} and each of them is numbered by a value of i indicating that the i^{th} factor is evaluated on the field $\phi_i(\xi)$. From (5.27) the reader may check that the covariance of $\mu_{ts_{\max}}$ is

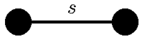
$$\int d\mu_{ts_{\max}}\, \phi_i(\xi)\phi_j(\xi') = \begin{cases} C_{t0}(\xi,\xi') & \{i,j\} = \{1,1\} \text{ or } \{2,2\} \\ C_{ts}(\xi,\xi') & \{i,j\} = \{1,2\} \,. \end{cases} \tag{5.28}$$

If we now proceed as in (5.25), we obtain that the second-order term is the integral over s of

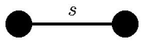
$$\mu_{ts_{\max}} * (\nabla V_{00}\,,\,\nabla V_{00})_s$$

$$= \quad \mu_{ts_{\max}} * \Big\{\Delta_s[V_{00}(\phi_1)V_{00}(\phi_2)]\Big\}\Big|_{\phi_i=\phi}$$

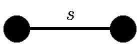
$$= \quad \mu_{ts_{\max}} \quad *$$



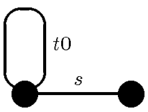
$$= \quad \Big[1+\frac{1}{2}\Delta_{ts_{\max}}+\dots\Big] \quad *$$



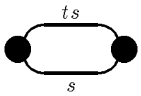
$$=$$



$$+ \quad 2\cdot\frac{1}{2}$$



$$+ \quad 2\cdot\frac{1}{2}$$



$$+ \quad \cdots\,. \tag{5.29}$$

Two remarks are in order. First, note the extra factors 2 in front of the second and third diagram. For the second diagram, this corresponds to the functional Laplacian acting either on the first or on the second vertex. For the third diagram, the action of the Laplacian has also two possibilities corresponding to the first $\delta/\delta\phi$ acting on the first or the second vertex. The second remark concerns the labels of the different lines: they refer to the relevant scales of the fluctuations. The label "ts" corresponds to a covariance $C_{ts} = -[C_{\infty t} - C_{\infty s}]$ involving scales between s and t. On the other hand, a line with a single label s represents a covariance $-\dot{C}_{\infty s}$ that is, involving only the scale s.

One conclusion from what we have done up to now is that the role of the convolutions with respect to Gaussian measures is to put in the loops of the Feynman graphs. In addition, we start to see that even when the iteration of (5.3) yields many successive convolutions with respect to different Gaussian measures, the magic of Gaussian calculus allows us to combine them into an overall convolution with respect to *one* big Gaussian measure. To understand this better it is perhaps convenient to continue our pedestrian analysis to one further order. Two iterations of the integral term in (5.3) yield a term corresponding to (5.11) which is

$$\int_0^t ds_1 \int_0^{s_1} ds_2\, \mu_{ts_1} * \Big(\nabla\mu_{s_1 0} * V_{00}\,,\, \nabla\mu_{s_1 s_2} * (\nabla\mu_{s_2 0} * V_{00}\,,\,\nabla\mu_{s_2 0} * V_{00})_{s_2}\Big)_{s_1}\;.$$

The derivatives with respect to Φ can be commuted through the convolutions and performed on V_{00} to obtain several terms, labelled as before by tree graphs. The term that corresponds to the first tree graph in (5.11) is

$$\int_0^t ds_1 \int_0^{s_1} ds_2 \; [\mu_{ts_1} * (\mu_{s_1 0} \otimes (\mu_{s_1 s_2} * (\mu_{s_2 0} \otimes \mu_{s_2 0})))] \\ * \Big(\nabla V_{00} \, , \, (\nabla\nabla V_{00} \, , \, \nabla V_{00})_{s_2} \Big)_{s_1} \Big|_{\phi_1=\phi_2=\phi_3=\phi} \; . \tag{5.30}$$

As for the second-order term, we see that $\mu_{ts_{\max}} \equiv [\mu_{ts_1} * (\mu_{s_1 0} \otimes (\mu_{s_1 s_2} * (\mu_{s_2 0} \otimes \mu_{s_2 0})))]$ is a new Gaussian measure—which integrates over variables $(\phi_i(\xi))_{i=1,2,3}$—with covariance

$$\int d\mu_{ts_{\max}} \, \phi_i(\xi)\phi_j(\xi') \;=\; \begin{cases} C_{t0}(\xi,\xi') & i=j \\ C_{ts_2}(\xi,\xi') & \{i,j\}=\{2,3\} \\ C_{ts_1}(\xi,\xi') & \{i,j\}=\{1,2\} \\ C_{ts_1}(\xi,\xi') & \{i,j\}=\{1,3\} \end{cases} \tag{5.31}$$

$$\equiv \; C_{ts_{\max}}(\xi,\xi') \; , \tag{5.32}$$

where $s_{\max}$ is the maximum value of the bond parameter (in this case the maximum of s_1 and s_2) found along any path connecting the vertices i, j with $i \neq j$. If $i=j$ $s_{\max} \equiv 0$.

This discussion can be extended to all orders to obtain

Proposition 5.2 *Let V_{00} be smooth functional of limited growth. Then in the sense of formal power series in powers of V_{00}, a solution to (5.1), or (5.3), is*

$$V_{t0} = \sum_N \frac{1}{N!} \sum_{T \text{ on } N} \int_0^t d^T s \, \mu_{ts_{\max}} * \left\{ \Delta^T_{\underline{s}} \prod_{i=1}^N V_{00}(\phi_i) \right\} \Big|_{\phi_i=\phi} \; , \tag{5.33}$$

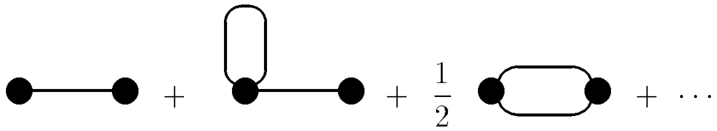
where $\mu_{ts_{\max}}$ is the Gaussian measure on the fields $\{\phi_i(\xi)\}_{\substack{\xi\in\Lambda \\ 1\le i\le N}}$ with covariance

$$C_{ts_{\max}}((\xi,i),(\xi',j)) = \int_{s_{\max}}^t ds \, \left[-\dot{C}_s(\xi,\xi') \right] \; . \tag{5.34}$$

Here $s_{\max}$ is the largest parameter s_{kl} encountered on the unique path in the tree graph T which leads from vertex i to vertex j. This solution is unique within the class of formal power series whose coefficients are smooth and have limited growth.

Proof. A direct verification that (5.33) is the unique formal-series solution with the prescribed initial value V_{00} is left as an exercise (Problem 5.5.4). We will be proving a closely related result when we come to Proposition 6.2. $\square$

The integration over $\underline{s}$ in (5.33) is not as simple as for the classical case (see comment at the end of Proposition 5.1 above). The reason is that the measure $\mu_{ts_{\max}}$ combines the different parameters s_{ij} in a non-factorizable manner. The net result is that in passing to unlabelled Feynman diagrams (that is to diagrams where all the lines corresponds to covariances C_{t0}), one obtains extra combinatorial factors. For example, the s-integration of (5.29) yields

$V_{t0} =$  (5.35)

(Exercise 5.5.1).

5.4 Convergence of the Formal Series

We first emphasize that the above formal series are expansions in powers of V_{00}. A more precise way to state this is to replace throughout $V_{00} \to zV_{00}$, in which case the N-tree term becomes the coefficient of z^N in the resulting expansion. When are these series convergent?

(i) If V_{00} is a local polynomial in ϕ with constant or globally bounded coefficients, then the tree expansion (5.17) converges[5] for t small depending on $\sup|\phi(\xi)|$. One way to understand this is to note that there is a $\frac{1}{N!}$ in (5.17) which is enough to control the number of tree graphs at order N (Cayley's Theorem says the number of tree graphs on $N > 2$ verices is N^{N-2}. Thus the problem reduces to showing that any tree graph on N vertices is bounded by $O(t^{N-1})$, which is an easy exercise or see the proof of Proposition 6.4. More generally, the tree expansion is convergent for t small depending on $\sup|\phi(\xi)|$ whenever V_{00} is analytic in ϕ in the sense that there exists a constant c such that

$$\sup_{\xi_1 \in \Lambda\,,\,|F|\le 1} \frac{1}{N!}\left|\int d\xi_2 \cdots \int d\xi_M \,\frac{\delta^M V_{00}(\phi)}{\delta\phi(\xi_1)\delta\phi(\xi_2)\cdots\delta\phi(\xi_M)} F(\xi_2,\cdots,\xi_M)\right|_{\phi=0} < c^N\,. \tag{5.36}$$

This type of result is standard. There is some discussion of it in [17].

[5]uniformly in Λ after division by the volume $|\Lambda| \equiv \int_\Lambda d\xi$

(ii) If $V_{00}(\phi)$ has *bounded derivatives* as in e.g., functionals of ϕ such as $\int_\Lambda d\xi$ $p(e^{i\phi(\xi)}, e^{-i\phi(\xi)})$, where p is a polynomial, then the series whose terms are given in Proposition 5.2 is convergent for t small by reduction to the case just considered with

$$\sup \left|\mu * \Delta^T \prod V_{00}\right| \;\leq\; \sup \left|\Delta^T \prod V_{00}\right| \tag{5.37}$$

whenever μ is a probability measure.

The dominance by tree expansions can not be expected to hold more generally when the derivatives are not bounded. For example, for $V_{00} = \lambda \int d\xi \phi^4(\xi)$, the tree expansion converges and for $|\phi(\xi)|$ bounded uniformly in ξ it is analytic in λ for $|\lambda| \ll 1$. Such analyticity obviously does not extend to the full series with loops because it would imply analogous properties for $Z_{t0} = \exp V_{t0}$. However, by (4.16) $Z_{t0}(\phi = 0) = \int d\mu_{t0} \exp \lambda \int d\xi\, \phi^4(\xi)$ which is divergent and forces the series in powers of λ to be divergent by the monotone convergence theorem.

On the other hand when V_{00} is a trigonometric polynomial there is an important smoothing effect of the convolution in (5.3). The convolution with respect to $\mu_{ts_{\max}}$ can be performed explicitly using (2.12) which implies, for any Gaussian measure with covariance C:

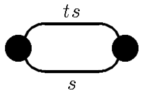
$$\mu_C * \prod_{i=1}^{N} e^{i\beta\phi_i(\xi_i)} \;=\; \exp\left[-\frac{1}{2}\sum_{i,j} C(\xi_i, \xi_j')\right] \prod_{i=1}^{N} e^{i\beta\phi_i(\xi_i)}\,. \tag{5.38}$$

We set $C = C_{ts_{\max}}$ which is positive-definite[6], so that the exponent is positive. This shows that the convolution has a damping effect. Correspondingly, bounds better than (5.37) can be obtained. In fact, formula (5.38) applies exactly when the initial data is the result of the Sine-Gordon transformation from classical statistical mechanics ($V_{00} = \int z d\xi : \exp i\phi(\xi) :$). In this case, the expansion (5.33) becomes an expansion in powers of the activity z which must therefore be the Mayer expansion for the logarithm of the partition function if $t \to \infty$. However in the statistical mechanical framework, rather than using the Sine-Gordon transformation to "import" results from the effective-action formalism, it is better to formulate the flow equations and these expansions directly in the statistical mechanics systems. This is because the Sine-Gordon transformation requires positive definiteness of the potential whereas stability in statistical mechanics is more general, for example any repulsive potential ought to be allowed. This study of the Mayer coefficients is the subject of the next section.

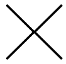
[6]It is not completely obvious that this covariance is positive-definite. See Proposition 6.18

5.5 Problems

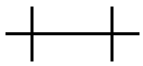
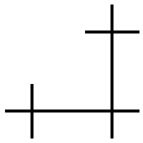
5.5.1 Show that the s-integration of (5.29) produces (5.35), that is, show that

$$\int_0^t ds \quad$$  $$\quad = \frac{1}{2} \quad$$  $$\quad . \qquad (5.39)$$

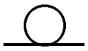
5.5.2 Follow the discussion of Sections 5.2–5.3 through for the particular example of $V_{00} = \int d\xi\, \phi^4(\xi)$. Obtain the first few *unlabelled* (i.e. $\underline{s}$-integrated) Feynman diagrams using in addition the standard graphic convention

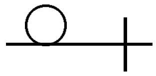
$$V_{00} \equiv$$  $$.$$

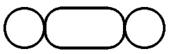
For the classical part, the answer is

$$V_{t0}^{\rm cl} \equiv$$  $$+ \frac{4\cdot 4}{2!}$$  $$+ 3 \cdot \frac{4\cdot(4\cdot 3)\cdot 4}{3!}$$  $$+ \cdots .$$

In the last term, the factor 3 is due to the multiplicity of the diagram (all the 3 possible trees with three vertices give the same contribution). The fully quantum solution requires more combinatorics. One should obtain

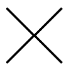
$$V_{00} \equiv V_{0t}^{\rm cl} + \frac{4\cdot 3}{2}$$  $$+ \frac{4!}{2!\cdot 2^2}$$ 

$$+ \frac{4\cdot 4}{2!}\cdot\frac{3\cdot 3\cdot(2)}{2}\cdot\frac{1}{2}$$  $$+ 2\cdot\frac{4\cdot 4}{2!}\cdot\frac{3\cdot 2}{2}$$ 

$$+ \frac{4\cdot 4}{2!}\cdot\frac{3\cdot 2\cdot 3\cdot 2\cdot(3\cdot 2\cdot 2)}{3!\cdot 2^3}\cdot\frac{1}{2}$$  $$+ \cdots . \qquad (5.40)$$

Note the different sources of the combinatoric factors: (i) factors due to multiplicity of diagrams (the factor 2 in the fifth term); (ii) factors due to the underlying tree expansion (e.g. the factor $4\cdot 4/2!$ in all the two-vertex diagrams); (iii) factors due to the application of Wick's theorem (in the form of Exercise 4.3.3); and (iv) factors due to the s-integration (the factors 1/2 in the diagrams with a central loop, see Exercise 5.5.1). The factors (iii) include a denominator $N!\cdot 2^N$ and, in the numerator, we have surrounded by brackets the contributions due to the different ways in which the Laplacians can act over the vertices to produce the given diagram. For instance, in the factor $(3 \cdot 2 \cdot 2)$ of the last diagram (which comes from the action of Δ^3), 3 counts which Laplacian goes to link the two vertices, one of the 2s counts the two possibilities for this link (first $\partial/\partial\phi$ acting on the first or the second vertex), and the remaining 2 counts which of the other Laplacians form a tadpole on vertex 1.

Alternatively, the formal series (5.40) can be written in the form

$V_{00} \;=\; V^{\rm cl}_{0t} + 6\, C_{t0}(0,0)$ $+\, 3\, C^2_{t0}(0,0) + 36$ 
$48\, C_{t0}(0,0)$  $+\, 36\, C^2_{t0}(0,0)$  $+ \;\cdots\;,$

where the factors $C_{t0}(0,0)$ come when a vertex is joined by a line to itself (tadpole graphs). This type of contributions is absent if we start with $V_{00} = \int d\xi : \phi^4(\xi):_{C_{\infty t}}$ where normal ordering $::$ is defined by $:e^{i\phi(x)}:_C \equiv e^{\frac{1}{2}C(x,x)} e^{i\phi} \equiv \sum \frac{(i)^N :\phi^N:_C(x)}{N!}$. Show that $\mu_{t,0} *: e^{i\phi(x)}:_{C_{\infty,0}} =: e^{i\phi(x)}:_{C_{\infty,t}}$ and deduce that $\mu_{t,0} *: \phi(x)^N:_{C_{\infty,0}} =: \phi(x)^N:_{C_{\infty,t}}$.

5.5.3 Now let us review the standard (much easier method) to obtain the series (5.40). From the discussion of Section 4 (in particular Lemma 4.3 and Exercise 4.3.3), one concludes that

$$\exp V_{t0} \;=\; \exp\left[\frac{1}{2}\Delta_{t0}\right] \exp V_{00}\;. \tag{5.41}$$

The material of the present section should suffice to compute the diagrammatics associated to the first few powers $(\Delta/2)^N V^M$ in the right-hand side. To obtain V_{t0} one must take the logarithm—in the sense of formal power series—of the resulting expression; this is easy: it corresponds to keeping only the connected diagrams. In particular convince yourself that in this method the combinatorics of each diagram is the same as counting how many ways there are to pair labelled legs on labelled vertices in order to produce a diagram with a given topology.

5.5.4 Prove Proposition 5.2 by showing that (5.33) solves equation (5.1) in the sense of formal power series, and moreover that it is the only formal series with V_{00} as initial condition. Some guidelines:

(i) By substituting in (5.1) a series of the form $V_{t0} = \sum_N g_N(t)$ and equating powers of V, show that the g_N's must satisfy the coupled system of equations

$$\frac{\partial g_N(t)}{\partial t} = \frac{1}{2}\left[\Delta_t g_N(t) + \sum_{K=1}^{N} (\nabla g_K(t)\,,\, \nabla g_{N-K}(t))_t\right]\;. \tag{5.42}$$

(ii) Show that $g_N(t) = \frac{1}{N!}\sum_{T \text{ on } N} \int_0^t d^T s\, \mu_{ts_{\max}} * \left\{\Delta_{\underline{s}}^T \prod_{i=1}^N V_{00}(\phi_i)\right\}\Big|_{\phi_i=\phi}$ solves (5.42). Note that when d/dt acts on $\mu_{ts_{\max}}$ one obtains the linear part of the equation, because $(d/dt)(\mu_{ts_{\max}} * F) = (1/2)\Delta_{C_t}(\mu_{ts_{\max}} * F)$ (Exercise 4.3.1). Then show that the second term in (5.42) shows up when d/dt acts on the upper limits of the integrals $\int_0^t d^T s$. For an answer: either (a) look to the proof of Proposition 6.2 down the road, where similar considerations play a role; or (b) look up the proof in reference [17, Theorem 3.4].

(iii) Show uniqueness of the solution proceding by induction on N. [Hint: As the non-linear part of equation (5.42) depends on g_L's with $L < N$, by induction one concludes that two solutions f_N and g_N obeying the same initial condition must satisfy the linear equation $(d/dt)(f_N - g_N) = (1/2)\Delta_t(f_N - g_N)$.]

6 The Mayer Expansion

This section is an interlude in the main development. Following [17, 16] we shall formulate flow equations directly in the context of the grand canonical ensemble of statistical mechanics and the coefficients of the Mayer expansion and conclude with a brief discussion of the relationship with the effective potential and Gaussian integrals. The resulting formulas for Ursell coefficients are improvements over the tree graph formulas discussed in my Les Houches Lectures [9] and in particular replace the rather ugly appendix on tree graph formulas in those notes. These types of formulas have an evolution [35, 11, 4, 5, 17, 23] in which they have become progressively neater.

First, we recall the Mayer expansion for a system of particles interacting by two-body forces $u(\xi, \xi')$. Let Z be the Grand Canonical partition function:

$$Z \equiv \sum_{N \geq 0} \frac{1}{N!} z^N \int d^N\xi \, e^{-\beta U(N,\xi)} \, ; \tag{6.1}$$

with

$$U(N, \underline{\xi}) \equiv \begin{cases} \sum_{(ij) \subset N} u(\xi_i, \xi_j) & N \geq 2 \\ 0 & N = 1 \, , \end{cases} \tag{6.2}$$

where $(ij) \subset N$ means that i and j are summed over pairs i, j with $1 \leq i < j \leq N$. (N is identified with $\{1, 2, \ldots, N\}$. More generally, we will define $U(X, \underline{\xi})$ for any set $X \subset \{1, 2, \ldots\}$). The Mayer expansion is the *formal* expansion for $\log Z$ in powers of z

$$\frac{1}{|\Lambda|} \log Z \sim \sum_{1}^{\infty} b_N z^N, \tag{6.3}$$

where $|\Lambda| = \int_\Lambda d\xi$.

6.1 The Ursell Coefficients

The Mayer coefficients b_N in (6.3) will ultimately be obtained by a formula of the form

$$b_N = \frac{1}{|\Lambda|} \frac{1}{N!} \int d^N\xi (e^{-\beta U})_c(N, \underline{\xi}), \tag{6.4}$$

where the coefficients $(e^{-\beta U})_c$ are called *Ursell coefficients*. Their existence and properties will be the subject of the next sections. Thus these coefficients are to be defined so that

$$\log Z \sim \sum_{N\geq 1} \frac{z^N}{N!} \int d^N\xi \, (e^{-\beta U})_c \,, \tag{6.5}$$

(in notational correspondence with (6.1)).

6.1.1 Equivalent Definitions

Perhaps the best way [56] to get at the Ursell coefficients is to define them by recursively solving: For each set $X \subset \{1, 2, \ldots\}$

$$\begin{aligned} e^{-\beta U(X)} &= \sum_{k=1}^{|X|} \frac{1}{k!} \sum_{X_1,\ldots,X_k} \prod_{i=1}^{k} (e^{-\beta U})_c(X_i) \\ &\equiv \sum_{\{X_1,\ldots,X_k\}} \prod_{i=1}^{k} (e^{-\beta U})_c(X_i) \end{aligned} \tag{6.6}$$

where in the first equation the inner sum ranges over all k-tuples of disjoint non-empty subsets $X_1, \ldots, X_k \subset \{1, \ldots, N\}$ forming a partition of the set X. We have deliberately omitted the $\underline{\xi}$ argument because it is useful to consider (6.6) as a procedure for passing from one set of coefficients $\phi(X)$ labeled by subsets $X \subset \{1, 2, \ldots\}$ to another set of coefficients $\phi(X)$ also labeled by subsets.

The proof that (6.6) implies, as a formal power series, (6.5) is standard and very simple (Exercise 6.5.1); the fact that it uniquely determines $(e^{-\beta U})_c$ can be seen recursively starting with $N = 1$. The identity (6.6) implies another popular expression for the Ursell coefficients. We first notice that

$$e^{-\beta U(N)} = \prod_{(ij)\subset N} \left[\left(e^{-\beta u(\xi_i,\xi_j)} - 1\right) + 1\right] \tag{6.7}$$

$$= \sum_{G \text{ on } N} \prod_{(ij)\in G} \left(e^{-\beta u(\xi_i,\xi_j)} - 1\right). \tag{6.8}$$

The sum is over all graphs with vertices on $N \equiv \{1, \ldots, N\}$ (the graphs may contain isolated vertices not belonging to any bond (ij)). We perform this sum by first summing over all graphs G compatible with a partition $\{X_1, \ldots, X_k\}$, and then summing over partitions. A graph G is said *compatible* with the partition $\{X_1, \ldots, X_k\}$ iff G is a union of maximal connected components $G_1, \ldots, G_k$, where G_i is a connected graph on

$X_i, i = 1, \ldots, k$. In this way we obtain (for $N \geq 2$):

$$e^{-\beta U(X,\underline{\xi})} = \sum_{\{X_1,\ldots,X_k\}} \prod_{i=1}^{k} \left[\sum_{\substack{G \text{ on } X_i \\ \text{connected}}} \prod_{(ij)\in G} (e^{-\beta u(\xi_i,\xi_j)} - 1) \right]. \tag{6.9}$$

Comparing with (6.6) we see that

$$(e^{-\beta U})_c(N, \underline{\xi}) = \begin{cases} \displaystyle\sum_{\substack{G \text{ on } N \\ \text{connected}}} \prod_{(ij)\in G} (e^{-\beta u(\xi_i,\xi_j)} - 1) & \text{for } N \geq 2 \\ 1 & \text{for } N = 1 \\ 0 & \text{for } N = 0\,, \end{cases} \tag{6.10}$$

This expression constitutes another complete characterization of the Ursell coefficients, equivalent to (6.6). In contrast with (6.8), the sum in (6.10) is only over *connected* graphs on $\{1, 2, \ldots, N\}$. Expression (6.10) and all of our formulas until the end of Section 6.1 only require that U be of the form $U(N) = \sum_{(ij)\subset N} u(ij)$; the fact that $u(ij)$ might have the structure $u(\xi_i, \xi_j)$ plays no part in their development.

We shall now describe another way, taken from [17], to obtain the Ursell coefficients. It follows the general philosophy advocated in these notes, of considering continuous flows and characterizing objects as solutions of differential equations. Moreover, in Section 6.4 we shall see that it is in direct correspondence with solutions of the effective-action equation (5.3).

As a motivation, we notice that expressions of the form (6.10) are seldomly useful to prove uniform convergence of the Mayer expansion. In many situations there are too many connected graphs G involved: the convergence depends crucially of delicate cancellations which are lost if absolute values are used to bound the contribution of each graph. There is, therefore, the need for expressions involving less, simpler graphs. These are the tree-graph identities. Previous ways to obtain the tree-graph identities [9, and references therein] were based in the use of decoupling parmeters to estimate—via integration by parts—the effect of succesive bond removals that partition the sites into decoupled clusters. Here, instead, a single parameter t will be introduced and the effect of infinitesimal variations (differential equations) will be estimated.

Suppose, then, that to each pair $(ij) \subset \{1, 2, \ldots\}$ we are given a continuously differentiable function $\beta u_{t0}(ij)$ of t. If we let $F_t(X) = (e^{-\beta U_{t0}})_c(X)$ then a short computation which will be carried out shortly shows that F_t satisfies the system, one equation for each X,

$$\frac{d}{dt}F_t(X) = \frac{1}{2}\sum_{i,j\in X}(-\beta\dot{u}_t(ij)) \left\{ F_t(X) + \sum_{I\subset X}\sum_{\substack{i\ni I\\ j\in X\setminus I}} F_t(I)F_t(X\setminus I)\right\}. \tag{6.11}$$

with the initial condition

$$F_{t=0}(X) = \begin{cases} 1 & \text{if } |X| = 1 \\ 0 & \text{if } |X| \neq 1\ . \end{cases} \tag{6.12}$$

If $X = \emptyset$ or $|X| = 1$, equation (6.11) is to be understood as $\frac{d}{dt}F_t(X) = 0$ and from the initial conditions one concludes that

$$F_t(X) = \begin{cases} 0 & \text{if } X = \emptyset \\ 1 & \text{if } |X| = 1\ . \end{cases} \tag{6.13}$$

The system (6.11)–(6.12) characterizes the Ursell coefficients for all $X \subset \Lambda$ because the integrating both sides of the equations we obtain a set of equations that can be recursively and uniquely solved . All this discussion can be summarized in the following

Proposition 6.1 *The equations (6.11)–(6.12) have a unique solution which for every finite subset $X \subset \{1, 2, \ldots\}$ is the Ursell coefficient given by:*

$$\begin{aligned} F_t(X) &= \left(e^{-\beta U_{t0}}\right)_c(X) \\ &= \begin{cases} \displaystyle\sum_{\substack{G \text{ on } X\\ \text{connected}}}\prod_{(ij)\in G}\left(e^{-\beta u_{t0}(ij)} - 1\right) & \text{if } |X| > 1 \\ 1 & \text{if } |X| = 1\ ; \end{cases} \end{aligned} \tag{6.14}$$

or by recursive solution of

$$e^{-\beta U_{t0}(X)} = \sum_{\{X_1,\ldots,X_k\}}\prod_1^k F_t(X_j), \tag{6.15}$$

where $U_{t0}(X) \equiv \sum_{(ij)\subset X} u_{t0}(ij)$, and the sum of the RHS is over all partitions $\{X_1, \ldots, X_k\}$ of X, with k arbitrary. If $k = 0$, $\prod_1^k F_t(\phi) \equiv 1$.

Proof. The proof is complete except for showing that the Ursell coefficient defined by (6.15) solves (6.11), (6.12).

Indeed, if we denote by $f(t)$ the right-hand side of (6.15), the flow equation ((6.11)) implies

$$\frac{df}{dt} = \sum_{\{X_1,\dots,X_k\}} \prod_{\ell=1}^{k} F_t(X_\ell) \sum_{(ij)\subset X} -\beta \dot{u}_t(ij), \tag{6.16}$$

where the terms for which $(ij) \subset X_\ell$ for some ℓ come from the first term in the flow (6.11) whilst the terms with $i \in X_\ell, j \in X_{\ell'}, \ell \neq \ell'$ come from the second term in the flow (6.11). This equation (6.16) can be rewritten as

$$\frac{df}{dt} = \left(\sum_{ij\subset X} -\beta \dot{u}_t(ij) \right) f \ , \tag{6.17}$$

and the initial conditions (6.12) imply $f(0) = 1$. The only solution to this linear ODE problem is $f(t) = e^{-\beta U_{t0}(X)}$. By the equivalence between (6.15) and (6.14), this concludes the proof. □

6.1.2 Tree-Graph Identity

The next task is to solve (6.11)–(6.12) again, but this time using the tree graph generating expansion analogous to the one we have already seen for the effective action when we solved (5.42).

Proposition 6.2 (Tree-Graph Identity) *For any finite set $X \subset \{1, 2, \dots\}$ with $|X| > 1$,*

$$\begin{aligned}
\left(e^{-\beta U_{t0}}\right)_c (X) &= \sum_{T \text{ on } X} \int_0^t d^T s \prod_{(ij)\in T} (-\beta \dot{u}_{s(ij)}(ij)) \exp\left[-\beta U_{ts_{\max}}(X)\right], \\
U_{ts_{\max}}(X) &= \sum_{(ij)\subset X} \int_{s_{\max}}^t ds\, \dot{u}_s(ij) \\
&= \int_0^t d\tau \sum_{C\in\gamma(\tau,\underline{s},T)} \dot{u}_\tau(C),
\end{aligned} \tag{6.18}$$

where $s_{\max} = s_{\max}(ij)$ is the largest parameter s_{kl} encountered on the unique path in the tree graph T which leads from vertex i to vertex j. T is summed over all tree-graphs on the set of vertices X, $\underline{s} \equiv (s(ij) : (ij) \in T)$ is a set of parameters, one for *each bond (ij) in T, which are integrated from 0 to t by $\int_0^t d^T s$, the family $\gamma(\tau, \underline{s}, T)$ is the partition of X into clusters $\{C_1, C_2, \dots, C_k\}$, where each C_i consists of vertices connected by $T \setminus \{(ij) \in T : s(ij) \geq \tau\}$.*

The second formula for $U_{ts_{\max}}$ follows from the first by merely unravelling the definitions although it might give you a headache trying to see it. This formula shows that $U_{ts_{\max}}$ is a stable interaction when $\dot{u}_s$ is stable.

Proof. By the last proposition it suffices to show that the right-hand side $\tilde{F}_t(X)$ of (6.18) solves the flow (6.11) and satisfies the initial condition (6.12). This last fact is obvious ($\tilde{F}_t(X) = 1$ by definition when $|X| = 1$). To check Equation (6.11) we notice that when $\frac{d}{dt}$ acts on $\tilde{F}$ it can differentiate the upper limit of the integral in the exponent resulting in a term which matches the linear term in (6.11). The other possibility is for $\frac{d}{dt}$ to act on the upper limit of the $\int_0^t ds(ij)$ inside $\int_0^t d^T s$, in which case we obtain a sum over $ij \subset X$ of terms

$$\sum_{T_i, T_j} \int_0^t d^{T_i} s \int_0^t d^{T_j} s \, I(T_i) I(T_j) (-\beta \dot{u}_t(ij))$$
$$\exp\left[-\beta \sum_{kl \subset X} \int_{s_{\max}}^t d\tau \; \dot{u}_\tau(kl)\right], \tag{6.19}$$

where the sum over T becomes a sum over T_i and T_j which are the components of $T \setminus \{(ij)\}$ that contain i and j respectively and

$$I(T_i) = \prod_{k\ell \in T_i} \left(-\beta \dot{u}_{s(k\ell)}(k\ell)\right), \tag{6.20}$$

and likewise for $I(T_j)$. The main point is that $s(ij)$ is now set equal to t and by referring to the definition of $s_{\max}$ we see that the exponential in (6.19), call it $\exp(-\beta W(T, \underline{s}(T))$, factors and (6.19) becomes

$$-\beta \dot{u}_t(ij) \sum_{T_i, T_j} \left(\int_0^t d^{T_i} s I(T_i) e^{-\beta W(T_i, s(T_i))}\right) \left(\int_0^t d^{T_j} s I(T_j) e^{-\beta W(T_j, s(T_j))}\right)$$
$$= -\beta \dot{u}_t(ij) \sum_{\substack{I \ni i \\ X \setminus I \ni j}} \tilde{F}_t(I) \tilde{F}_t(X \setminus I) \,. \tag{6.21}$$

On recalling that we have to sum over the bond $(ij) = (ji)$ we see that this term matches the second term in the flow (6.11) so we have proved that the right-hand side $\tilde{F}(X)$ of (6.18) satisfies the flow. $\square$

6.1.3 Tree-Graph Domination

The preceding tree-graph identity yields a basic tool for the study of the convergence of Mayer expansions and many related expansions: the tree-graph domination inequalities.

Proposition 6.3 (Tree-Graph domination)

(i) If $u(ij) \geq 0$ for all (ij), then

$$|(e^{-\beta U})_c(X)| \;\leq\; \sum_T \prod_{(ij)\in T} |e^{-\beta u(ij)} - 1|. \tag{6.22}$$

where T is summed over all tree-graphs on the set of vertices X.

(ii) If $u_{t0}(ij) = \int_0^t d\tau\, \dot{u}_\tau(ij)$, where $\dot{u}_t$ is stable*, that is, there exists $\dot{B}_\tau \geq 0$ such that for all finite sets C,*

$$\sum_{(ij)\subset C} \dot{u}_\tau \geq -\dot{B}_\tau |C|, \tag{6.23}$$

then, for any finite set X,

$$(a)\quad \left|\left(e^{-\beta U_{t0}}\right)_c(X)\right| \;\leq\; \sum_{T \text{ on } X} \prod_{(ij)\in T} \left[\int_0^t ds\, |\beta \dot{u}_s(ij)|\right] e^{\beta |X| B_{t0}} \tag{6.24}$$

$$(b)\quad \left|\left(e^{-\beta U_{t0}}\right)_c(X)\right| \;\leq\; \sum_{T \text{ on X}} \prod_{(ij)\in T} \left[\int_0^t ds\, |\beta \dot{u}_s(ij)| e^{2\beta B_{ts}}\right], \tag{6.25}$$

where $B_{ts} \equiv \int_s^t \dot{B}_\tau d\tau$.

Remarks: Inequality (6.22) was the first one proved [55, 48, 52]. It can be more dramatically stated in the form

$$\left|\sum_G \prod_{(ij)\in G} (e^{-\beta u(ij)} - 1)\right| \leq \sum_T \prod_{(ij)\in T} |e^{-\beta u(ij)} - 1|\,, \tag{6.26}$$

which shows the large amount of cancellations presented in the original expression (6.10) for the Ursell coefficients. Inequality (6.24) yields bounds similar to Ruelle's criterium [56, page 84, Theorem 4.3.1]. Inequality (6.25) is an improvement over (6.24) that allows, for instance, to treat the Yukawa gas beyond the region of stability of the interaction.

Proof. We obtain part (i) by choosing $u_{t0}(ij) = t\,u(ij)$ with $0 \le t \le 1$ in Proposition 6.2. Then we bound the exponent in the proposition by dropping $\dot{u}_{k\ell}$ if $k\ell$ is not in T,

$$\begin{aligned} \exp\left[-\beta U_{ts_{\max}}\right] &\geq \sum_{(ij)\in T} \int_{s_{\max}}^{1} ds\, \dot{u}_s(ij) \\ &= \sum_{ij\in T} \int_{s(ij)}^{1} ds\, \dot{u}_s(ij) \end{aligned} \tag{6.27}$$

by definition of $s_{\max}$. When this is substituted into Proposition 6.2 we can explicitly evaluate $\int d^T s$ and the result is part (a).

Part (iia): Substitute the stability bound into Proposition 6.2.

Part (iib): We take advantage of the fact that when τ in the exponent of Proposition 6.2 is small compared with most of the parameters $s(ij)$, then many of the clusters $C \in \gamma(\tau, \underline{s}, T)$ have only one particle $|C| = 1$ and we can replace $-\dot{B}_\tau |C|$ by zero in the stability bound. This idea appeared in [36] and [6]. Thus in Proposition 6.2 we substitute

$$\begin{aligned} \sum_{C\in\gamma(\tau,\underline{s},T)} \dot{u}_\tau(C) &\geq -\dot{B}_\tau |\{i : i \in \text{ some } C \text{ with } |C| > 1\}| \\ &\geq -\dot{B}_\tau |\{i : s(ij) \leq \tau \text{ for some } j\}| \\ &\geq -\dot{B}_\tau \sum_{i\in X} \sum_{\substack{b\in T \\ b\ni i}} \theta(\tau - s(b)), \end{aligned} \tag{6.28}$$

where $\theta(t)$ is the step function, $\theta(t) = 1$ if $t \geq 0, \theta(t) = 0$ if $t < 0$ and b is a bond $k\ell$ in T. Since every bond contains two vertices,

$$\sum_{C\in\gamma(\tau,\underline{s},T)} \dot{u}_\tau(C) \geq -2\dot{B}_\tau \sum_{b\in T} \theta(\tau - s(b)). \tag{6.29}$$

Part (iib) follows immediately on putting this bound into Proposition 6.2. □

6.2 Convergence of Mayer Expansions

The tree-graph bounds for the Ursell coefficients immediately yield criteria for the absolute convergence of Mayer expansions. Indeed, as reviewed in Section 6.1, this expansion has the form

$$\frac{1}{|\Lambda|} \log Z_{t0} = \sum_{N\geq 0} \frac{z^N}{N!} \int d^N\xi\, (e^{-\beta U_{t0}})_c(N, \xi)\ , \tag{6.30}$$

and one is interested in the range of fugacities for which the series is analytic and moreover converges term-by-term to an analytic series as $|\Lambda| \to \infty$. Such properties hold at least if the series

$$\sum_{N\geq 0} \frac{|z|^N}{N!} \sup_{\xi_1} \int d\xi_2 \cdots d\xi_N \, |(e^{-\beta U_{t0}})_c(N,\xi)| \tag{6.31}$$

converges. If we now use the tree-graph bounds of Proposition 6.3 we readily conclude the following.

Proposition 6.4 *If either*

(i) $u(\xi_i, \xi_j) \geq 0$ *for all* (ij)*, and*

$$\sup_{\xi_i} \int d\xi_j \, |e^{-\beta u(\xi_i,\xi_j)} - 1| \equiv q < \infty \tag{6.32}$$

or

(ii) $u_{t0}(\xi_i, \xi_j)$ *is as in (ii) of Proposition 6.3, and*

$$\sup_{\xi_i} \int_0^t ds \, \left[\int d\xi_j |\beta \dot{u}_s(\xi_i,\xi_j)|\right] e^{2\beta B_{ts}} \equiv q < \infty \, ; \tag{6.33}$$

then the Mayer expansion (in finite or infinite volume) is convergent for

$$q|z|e < 1 \, . \tag{6.34}$$

Proof. The trick is to perform the integral in (6.18) over the vertices $\xi_1, \ldots, \xi_N$ of each tree graph, by starting from the outermost branches ("leaves") and working inwards. In this fashion each integral factors into a sequence of independent integrals and we obtain that the series (6.31) is majorized—both in cases (i) and (ii)—by

$$\sum_{N\geq 1} \frac{|z|^N}{N} \sum_{T \text{ on } N} q^{N-1} = \sum_{N\geq 1} \frac{|z|^N}{N} N^{N-2} q^{N-1} \, . \tag{6.35}$$

The last expression follows from the fact that a tree of N vertices has $N-1$ bonds and of Cayley's theorem (there are exactly N^{N-2} trees of N vertices). Condition (6.34) follows from Stirling's theorem. $\square$

An important particular case of (6.33) is when the functions $\{\dot{u}_s(\xi_i,\xi_j)\}_{i,j\in C}$ constitute a positive-definite form for all finite sets C. In this case

$$\sum_{i,j\in C} \dot{u}_s(\xi_i,\xi_j) \;=\; \frac{1}{2}\sum_{\substack{i,j\in C\\ i\neq j}} \dot{u}_s(\xi_i,\xi_j) \;\geq\; -\frac{1}{2}\sum_{i\in C}\dot{u}_s(\xi_i,\xi_i)\;, \tag{6.36}$$

hence we can choose

$$\dot{B}_\tau \;=\; \frac{1}{2}\sup_\xi |\dot{u}_\tau(\xi,\xi)|\;\;. \tag{6.37}$$

Corollary 6.5 *If $\{\dot{u}_t(\xi_i,\xi_j)\}_{i,j\in C}$ is positive-definite for all finite sets C, then the Mayer expansion (in finite or infinite volume) is analytic if*

$$|z|\beta\int_0^t ds\,\dot{A}(s)\exp\beta\int_s^t d\tau\,\dot{a}(\tau) \;<\; \frac{1}{e}\;, \tag{6.38}$$

where

$$\dot{A}(s) \;=\; \sup_\xi\int d\xi'\,|\dot{u}_s(\xi,\xi')| \tag{6.39}$$

and

$$\dot{a}(s) \;=\; \sup_\xi |\dot{u}_\tau(\xi,\xi')|\;\;. \tag{6.40}$$

6.3 Application: Convergence of the Mayer Expansion of the Yukawa Gas in Two Dimensions

We will derive the result, first obtained by Benfatto [6], that the two-dimensional Yukawa gas has a convergent Mayer expansion for $\beta < 4\pi$. This is a system which is not stable in the sense of Ruelle so this result cannot be directly deduced from Ruelle's criterion [56, page 84, Theorem 4.3.1]. The reason why the estimates here do better is that within the flow equation approach one can separate out the simplest parts of the Ursell coefficients and solve the flow equation for them exactly. This is in essence what was done above in the proof of part (iib) of Proposition 6.3 when we recognized that there is no need for a stability bound for clusters consisting of a single particle. This idea will recurr in the coming section on cluster expansions, but there perturbation theory will be used to calculate the relevant parts more accurately.

The particles are described by $\xi = (x,\epsilon)$ with $\epsilon = \pm 1$. The two-body potential is

$$u(\xi,\xi') \equiv \epsilon\epsilon' u(x-x') \tag{6.41}$$

with

$$u(x) = \frac{1}{(2\pi)^2} \int d^2k \hat{u}(k) e^{ik\cdot(x-x')} \tag{6.42}$$

for

$$\hat{u}(k) = \frac{1}{1+k^2} \; . \tag{6.43}$$

Since $u(x)$ is singular at $x = 0$ we introduce a cutoff at length scale $\ell \ll 1$ by replacing $\hat{u}$ with

$$\hat{u}_{1\ell}(k) = \frac{1}{1+k^2} - \frac{1}{\ell^{-2}+k^2}. \tag{6.44}$$

(We substract the length scales "up to scale l".) We now introduce the continuum parameter t varying from ℓ to 1, defining

$$\hat{u}_{t\ell}(k) = \frac{1}{t^{-2}+k^2} - \frac{1}{\ell^{-2}+k^2}. \tag{6.45}$$

We are interested in the case $t = 1$. From the results of the previous section we learn that, to show analyticity of the Mayer expansion uniformly in the cutoff, it is enough to obtain a bound of the type (6.33):

$$\sup_{\xi} \left[\int_0^1 ds \int d\xi' \, |\beta \dot{u}_s(\xi,\xi')| e^{2\beta B_{1s}} \right] \le q \tag{6.46}$$

with a constant q independent of ℓ. In particular, as $\dot{u}_s$ is positive definite, we can use Corollary 6.5 with

$$\begin{aligned} \int_s^1 d\tau \, \dot{a}(\tau) &= u_{1s}(0) = \int d^2k \hat{u}_{1s}(k) \\ &= \left(\frac{1}{2\pi}\right)^2 \int d^2k \left\{ \frac{1}{1+k^2} - \frac{1}{s^{-2}+k^2} \right\} = \frac{1}{4\pi} \log(s^{-2}) \, , \end{aligned} \tag{6.47}$$

and, from (6.45),

$$\dot{A}(s) = \sup_{\xi} \int d\xi' \, \dot{u}_s(\xi,\xi') = 2 \int dx |\dot{u}_s(x)| = 4s. \tag{6.48}$$

(The factor 2 in the middle term is due to the two values of the charge, $\epsilon = \pm 1$.) Therefore, from (6.38) we see that the Mayer expansion (6.3)–(6.4) converges *uniformly* as the cutoff ℓ decreases to zero provided

$$|z|(4\beta) \int_\ell^1 ds \left(s s^{-\frac{2\beta}{4\pi}} \right) \; < \; \frac{1}{e} \, , \tag{6.49}$$

that is, if $\beta < 4\pi$ and z is in the disk

$$e(2\beta)[1-\frac{\beta}{4\pi}]^{-1}|z| < 1. \tag{6.50}$$

By using the formula in Proposition 6.2 it is not difficult to verify, with these uniform bounds, that the connected parts (i.e. the derivatives of (4.3)) also have limits as $\ell \downarrow 0$ for β and z as above. Thus the model without cutoffs has a convergent expansion near $z = 0$. This is in fact a way towards the construction of the massive Sine-Gordon field theory for $\beta < 4\pi$ and $|z|$ small.

Open Problems. By recent results [43], which I shall discuss later, it is known that this model actually has a convergent Mayer expansion for $\beta < 8\pi$ provided one omits the first n terms which diverge successively of thresholds $\beta_n = 8\pi(1-\frac{1}{2n})$ which were investigated in [8, 50].

We would like to see a direct proof of this based on further insights into the Ursell coefficients. In [17] this was achieved for $4\pi \leq \beta < 16\pi/3$, but it seems to be a difficult problem for $\beta \geq 6\pi$. The same situation holds for the dipole gas (or Coulomb gas with $\beta > 8\pi$) and cutoffs at short distance to remove ultraviolet divergences: it is known that for the dipole gas the pressure is analytic [30, 18], but not by direct estimates on the Ursell coefficients. We shall be discussing this proof later as well. For the Coulomb gas at $\beta > 8\pi$ in two-dimensions, we are sure that the pressure is analytic but Dimock and Hurd have not yet proved it.

6.4 Effective Potentials as Mayer Expansions

What is the relation between the last section on the Mayer expansion and section 5.3 on the formal expansion for the action? In the case where the two-body potential u is positive definite and regular on the diagonal we can use the Sine-Gordon transformation discussed in Section 2 to rewrite the partition function (6.1),

$$Z = \int d\mu_{u/\beta}(\phi) \exp(\int z d\xi : e^{i\beta\phi} :). \tag{6.51}$$

where $: e^{i\beta\phi} :\equiv e^{\frac{1}{2}\beta u(\xi,\xi)} e^{i\beta\phi}$ inserts factors which cancel self-energies in the interaction. Then we choose a deformation $C_{\infty t} \equiv u_{\infty t}/\beta$ and continue with

$$= \int d\mu_{\infty t} \exp(V_{t0}), \tag{6.52}$$

where V_{t0} solves the flow (5.1) with initial data

$$V_{00} = \int z d\xi : e^{i\beta\phi} :_{\infty 0} , \tag{6.53}$$

where $: e^{i\beta\phi} :_{\infty t} \equiv e^{\frac{1}{2}\beta C_{\infty t}(\xi,\xi)} e^{i\beta\phi}$. The series solution (5.33) obtained by iterating the integral flow (5.3) is clearly an expansion in powers of z, as is the Mayer expansion. Its limit $t \to \infty$ yields an expansion for $\log Z$ that must therefore coincide with the Mayer expansion.

If we substitute into (5.1) the functional *Fourier* series

$$V_{t0}(\phi) = \sum \frac{1}{N!} \int z^N d^N \xi \, V_{t0}(N, \underline{\xi}) \prod_1^N : e^{i\beta\phi(\xi_j)} :_{\infty t} \ . \tag{6.54}$$

We obtain a system of flow equations and initial conditions for the coefficients $V_{t0}(N, \underline{\xi})$, which is the same as the equations (6.11)–(6.12) that characterized the Ursell coefficients $\left(e^{-\beta U_{t0}}\right)_c (N, \xi)$. Thus

$$V_{t0}(N, \xi) = \left(e^{-\beta U_{t0}}\right)_c (N, \underline{\xi}) \ . \tag{6.55}$$

6.5 Problems

6.5.1 Show that (6.6) implies that, as a formal power series,

$$\exp\left[\sum_{N\geq 1} z^N \int d^N \xi \ (e^{-\beta U})_c (N, \underline{\xi})\right] = \sum_{N\geq 0} \frac{z^N}{N!} \int d^N \xi \, e^{-\beta U(N,\underline{\xi})}. \tag{6.56}$$

For an answer see [9], page 176 or [56].

6.5.2 Check directly that if $\left(e^{-\beta U_{t0}}\right)_c$ is defined by (6.10) then it is the unique solution to the system (6.11)–(6.12). Hints: To check that it is a solution, a useful intermediate expression is:

$$\begin{aligned} \frac{d}{dt}(e^{-\beta U_{t0}})_c(X) \ &= \ \frac{1}{2} \sum_{i,j\in X} (-\beta \dot{u}_t(ij)) \sum_{\substack{G \text{ on } X \\ \text{connected}}} \prod_{(k\ell)\in G} \left(e^{-\beta u_{t0}(k\ell)} - 1\right) \\ &\quad - \frac{1}{2} \sum_{i,j\in X} (-\beta \dot{u}_t(ij)) \Big[\sum_{\substack{G \text{ on } X \\ \text{connected} \\ G \not\ni (ij)}} \prod_{(k\ell)\in G} \left(e^{-\beta u_{t0}(k\ell)} - 1\right) \end{aligned}$$

$$- \sum_{\substack{G \text{ on } X \\ \text{connected} \\ G \ni (ij)}} \prod_{\substack{(k\ell) \in G \\ (k\ell) \neq (ij)}} \left(e^{-\beta u_{t0}(k\ell)} - 1 \right) \Bigg] . \tag{6.57}$$

6.5.3 This exercise obtains a useful expression for truncated expectations. Let μ_C be a Gaussian measure with regular covariance C. Define the truncated expectations $\langle F_1; F_2; \ldots; F_N \rangle$ of smooth functionals $(F_i(\phi))_{i=1,2,\ldots,N}$ by

$$\langle F_1; F_2; \ldots; F_N \rangle \equiv \left[\frac{\partial^N}{\partial \lambda_1 \ldots \partial \lambda_N} \log \int d\mu_C \, e^{\sum \lambda_i F_i} \right]_{\underline{\lambda}=0} . \tag{6.58}$$

Work out explicitly the first few truncated expectations, $N = 1, 2, 3$ and prove that

$$\langle F_1; F_2; \ldots; F_N \rangle \quad = \quad \sum_{T \text{ on } N} \int_0^\infty d^T s \int d\mu_{C_{ts_{\max}}} \Delta_{\underline{s}}^T \prod_{i=1}^N F_i(\phi_i)$$

where the notation is as in Proposition 5.2.

7 Cluster Expansions

In Section 5 we studied the solution of the functional heat equation of Lemma 4.3 by writing $Z_t = e^{V_t}$ and finding the flow of V_t by iterating an integral equation for V_t. This method is effective in tracking the locality properties of Z_t: the L_1 norms $\|V_{t,N}\|$ appearing in (6.31) are small if Z_t resembles $\exp\left[z \int d\xi e^{i\beta\phi(\xi)}\right]$. However, any method based on the effective potential V will encounter difficulties if the initial V_0 has any kind of growth as $|\phi| \to \infty$ because a norm of V will not know the difference between $\pm V$ and one of them might be unstable, e.g., $\pm\lambda \int \phi^4$.

For these reasons another kind of expansion, called a cluster expansion [35], has been introduced in quantum field theory. These expansions still track locality but are able to distinguish stable and unstable potentials. In this section we will show that they can be understood in very close analogy to what we have done so far, one simply changes what is meant by exponential!

In this section, $\xi = (x, \alpha)$ where α is an index which takes on discrete values from a finite set Ω. It labels properties such as charge or components of a direction for fields connected with dipole gases, and $x \in \mathrm{R}^d$ is the particle coordinate. The term *unit block* (Δ) will mean a *closed* cube of side one centered on one of the points of a unit lattice $\mathbf{Z}^d \subset \mathrm{R}^d$. A *polymer*[7] X is a finite, non-empty union of closed unit blocks. It need not be connected. *The length scale on which the blocks are "unit" is chosen to be of the same order of magnitude as the length scale given by an ultra-violet cutoff in the measure $d\mu_t$.* This is because at this scale the fields become approximately constant so there are no significant fluctuations inside such a unit block.

We will represent solutions Z_t to the functional heat equation of Lemma 4.3 in the form of a *polymer gas*. These representations were first studied and used systematically in [37]:

$$Z(\Lambda, \phi) = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\substack{X_1,\ldots,X_N \subset \Lambda \\ X_i \cap X_j = \emptyset \text{ if } i \neq j}} \prod_{i=1}^{N} K(X_i, \phi), \tag{7.1}$$

The quantity $K(X, \phi)$ is called a *polymer activity*, and variational derivatives $\frac{\delta^N K(X,\phi)}{\delta\phi(\xi_1)\cdots\delta\phi(\xi_N)}$ vanish unless each $\xi_i \in X$. In this sense $K(X, \phi)$ depends on $\phi(\xi)$ only for $\xi \in X$, where we denote in this way the property that $\xi = (x, \alpha)$ and $x \in X$. From now on it should be assumed that all sums over sets extend over the class of *allowed sets*, which are unions of *cells* which are *open* unit blocks, *open* faces of unit blocks, ..., *open* edges, vertices.

[7] These polymers have no connection with the polymers of Section 3

The null set is not a cell but is also allowed. We extend the definition of $K(X, \phi)$ to arbitrary allowed sets with the convention $K(X, \phi) = 0$ if X is not a polymer.

The expansion (7.1) looks like an exponential, and in fact, it is one but in a different product

$$(A \circ B)(X) = \sum_{\substack{Y, Z \\ Y \dot{\cup} Z = X}} A(Y)B(Z) \tag{7.2}$$

defined on functions of allowed sets. Here $Z \dot{\cup} Y = X$ means that $Z \cup Y = X$ and $Z \cap Y = \emptyset$. Functions on allowed sets are a commutative algebra under the product (7.2) which has the identity

$$1(X) = \begin{cases} 1 \text{ if } X = \emptyset \\ 0 \text{ otherwise.} \end{cases} \tag{7.3}$$

If we also define

$$\Box(X) = \begin{cases} 1 \text{ if } X = \text{ is a cell} \\ 0 \text{ otherwise,} \end{cases} \tag{7.4}$$

($\Box$ = "the space") then we can rewrite the polymer gas compactly as

$$Z(\Lambda, \phi) = \mathcal{E}\text{xp}[\Box + K](\Lambda, \phi), \tag{7.5}$$

where $\mathcal{E}$xp is the exponential defined for A such that $A(\emptyset) = 0$ using a product

$$\mathcal{E}\text{xp}[K] = 1 + K + \frac{1}{2!} K \circ K + \cdots. \tag{7.6}$$

There is also a logarithm $\mathcal{L}\text{n}[1+A] = A - \frac{1}{2} A \circ A + \frac{1}{3} A \circ A \circ A \cdots$ defined on all A with $A(\emptyset) = 0$. It is important to realize that, when evaluated on a given set X, all these series are finite because no allowed set X can be indefinitely subdivided into proper subsets which are allowed. Thus there will be no analysis problems in manipulating these functions.[8]

If we substitute $Z_t = \mathcal{E}\text{xp}[\Box + K_t]$ into the functional heat equation of Lemma 4.3 we find that Z_t solves the heat equation *iff* the polymer activity K_t solves

$$\frac{\partial K_t}{\partial t} = \frac{1}{2}(K_{t\phi\phi} + K_{t\phi} \circ K_{t\phi}), \tag{7.7}$$

[8] This product is reminiscent of a product on functions of sequences invented by Ruelle [56]. There is a difference which is important: our product is designed to suppress multiple overlap of polymers which leads to artificial large field problems.

where

$$K_\phi \circ K_\phi = \int d\xi_1 \int d\xi_2 \, \frac{\delta K}{\delta\phi(\xi_1)} \circ \frac{\delta K}{\delta\phi(\xi_2)} [-\dot{C}(\xi_1, \xi_2)]. \tag{7.8}$$

As in Section 5 we can convert this to an equivalent integral equation

$$K_t = \mu_{t0} * K_0 + \frac{1}{2}\int_0^1 ds \, \mu_{ts} * (K_{s,\phi} \circ K_{s\phi}). \tag{7.9}$$

Iteration of this integral equation yields a *finite* series for K_t which is called a *cluster expansion.* This expansion was really invented by Glimm, Jaffe and Spencer in [35] but this presentation based on [18] reveals the structure more clearly than they probably understood at the time.

We can obtain a beautiful explicit formula for the result of iterating this integral equation by following the procedure of Section 5.3 and the result is

$$K_t(\phi) \;\; = \;\; \sum_N \frac{1}{N!} \sum_{T \text{ on } N} \int_0^t d^T s \, \mu_{ts_{\max}} * \Delta^T_{\underline{s}} \, \{K_0(\phi_1) \circ \cdots \circ K_0(\phi_N)\}|_{\phi_i = \phi} \tag{7.10}$$

The notation is as in Proposition 5.33. However if all we want is estimates then there is a simpler procedure which we now describe.

Since a cluster expansion is finite, convergence is not an issue, but one still wants to measure the extent to which locality is preserved under the evolution. In the present context the locality question means: to what extent $K_t(X)$ remains small whenever the polymer is spread out or large, if it is true at $t = 0$? Another point to notice is that the *partition function* $Z(\Lambda)$ loses its meaning as Λ becomes an infinite set but the collection $(K_t(X), X$ ranges over polymers) remains a good description of the evolution in the infinite volume limit.

These considerations are embodied in the definition of the following norms:

$$\|h^N K_N(X,\phi)\|_{\underline{\Delta}} \equiv \sup_{F:|F|\le 1} \int_{\Delta_1} h d\xi_1 \cdots \int_{\Delta_N} h d\xi_N \, \frac{\delta^N K(X,\phi)}{\delta\phi(\xi_1)\cdots\delta\phi(\xi_N)} F(\xi_1, \ldots, \xi_N), \tag{7.11}$$

where $\Delta_1, \ldots, \Delta_N$ is an N-tuple of closed unit blocks and F ranges over all continuous functions of N arguments $\xi_1, \ldots, \xi_N$. Note that variational derivatives of K are measures. Now we want to impose analyticity in ϕ in a strip of radius $h > 0$ about the real-ϕ axis. Roughly speaking, for all N,

$$\sum_{\Delta_1, \ldots, \Delta_N} \|K_N(X,\phi)\|_{\underline{\Delta}} \le C \frac{N!}{h^N} \frac{g(X,\phi)}{\Gamma(X)}, \tag{7.12}$$

where $\Gamma(X)$ is a weight on sets which quantifies locality. We call it a *large set regulator.* We will choose it later, but for the present section it can be anything that satisfies

$$\Gamma(X) \geq 1, \quad \Gamma(X \cup Y) \leq \Gamma(X)\Gamma(Y), \tag{7.13}$$

e.g., $\Gamma(X) = 1$. The function $g(X, \phi)$ quantifies the growth in K as $|\phi| \to \infty$. It is called a *large-field regulator* and can be any function that is integrable with respect to $d\mu(\phi)$ and satisfies

$$g(X, \phi) \geq 1, \quad g(X \dot{\cup} Y, \phi) \geq g(X, \phi) g(Y, \phi). \tag{7.14}$$

The large-field regulator will be explained further at the end of this section. We incorporate all the inequalities (7.12) into one norm with the following definitions for $\| \, \|_{g,\Gamma,h}$,

$$\begin{aligned} \|h^N K_N(X)\|_g &\equiv \sum_{\underline{\Delta}} \sup_{\phi} \|h^N K_N(X, \phi)\|_{\underline{\Delta}} \, g^{-1}(X, \phi) \\ \|h^N K_N(X)\|_{g,\Gamma} &\equiv \sup_{x} \sum_{X \ni x} \|h^N K_N(X)\|_g \Gamma(X) \\ \|K\|_{g,\Gamma,h} &\equiv \sum_{0}^{\infty} \frac{1}{N!} \|h^N K_N\|_{g,\Gamma}. \end{aligned} \tag{7.15}$$

The covariance C will be measured by

$$\|C\|_\Gamma \equiv \sup_{\Delta} \sum_{\Delta'} |C(\Delta, \Delta')| \Gamma(\Delta \cup \Delta'), \tag{7.16}$$

where

$$C(\Delta, \Delta') \equiv \sup_{\xi \in \Delta, \xi' \in \Delta'} |C(\xi, \xi')|. \tag{7.17}$$

Now we obtain the following theorem by taking the $\| \, \|_{g,\Gamma,h}$ norm of the integral flow equation (7.9) which reduces it to a one-variable inequality for the norm, which can be solved explicitly.

Theorem 7.1 *If g_t is a family of large field regulators that are integrable in the sense*

$$\mu_{ts} * g_s \leq g_t, \quad 0 \leq s \leq t,$$

and at $t = 0$, for some $h > h' > 0$,

$$\|K_{t=0}\|_{g_0,\Gamma,h} \leq \frac{(h - h')^2}{16 \int_0^t \|\dot{C}_s\|_\Gamma ds},$$

then

$$\mu_0 * \mathcal{E}\mathrm{xp}[\Box + K_0] = \mu_t * \mathcal{E}\mathrm{xp}(\Box + K_t)$$

with

$$\|K_t\|_{g_t,\Gamma,h'} \leq \|K_{t=0}\|_{g_0,\Gamma,h}.$$

Proof. Rename the h and h' in the theorem h_t, h_0 respectively. For $p = 0, 1, \ldots$, $\frac{\partial^p}{\partial h^p}\|K_t\|_h \equiv \frac{\partial^p}{\partial h^p}\|K_t\|_{g_s,\Gamma,h}$ are (semi)norms. By taking (semi)norms of both sides of the integral equation

$$K_t \quad = \quad \mu_{t0} * K_0 + \frac{1}{2}\int_0^t ds\, \dot{C}\mu_{ts} * \{K_{s,\phi} \circ K_{s,\phi}\} \tag{7.18}$$

that corresponds to the flow (7.9), we obtain, for any h,

$$\|K_t\|_h \quad \leq\leq \quad \|K_0\|_h + \int_0^t ds\, \|\dot{C}_s\|_\Gamma \left(\frac{\partial}{\partial h}\|K_s\|_h\right)^2. \tag{7.19}$$

The notation $\leq\leq$ means that the bound holds with $\frac{\partial^p}{\partial h^p}$ applied to each side. For more details on this step, see p. 380 of [18]. Iterating (7.18) generates a finite series because a polymer X can only be subdivided into finitely many proper subsets. This series is $\leq\leq$ bounded by the series that results from iterating (7.19) (which is an infinite series). But this series in its turn is $\leq\leq$ dominated by the series generated by

$$k_t(h) \quad = \quad k_0(h) + \int_0^t ds\, \|\dot{C}_s\|_\Gamma \left(\frac{\partial k}{\partial h}\right)^2; \tag{7.20}$$

$$k_0(h) \quad = \quad \|K_0\|_{g_0,\Gamma,h}, \tag{7.21}$$

This integral equation generates a solution, analytic near $h = 0$, to

$$\frac{\partial k}{\partial \tau} = \left(\frac{\partial k}{\partial h}\right)^2, \tag{7.22}$$

where

$$\tau = \int_0^t ds\, \|\dot{C}_s\|_\Gamma. \tag{7.23}$$

But for τ small, the reader can check that (7.22) has a solution which is analytic in h, namely

$$k(\tau, h) = k(\tau = 0, h_{cl}) - \frac{(h - h_{cl})^2}{4\tau}, \tag{7.24}$$

where h_{cl} is determined by requiring that the right hand side of (7.24) be stationary in h_{cl}. [This solution was obtained by applying the action principle to the Hamilton-Jacobi equation (7.22)]. Using the hypotheses on h_0, h_t, Rouchés theorem implies that there is a unique solution $h_{cl}(\tau, h)$ such that $h_{cl}(\tau = 0^+, h) = h$, which is analytic for $|h| \leq h_t$ and real for h real. Also $|h_{cl}| \leq h_0$. Therefore (7.24) implies $k(\tau, h_t) \leq k(0, h_0)$, which is the bound claimed in the theorem. The solution we have found is unique within the class of functions analytic in h about $h = 0$ because (7.22) determines the power series. More details can be found on page 381 of [18].

A choice for the large field regulator g: In order to apply Theorem 7.1 we must choose a large field regulator g and a large set regulator Γ. Good choices of these objects will be discussed in the next section but here we will illustrate the role of g in bounding a field pointwise in x with the aid of stability in a model with a ϕ^4 interaction. For simplicity we take $\Gamma = 1$.

Consider

$$E_0(X) \equiv e^{-\frac{\lambda}{4!}\int_X \phi^4 dx}\phi(x) \tag{7.25}$$

if X is a unit block, otherwise $E(X) = 0$. $x \in X$ or more generally x is near X. The norm $\|E\|_{g,\Gamma,h}$ involves all derivatives of E but in particular we first need to see that (7.25) is small in supremum norm. However ϕ can be large just in a small neighborhood of one point $x \in X$ without the factor $\int \phi^4$ in the exponent being large. The remedy is to recognize that the fields which are typical for the measure $d\mu_t$ cannot make such fluctuations because the ultra-violet cutoff in $d\mu_t$ makes them *stiff.* This is where the large field regulator $g(X, \phi)$ which is really a large gradient regulator comes in. *Suppose the dimension of space-time* $d = 1$ and let

$$g(X, \phi) = \exp \frac{\kappa}{2}\int_X \left(\frac{d\phi}{dx}\right)^2 dx \tag{7.26}$$

Then from

$$\begin{aligned} |\phi(x)| &\leq \frac{1}{|X|}\int_X dy\, |\phi(y) + (\phi(x) - \phi(y))| \\ &\leq \frac{1}{|X|}\int_X dy\, |\phi(y)| + \frac{1}{|X|}\int_X dy \int_y^x dz\, |\frac{d\phi}{dz}| \\ &\leq \left[\frac{1}{|X|}\int_X dy\, |\phi|^4\right]^{\frac{1}{4}} + \left[|X|\int_X dz\, |\frac{d\phi}{dz}|^2\right]^{\frac{1}{2}}, \end{aligned} \tag{7.27}$$

we find that if $\kappa > 0$ is held fixed and λ is made small then $|E(X, \phi)|g^{-1}(X, \phi) \leq O(\lambda^{-\frac{1}{4}})$. In other words the factor of ϕ is of order $\lambda^{-\frac{1}{4}}$ when the exponent $exp[-\frac{\lambda}{4!}\int \phi^4]$

and the regulator g are taken into account. (7.27) is an example of a *Sobolev estimate*. Sobolev estimates hold for $d > 1$ but one needs one extra derivative for each dimension[9]. Therefore for dimension d, we choose

$$\begin{aligned} g(X,\phi) &= \exp\left[-\tfrac{\kappa}{2}\|\nabla\phi\|^2_{q,X} - Const.|X|\right] \\ \|\phi\|^2_{q,X} &\equiv \sum_{1\le|\alpha|\le q}\int_X |\partial^\alpha\phi(x)|^2 dx, \\ q &\ge d+1, \end{aligned} \tag{7.28}$$

where $\partial^\alpha\phi$ denotes a partial derivative of ϕ of order $|\alpha|$. Then for $h = O(\lambda^{-\frac{1}{4}})$ and any $\kappa > 0$ held fixed.

$$\|E\|_{g,\Gamma,h} = O(\lambda^{\frac{1}{4}}), \qquad \text{as } \lambda \to 0. \tag{7.29}$$

The ϕ derivatives which are part of the definition (7.15) of the norm have to be estimated in variation norm and this is done using e.g., $1/|\Delta|\int_\Delta d\xi\, f(\xi)\phi^3(\xi) \le \{1/|\Delta|\int_\Delta d\xi\, \phi^4(\xi)\}^{\frac{3}{4}}$ for $|f| \le 1$, to estimate factors resulting from derivatives of the exponential in (7.25) in terms of the exponent.

The fact that the fields ϕ actually are *stiff* is captured by the hypothesis $\mu_{ts} * g_s \le g_t$ in Theorem 7.1. Let us take g_s as in (7.28), and let $t = s + ds$ so that the covariance of μ_{ts} is infinitesimal. Then,

$$\mu_{ts} * g_s = g_s + \frac{1}{2}(g_s)_{\phi\phi}ds. \tag{7.30}$$

We find that if we let κ in (7.28) and the Const. $\equiv v$ depend on $s, \kappa = \kappa_s, v = v_s$, then $\mu_{ts} * g_s \le g_t$ holds for $t = s + ds$ provided

$$\frac{d\kappa_s}{ds} \ge a_s\kappa_s^2 \quad ; \quad \frac{dv_s}{ds} \ge b_s\kappa_s, \tag{7.31}$$

where the a_s and b_s are determined by the partial derivatives of the covariance $\dot{C}_s$ up to order s in x and in y, e.g., $\partial_x^\alpha\partial_y^\beta\dot{C}_s(x,y), |\alpha| \le s, |\beta| \le s$, which have to exist as continuous functions for $x \simeq y$ and to be integrable in $x - y$. These are natural conditions because, for $t = s + ds$, $\partial_x^\alpha\partial_y^\alpha\dot{C}_s(x,y)ds$ is the covariance of $\partial_x^\alpha\zeta(x)$ which occurs in $(\mu_{ts} * g_s)(\phi) = \int d\mu_{ts}(\zeta)g_s(\phi+\zeta)$. The regularity of these derivatives of $\dot{C}$ is the condition that guarantees that $d\mu_{ts}$ lives only on *stiff* fields ϕ.

[9]For this crude argument. It is possible to get away with $\frac{1}{2}+\epsilon$ derivatives per dimension

7.1 Manipulations with $\mathcal{E}$xp

We illustrate some of the properties of $\mathcal{E}$xp which we will need in the next section and also show how to rewrite Gibbs factors in the form $\mathcal{E}\mathrm{xp}[\Box + K]$ along with bounds on K.

Let $\phi(\xi) = \frac{\partial}{\partial x_\mu}\phi(x)$,

$$\begin{aligned} Z_0(\Lambda, \phi) &= e^{-\int_\Lambda \lambda\phi^4(\xi)d\xi} \\ \int d\xi(\cdot) &\equiv \sum_{\mu=1}^{d} \int dx(\cdot). \end{aligned} \tag{7.32}$$

We can rewrite this in terms of $\mathcal{E}$xp by defining

$$F(X) = \begin{cases} e^{-\lambda\int_X \phi^4 d\xi} & \text{if } X \text{ is a unit block} \\ 0 & \text{otherwise,} \end{cases} \tag{7.33}$$

$$R(X) = e^{F(X)} - 1, \tag{7.34}$$

then

$$\begin{aligned} Z_0(\Lambda) &= \prod_{X\subset\Lambda} e^{F(X)} \\ &= \prod_{X\subset\Lambda} (1 + R(X)) \end{aligned} \tag{7.35}$$

(which defines R).

Now we expand the product. It does not immediately give $\mathcal{E}\mathrm{xp}[\Box + R]$ because two different sets X can touch, therefore we group them together whenever this happens by

Definition 6.1.1. Given any function $K(X)$ on polymers, let

$$K^+(X) = \sum_{\{X_i\}\to X} \prod_i K(X_i),$$

where $\{X_i\} \to X$ means that $\{X_i\}$ is summed over all *sets*[10] $\{X_1, \ldots, X_N : \cup X_i = X, N > 1 \text{ arbitrary}\}$ such that $X_1, \ldots, X_N$ are distinct and the overlap graph is connected. The *overlap graph* is the graph on vertices $\{1, \ldots, N\}$ whose bonds are those pairs (ij) such that $X_i \cap X_j \neq \emptyset$.

[10] So the order of $X_1, \ldots, X_N$ does not distinguish $\{X_1, \ldots, X_N\}$

In other words $K^+(X)$ is a sum over all ways of filling a set X with polymers so that the polymers overlap to form a connected clump. Under these circumstances the norm of the clump of polymers is roughly speaking the product of the norms of the individual polymers except that if several polymers overlap then the resulting function of ϕ will grow as the product of the growths in ϕ of the polymers that overlap. This is what we see in

Proposition 7.2 *For any functional K*

(i)

$$\prod_{Y \subset X} [1 + K(Y)] = \mathcal{E}\text{xp}[\Box + K + K^+](X)$$

(ii) For $\gamma > 1$ there is C_γ such that for any Γ

$$\|K^+\|_{g^\tau,\Gamma,h} \leq \sum_{N \geq 2} C_\gamma^N \|K\|_{g,\gamma\Gamma,h}^N$$

where $(\gamma\Gamma)(X) = \gamma^{|X|}\Gamma(X)$, and τ is the maximum overlap of sets in the support of R:

$$\tau = \sup_x \#\{X : X \ni x \text{ and } R(X) \neq 0\}.$$

Proof. Part (i) is left to the reader, and for part (ii), see [18], Lemma 5.1. $\Box$

Thus

$$e^{-\int_\Lambda \lambda\phi^4(\xi)\, d\xi} = \mathcal{E}\text{xp}[\Box + R + R^+](\Lambda) \tag{7.36}$$

and the proposition is the estimate we need to show that $\|R^+\|_{g,\Gamma,h}$ is (very) small when $\|R\|_{g,\Gamma,h}$ is small.

There is another proposition which is not of immediate relevance but which is of the same nature and is needed later, so we include it now. By definition of $\mathcal{E}\text{xp}$ we have $\mathcal{E}\text{xp}[A + B] = \mathcal{E}\text{xp}[A] \circ \mathcal{E}\text{xp}[B]$, but when we want an ordinary product of two $\mathcal{E}\text{xp}$'s then the result is

Proposition 7.3 *(i)*

$$\mathcal{E}\text{xp}[\Box + A]\mathcal{E}\text{xp}[\Box + B] = \mathcal{E}\text{xp}[\Box + A + B + A \vee B],$$

where

$$(A \vee B)(X) = \sum_{\{X_i\},\{Y_j\}\to X} \prod_i A(X_i) \prod_j B(Y_j),$$

where $\{X_i\}, \{Y_j\} \to X$ *means that the sum is over families of sets* $\{X_1, \ldots, X_N, Y_1, \ldots, Y_M\}$ *with* $N \geq 1$, $M \geq 1$, *such that* $(\cup X_i) \cup (\cup Y_j) = X$, *the* X*'s are disjoint, the* Y*'s are disjoint, and the overlap graph on* $\{1, \ldots, N, 1', \ldots, M'\}$ *is connected.*

(ii) *For* $\gamma > 1$ *there is* C_γ *such that for any* Γ,

$$\|A \vee B\|_{g_A g_B,\Gamma,h} \leq \sum_{N,M\geq 1} C_\gamma^{N+M} \|A\|^N_{g_A,\gamma\Gamma,h} \|B\|^M_{g_B,\gamma\Gamma,h},$$

where $(\gamma\Gamma)(X) = \gamma^{|X|}\Gamma(X)$.

Proof. A simple variation on the proof of Proposition 7.2. □

8 The Renormalization Group (RG)

This section is based on [18, 42].

The RG makes its appearance when the parameter t that enters into the deformation of the covariance is tied to the change of a length scale ℓ which parametrizes the smoothness of the field ϕ. E.g., we take $\ell = \ell(t) = \ell_0 e^t$ and consider

$$Z_t = \int d\mu_t Z_t(\phi), \tag{8.1}$$

where $d\mu_t$ has covariance

$$\begin{aligned} \int d\mu_t \phi(x)\phi(y) &= C_{\ell(t)}(x-y) \\ C_\ell(x) &= \ell^{-(d-2)} C_1\left(\frac{x}{\ell}\right), \end{aligned} \tag{8.2}$$

and the Fourier transform of C_ℓ is

$$\hat{C}_\ell(k) = \frac{F(\ell^2 k^2)}{\sigma k^2}, \tag{8.3}$$

where $F(t)$ is a smooth decreasing function of $t \geq 0$ which decays rapidly as $t \to \infty$, together with all derivatives and $F(0) = F'(0) = 1, \sigma > 0$ is the dielectric constant or Field Strength Renormalization.

The function F imposes an ultra-violet cutoff in such a way that fields $\phi(x)$ that are typical for the measure $d\mu_t$ are functions which are self-similar on all scales down to $\ell(t)$ and at this scale they become approximately constant. This picture motivates, for example, the choice of block size in the cluster expansion of the last section. "Unit block" should have side 1 in a unit of length comparable to $\ell(t)$ because a block should be an object with the property that the field is essentially constant within it. If the block size is always kept comparable with $\ell(t)$, then the cluster expansion of the last section will provide error estimates that scale according to simple dimensional analysis.

The collection of fields $(\phi(x), \nabla_\mu\phi(x), \nabla_\mu\nabla_\nu\phi(x), \dots) \equiv (\phi(x), \phi(x,\mu), \phi(x,\mu,\nu), \dots)$ will be denoted $\phi(\xi)$ so in this section $\xi = (x, \alpha)$ with $\alpha = \emptyset$, μ, (μ,ν), or $\dots$ up to some permanently fixed finite number of derivatives.

For the rest of this section,

$$\ell_0 = 1.$$

I will write $d\mu_\ell$ sometimes instead of $d\mu_t$ where $\ell = e^t$.

Conventions. $a \gg 1$ means a sufficiently large depending only on the dimension d unless I explicitly mention dependence on other parameters. Similarly for $a \ll 1$. I will use $C, C_1, C_2, \ldots,$ for constants which depend only on dimension d. If a constant depends on a parameter, e.g., λ, I will indicate it by $C(\lambda)$ unless λ has already been fixed once and for all.

Periodic Boundary Conditions. For simplicity I shall assume Λ is a torus, whose sides are equal and equal to a power of L where L is an integer that will be produced below. The Fourier transform becomes a Fourier series and in the series for C_ℓ the $k = 0$ term is omitted.

For the rest of this section I will be describing how to choose the large field regulator g and the large set regulator Γ. The choice of g will have the good property that it is (almost) invariant under the flow. I will concentrate on the simplest case where the perturbation K depends on ϕ only through $\nabla\phi$ but these choices of g and Γ are of much more general use.

The basic example of a system where the interaction is a functional of $\nabla\phi$ is the dipole gas we discussed in Section 2.1. We have seen in Section 7.1 how to rewrite the interaction in (2.18) with the external field $f = 0$ in the form $\mathcal{E}\text{xp}[\Box + K](\Lambda)$ required for this section and Proposition 7.2 shows that the norm of the resulting K is small when the activity $\tilde{z}$ is small. Given such an initial $K_{\ell=1}$ we may consider a perturbation $K(\underline{\epsilon}) = K_1 + \sum \epsilon_i Q_i$ by some non-translation invariant functionals Q_i. Each Q should be a local observable by which I mean that there is a set Y (where the observable is located) and $Q(X)$ vanishes unless $X \supset Y$. The expectation of observables can be obtained by differentiating $\frac{1}{Z}\int d\mu_{\ell=1}\mathcal{E}\text{xp}[\Box + K(\epsilon)](\Lambda)$ with respect to $\underline{\epsilon}$ and setting $\underline{\epsilon} = 0$. I say "can" but no one has quite done it this way yet using the method I am about to describe and I will be discussing the trajectory only at $\underline{\epsilon} = 0$ whereas I should be considering an infinitesmal neighborhood. Work on this is now in progress. Dimock and Hurd [41] have included an external field $\int dx J(x)\phi(x)$ in the interaction in (2.18) and followed the evolution of K with a norm that includes derivatives with respect to J at $J = 0$. From this they have obtained the asymptotics of correlation functions. The other type of result that comes out of the methods I am about to describe is that for the dipole gas the pressure is analytic in the activity at small activity [30, 18]. Results on correlations have already been obtained by different methods by Gawedzki and Kupiainen e.g. [32] and also by Feldman et al. [26]. In the meantime I am reduced to asking you to trust me that the forthcoming scheme is good for more than just an elaborate calculation of an object without physical significance, the partition function without any external fields!

I will prove, for an initial K subject to certain symmetries, that

1. that there is a trajectory σ_ℓ for the σ in the covariance (8.3) in $d\mu$, a vacuum energy E_ℓ and a flow K_ℓ such that

$$e^{E_\ell}\int d\mu_{\ell,\sigma_\ell}\mathcal{E}\mathrm{xp}[\Box + K_\ell](\Lambda)$$

   is fixed and

$$\|K_\ell\|_{g,\Gamma,h} \to 0$$

   .

2. that if $\ell' > \ell$ then $K_{\ell'}(X)$ is a functional only of $K_\ell(Y)$ for $Y \subset X$ in the sense of $\underline{\epsilon}$ dependence.

g appearing in the norm of K has a slight dependence on ℓ which stabilizes as $\ell \to \infty$. This is natural and comes about because part of K is being withdrawn into the measure $d\mu_{\ell,\sigma_\ell}$. This withdrawal of a factor of the form $e^{-\frac{1}{2}\delta\sigma_\ell(\nabla\phi)^2}$ affects the growth of K at large $\nabla\phi$. Item (2) would be important when observables are analysed.

8.1 The Choice of Large Field Regulator

The choice of $g = g(X,\phi)$ in the norm $\| \|_{g,\Gamma,h}$ is made in such a way as to (i) satisfy the requirements (7.14), (ii) (essentially) satisfy the requirement $\mu_{ts} * g_s \le g_t$ in the cluster expansion Theorem 7.1 in such a way that under rescaling g_t returns to g_s, (iii) $|\partial^\beta\phi(x)|$, $1 \le |\beta| \le d+2$, are dominated by $g(X,\phi)$ if x is in X.

Let $\partial^\beta\phi(x)$ denote any partial derivative of $\phi(x)$ of order $|\beta|$. The Sobolev norm of order s is defined by

$$\|\phi\|_{s,X,\ell} \equiv \sum_{1\le|\beta|\le s}\left(\int_X |\partial^\beta\phi|^2\right)\ell^{2(|\beta|-1)}. \tag{8.4}$$

It has the property that if s is large enough depending on the dimension, then there is C_s such that for $\ell \ge 1$, for all multi-indices $1 \le |\beta| \le d+2$,

$$|\partial^\beta\phi(x)| \le C_s\|\phi\|_{s,X,\ell} \tag{8.5}$$

provided x is in the polymer X.

The regulator is defined to be

$$g_{\epsilon,\ell}(X,\phi) = \exp\left(\frac{\epsilon}{2}\|\phi\|_{s,X,\ell} + \frac{\epsilon\ell}{2c}\int_{\partial X}|\nabla\phi|^2\right). \tag{8.6}$$

c depends only on d and is fixed by the following lemma.

Lemma 8.1 *(a)*

$$|\partial^\beta\phi(x)| \leq \frac{C_s}{\sqrt{\epsilon}}\, g_{\epsilon,\ell}(X,\phi) \tag{8.7}$$

for any x in X, $\ell \geq 1$, $\epsilon > 0$, $1 \leq |\beta| \leq d+2$.

(b) If $\{X_i\}$ is any set of polymers which have at most $\tau \equiv \sup_x \#\{i : X_i \ni x\}$ overlapping at any point, then if c is sufficiently large depending only on d,

$$g_{\epsilon,\ell}(\cup X_i,\phi) \geq \prod_i g_{\epsilon/\tau,\ell}(X_i,\phi). \tag{8.8}$$

(c) Fix any $L > 1$. For $\epsilon > 0$ small depending on L, there exists $v(\epsilon) = 0(\epsilon)$ as $\epsilon \to 0$ such that

$$\mu_{\ell'\ell} * g_{\epsilon,\ell} \leq g_{\epsilon,\ell'}\left(\frac{\ell'}{\ell}\right)^{v(\epsilon)|X|} \tag{8.9}$$

for $1 \leq \ell \leq \ell' \leq L$, and all $\sigma \geq \frac{1}{2}$ (σ appears in (8.3)).

From part (c) we see that g does not quite satisfy the right hypothesis $\mu_{ts} * g_s \leq g_t$ to apply Theorem 7.1 but

$$\bar{g}_{\epsilon,\ell}(X) \equiv g_{\epsilon,\ell}(X)\ell^{v(\epsilon)|X|} \tag{8.10}$$

does.

From now on, I will write

$$\|K\|_{\epsilon,\Gamma,h} \equiv \|K\|_{g_\epsilon,\Gamma,h}. \tag{8.11}$$

Proof of Lemma 8.1. See [42, Appendix A]. There are three ideas, one is described in the discussion of large field regulators in Section 6, namely that κ evolves according to an equation of the form $\frac{d\kappa}{dt} = \text{Const. } \kappa^2$ so κ can be dominated by $\kappa_1(\ell/\ell_1)^\epsilon$ for any

$\epsilon > 0$ for $1 \leq \ell \leq L$ if κ_1 is small. The second idea is that the coefficient of $(\nabla\phi)^2$ need not increase at all because we can transfer its growth to a higher derivative by integration by parts. This generates the boundary integral. The third idea which is needed in part (b) is to dominate boundary integrals by volume integrals by a Sobolev inequality. $\square$

Comments.

(1) We shall later slightly change the deformation so we will need the fact that part (c) holds for $0 < \epsilon \ll 1$ depending on L and constants A, B:

$$\begin{aligned} A &= \sup_{\alpha, |\alpha| \leq 2s} \sup_{1 \leq \ell \leq L} |\partial^\alpha \dot{C}_\ell(x)| \\ B &= \sup_{\alpha, |\alpha| \leq 2s} \sup_{1 \leq \ell \leq L} \int |\partial^\alpha \dot{C}_\ell(x)| dx, \end{aligned} \tag{8.12}$$

regardless of what covariance C_ℓ is being deformed and regardless of what the deformation is. In particular, these conditions hold for the standard deformation with σ fixed discussed above.

(2) The ℓ dependence in (8.4) is chosen to make the norm rescale to itself after a RG step (defined below). However Lemma 8.1 is valid with any positive powers of ℓ in the Sobolev norm.

8.2 Reblocking B

In order to keep the block size comparable with the scale $\ell(t)$, which is increasing as the covariance is deformed, we must periodically stop the deformation and rewrite $\mathcal{E}\mathrm{xp} \equiv \mathcal{E}\mathrm{xp}_1$ on a new scale ($\mathcal{E}\mathrm{xp}_L$) where L is an integer > 1 so that the new polymers (L-polymers) are based on blocks L times larger. Suppose X is a 1-polymer. Let $\bar{X}$ be the smallest L-polymer that contains X. Then

$$\mathcal{E}\mathrm{xp}_1(\square + K) = \mathcal{E}\mathrm{xp}_L(\square + \mathcal{B}K), \tag{8.13}$$

where on the right hand side $\square$ and $\mathcal{B}K$ are defined on sets associated with L-blocks and for any L-polymer U

$$\mathcal{B}K(U) \equiv \sum_{\{X_i\}} \prod_i K(X_i), \tag{8.14}$$

and $\{X_i\}$ is summed over all sets $\{X_1, \ldots, X_N : N \text{ arbitrary}\}$ of 1-polymers such that

$$\begin{array}{rl} \text{(i)} & \cup \bar{X}_i = U; \\ \text{(ii)} & \text{the overlap graph on } \{X_i\} \text{ is connected.} \end{array} \tag{8.15}$$

The *overlap graph* is all pairs $\{(ij) : \bar{X}_i \cap \bar{X}_j \neq \phi\}$.

This $\mathcal{B}$ at first sight is not very prepossessing but with the right choice of large set regulator Γ it has a good property that captures Wilson's observation that only the local parts of an interaction are relevant.

Choice of Γ. We want to choose $\Gamma(X)$ in such a way that the activities $K(X, \phi)$ will have to be very small if X is not close to being a unit block, i.e., local. To quantify the suppression of polymers with many blocks we include a factor $A^{|X|}$ where $A \geq 1$, and $|X|$ is the number of blocks in the polymer X. To quantify the suppression of polymers with widely separated blocks we include a factor

$$\Theta(X) = \inf_T \prod_{b \in T} \theta(|b|), \tag{8.16}$$

where T runs over tree graphs connecting the unit blocks that comprise X by bonds $b \in T, |b|$ is the length [11] of such a bond. $\theta(s)$ is an increasing function of s, s a non-negative integer, such that $\theta(0) = \theta(1) = 1$ and

$$\theta\left(\left\{\frac{s}{L}\right\}\right) \leq A^{-1}\theta(s), \quad s \geq 2, \tag{8.17}$$

where $\{x\}$ is the smallest integer $\geq x$. (As $s \to \infty, \theta(s) \sim s^p$ with $p = \log_L A$). The point of this requirement is that whenever a polymer consisting of two or more blocks which are not neighbors is reblocked, then on the scale of the new blocks the distance between them will be smaller and so the reblocked polymer will be smaller in the norm appropriate to the new blocks. In this way we capture Wilson's idea that non-local parts of the interaction are irrelevant. If a polymer consists of many blocks then when reblocked it will have fewer blocks, therefore there will be fewer factors of A in the norm of the new polymer so again it will look smaller. These ideas are summarised in the following proposition.

Given K define K_S, the part of K that lives on small sets, by

$$K_S(X) \equiv \begin{cases} K(X) & \text{if } X \text{ connected and } |X| \leq 2^d \\ 0 & \text{otherwise,} \end{cases} \tag{8.18}$$

and let K_B be defined by $K = K_S + K_B$. Let Γ be constructed as above. Fix[12] η so that $1 < \eta^{2(2^d)} < \frac{3}{2}$, and set $(\eta\Gamma)(X) = \eta^{|X|}\Gamma(X), (\eta^{-1}\Gamma)(X) = \eta^{-|X|}\Gamma(X)$. Then

[11] All lengths are measured by $|(x_1, \ldots, x_d)| = \sup_i |x_i|$ where x_i is measured in units of block size.

[12] This mysterious η is best ignored, i.e., set equal to one, the first time through this argument. It is there mostly because I want to reserve some decay in the size of the polymer to cancel a small growth in the vacuum energy part $v(\epsilon)$ of the large field regulator. See (8.9)

Proposition 8.2 *For all $\epsilon \geq 0, \underline{h} \geq 0$, if $\|K\|_{\epsilon,\eta^{-1}\Gamma,\underline{h}} \ll 1$ depending only on η,*

$$\|\mathcal{B}K\|_{\epsilon,\eta\Gamma,\underline{h}} \leq 2L^d \left\{ \|K_S\|_{\epsilon,\eta^{-1}\Gamma,\underline{h}} + A^{-1}\|K_B\|_{\epsilon,\eta^{-1}\Gamma,\underline{h}} \right\}. \tag{8.19}$$

Proof: See Lemmas 3.1 and 3.2 in [18]. The main point is that the small sets are those sets that do not become smaller as measured by Γ when they are minimally covered by L-blocks. To understand the growth by L^d of the small set part, note that a single block can occupy any of L^d positions and still be covered by the same L-block. □

Now we choose η *once and for all*, and let

$$A = L^{-2d},$$

so that the K_B contribution to $\mathcal{B}K$ is suppressed by a factor L^{-d}.

8.3 Renormalization (Dipole Systems)

We have seen in Proposition 8.2 that the reblocking operation $\mathcal{B}$ automatically suppresses the part K_B of K that lives on big sets but the local parts K_s that live on *small sets* defined by

$$X \in \mathcal{S} \Longleftrightarrow X \text{ is connected and } |X| \leq 2^d \tag{8.20}$$

grow by a volume factor L^d under $\mathcal{B}$. Thus the choice of Γ has captured Wilson's idea that only local parts of the interaction can be relevant.

To examine the local parts carefully we define a localization operator $\mathcal{L}$. I am borrowing this terminolgy from an analogous constuction which appears in many papers by Gallavotti et al., e.g., see section 8 in [29]. Let

$$(\mathcal{L}K)(X) = \begin{cases} 0 & \text{if } X \notin \mathcal{S} \\ \displaystyle\sum_{N=0}^{P} \frac{1}{N!} \left[\int \cdots \int \frac{\delta^N K(X, \phi = 0)}{\delta\phi(\xi_1)\cdots\delta\phi(\xi_N)}\, \phi(\xi_1)\cdots\phi(\xi_N) \right]_{\text{loc}} & \text{if } X \in \mathcal{S} \end{cases} \tag{8.21}$$

$[M]_{\text{loc}}$ means that the monomial M is replaced by a local polynomial of the form

$$[M]_{\text{loc}} = \sum_{\underline{\alpha}} c_{\underline{\alpha}} \int_X \partial^{\alpha_1}\phi(\xi)\cdots\partial^{\alpha_N}\phi(\xi)\, d\xi \tag{8.22}$$

such that $M - [M]_{\text{loc}}$ has dimension more negative than the $(p+1)^{\text{th}}$ term in the Taylor expansion of K in ϕ about $\phi = 0$. The dimension is computed by assigning $L^{-\frac{d-2}{2}}$ to each $\phi(x)$ and L^{-1} to each gradient. For this computation do not assign dimensions to the integrals in ξ or kernels in ξ. To see that $[M]_{\text{loc}}$ exists insert

$$\phi(\xi) \equiv \partial^\alpha \phi(x) = \partial^\alpha \phi(z) + \int_z^x d(\partial^\alpha \phi) \tag{8.23}$$

for each $\phi(\xi)$ in M and consider $M = \frac{1}{|X|} \int_X M\, dz$. This contains a part which comes solely from the $\partial^\alpha \phi(z)$ terms in (8.23) and this part is local as in (8.22). Everything else in this expression for M is of dimension improved by one because it has at least one more gradient arising from the d in (8.23). We can take all these terms and use the same method to write them as local polynomials plus remainders of improved dimension until all remainders have sufficiently negative dimension. This process generates dependence on fields $\partial^\beta \phi(x)$ where β belongs to a range, Range(β), determined by the choice of p. If, for example, $p = 3$, then $M - [M]_{\text{loc}}$ contains $\nabla\phi\partial^\beta\phi$ with $|\beta| = d + 2$ whose dimensions are $L^{-(2d+1)}$ which is one more negative than $(\nabla\phi)^4$ which is the dominant term in the $(p+1)^{\text{th}}$ term in the Taylor expansion of K in ϕ about $\phi = 0$. This is the reason for the hypotheses we will impose on K dependence on fields $\partial^\beta\phi$.

I make the following assumptions on $K(X, \phi)$:

(i) $K(X, \phi)$ depends on ϕ only through $\partial^\beta \phi(x)$, $1 \le |\beta| \le d + 2$;

(ii) $K(X, \phi) = K(X, -\phi)$;

(iii) K is invariant under rotations and parity, i.e., if r is a reflection in a coordinate hyperplane or a rotation by 90°, then

$$K(rX, r\phi) = K(X, \phi).$$

These assumptions hold for the dipole gas, for example. If we have (i) and (ii), then we can and will define $\mathcal{L}$ with $p = 3$ (which will be the same as $\mathcal{L}$ with $p = 2$ because the 3^d variational derivative of K vanishes at $\phi = 0$). (iii) is useful because it guarantees that $\mathcal{L}K$ will lead to a renormalization of σ. Without it one would have to allow σ to become a bilinear form $\sigma_{\mu\nu}k_\mu k_\nu$ instead of σk^2.

Now I will summarize a sequence of operations that will constitute one R. G. transformation. Suppose this operation has already been accomplished some number of times, and as a result we have written the original partition function in the form

$$e^{-E^j} \int d\mu_{L,\sigma^{(j)}} \mathcal{E}\text{xp}_1 \left[\Box + K_L^{(j)}\right] (\Lambda^{(j)}), \tag{8.24}$$

where $d\mu_{L,\sigma^{(j)}}$ has covariance $F(L^2k^2)/(\sigma^{(j)}k^2)$. The subscript L on $d\mu$ means that the length scale of the ultra-violet cutoff in the covariance of $d\mu$ is $\ell = L$. The subscript 1 on $\mathcal{E}$xp means that it is defined on *unit* blocks. $\Lambda^{(j)}$ is the original torus Λ rescaled by L^{-j}. L will be specified below. An R. G. transformation will consist of:

(1) Renormalization $\mathcal{E}$ in which the $\mathcal{L}K$ parts of K are extracted and cancelled by a shift in the covariance of $d\mu_{L,\sigma^{(j)}}$. To do this, let X be any allowed set as defined below (7.2) and set

$$\Omega(X) \equiv \sum_{Y \subset X} \mathcal{L}K_L(Y). \tag{8.25}$$

Ω is quadratic in $\nabla\phi$ by our choice of $\mathcal{L}$. Then, dropping indices j and $\sigma^{(j)}$,

$$e^{-E} \int d\mu_L \mathcal{E}\mathrm{xp}_1[\Box + K_L](\Lambda) = e^{-(E+\delta E)} \int d\mu_{L,\Omega} \mathcal{E}\mathrm{xp}_1[\Box + \mathcal{E}K_L](\Lambda), \tag{8.26}$$

where $d\mu_{L,\Omega}$ is the new Gaussian measure defined by

$$\begin{aligned} d\mu_{L,\Omega} &= e^{\delta E} d\mu_L e^{\Omega(\Lambda)} \\ e^{-\delta E} &= \int d\mu_L e^{\Omega(\Lambda)}, \end{aligned} \tag{8.27}$$

so that $d\mu_{L,\Omega}$ is normalized and $\mathcal{E}K$ is a new set of polymer activities defined by solving

$$\mathcal{E}\mathrm{xp}[\Box + \mathcal{E}K_L] = e^{-\Omega} \mathcal{E}\mathrm{xp}[\Box + K_L]; \tag{8.28}$$

$E^{(j+1)} = E^{(j)} + \delta E$.

(2) Reblocking $\mathcal{B}$ in which we change the blocks from unit to L-blocks using the operation $\mathcal{B}$ defined in Section 7.2,

$$\mathcal{E}\mathrm{xp}_1[\Box + \mathcal{E}K_L] = \mathcal{E}\mathrm{xp}_L[\Box + \mathcal{B}\mathcal{E}K_L]. \tag{8.29}$$

(3) Rescaling $\mathcal{R}$ in which the change of variables $x = Lx', \phi(x) = L^{1-d/2}\phi'(x')$ brings L-blocks back to unit blocks and rescales the Gaussian measure so that $L \to 1$:

$$\begin{aligned} &e^{-E^{(j+1)}} \int d\mu_{L,\Omega} \mathcal{E}\mathrm{xp}_L \left[\Box + \mathcal{B}\mathcal{E}K_L^{(j)}\right] (\Lambda^{(j)}) \\ &\quad = e^{-E^{(j+1)}} \int d\mu_{1,\mathcal{R}\Omega} \mathcal{E}\mathrm{xp} \left[\Box + K_1^{(j+1)}\right] (\Lambda^{(j+1)}), \end{aligned} \tag{8.30}$$

where

$$K_1^{(j+1)} = \mathcal{R}\mathcal{B}\mathcal{E}K_L^{(j)}. \tag{8.31}$$

(4) Integrating Out in which Theorem 7.1 is used to advance $\ell = 1$ to $\ell = L$ by deforming the covariance of $d\mu_{1,\mathcal{R}\Omega}$ so that it becomes $d\mu_{L,\sigma^{(j+1)}}$ which is the same Gaussian measure which we started with at the outset of this RG transformation in (8.24) except that $\sigma^{(j)}$ has become $\sigma^{(j+1)}$.

$$\int d\mu_{1,\mathcal{R}\Omega}\mathcal{E}\mathrm{xp}_1\left[\Box + K_1^{(j+1)}\right](\Lambda^{(j+1)}) = \int d\mu_{L,\sigma^{(j+1)}}\mathcal{E}\mathrm{xp}_1\left[\Box + K_L^{(j+1)}\right](\Lambda^{(j+1)}), \quad (8.32)$$

and so (8.24) with j has become (8.24) with $j+1$.

This completes the RG transformation (steps (1)-(4)).

The integration out involves a deformation of the covariance which is not quite rescaling because the initial covariance, that of $d\mu_{1,\mathcal{R}\Omega}$, is not quite equal to $C_{\ell=1} = \mathcal{R}C_{\ell=L}$. This is of course the way in which anomalous dimensions can appear in other more complicated models. In this model there is no need to use a deformation which is not quite canonical rescaling but it is attractive because it restores the covariance to a simple form from which the role of σ as a dielectric constant or eqivalently as a field strength renormalisation is immediately apparent. If we simply rescaled the covariance of $d\mu_{1,\mathcal{R}\Omega}$[13] then σ would be embedded in the covariance in a more complicated way and one would need to prove a small lemma to show that the asymptotic decay of such a covariance is of the form $\frac{1}{\sigma}\|(x-y)\|^{2-d}$ as befits a dielectric constant. I am indebted to Pronob Mitter for education on this point.

Now let us consider how to choose the parameters L, κ in the large field regulator g_κ and h in the norm $\|\ \|_{g_\kappa,\Gamma,h} = \|\ \|_{\kappa,\Gamma,h}$. We will construct a constant C_1 which determines these parameters and some others according to:

Parameters

(A) Choose L large so that $\frac{C_1}{L^d} < \frac{1}{2}$.

(B) Let $\kappa_j = \kappa_0 \sum_0^j 2^{-n}$. This will be a sequence of numbers that limits the large field behavior of the polymer activities produced after j RG steps, roughly speaking by $|K^{(j)}| \leq e^{\frac{1}{2}\kappa_j \int (\nabla\phi)^2}$

Choose κ_0 so small that for all $j \geq 0$

$$L^{v(\kappa_j)} \leq \eta,$$

[13] as was done in [18]

where $v(\kappa)$ appears in Lemma 8.1, part (c).

Define $\epsilon_j = \frac{\kappa_{j+1}-\kappa_j}{\tau}$ where τ is the largest number of small sets that can overlap at a point, $\tau = \sup_x \#\{X \in \mathcal{S} : X \ni x\}$. It is very important to this argument that this number τ is finite.

(C) Choose h large so that

$$\kappa_0 h^2 L^{-3d-2} \geq 1.$$

Note that if $j \gg 1$, then $\epsilon_j h^2 \leq 1$. Recall that h is the radius of analyticity of the polymer activities

Having set these parameters, I make the following

Inductive Assumptions $(I^{(j)})$

(i) $\|K_L^{(j)}\|_{\kappa_j,\Gamma,h} \leq \left(\frac{C_1}{L^d}\right)^j$;

(ii) $\sigma^{(j)} \geq 1 - \frac{1}{8}\sum_0^j 2^{-n}$ where $\sigma^{(j)}$ appears in the Gaussian measure covariance (8.3).

Note that for $j \gg 1, (\epsilon_j h^2)\|K^{(j)}\|_{\kappa_j,\Gamma,h} \leq 1$, indeed, as small as you like.

Theorem 8.3 *If $j \gg 1$ depending on these choices of L, κ_j, h, then $I^{(j)} \Longrightarrow I^{(j+1)}$ and*

$$\begin{aligned}
\int d\mu_{L,\sigma^{(j)}} \mathcal{E}\mathrm{xp}_1[\Box + K_L^{(j)}](\Lambda^{(j)}) &= e^{-\delta E^{(j)}} \int d\mu_{L,\sigma^{(j+1)}} \mathcal{E}\mathrm{xp}_1[\Box + K_L^{(j+1)}](\Lambda^{(j+1)}); \\
\delta E^{(j)} &= \int d\mu_{L,\sigma^{(j)}} \exp\left(\sum_{Y \subset \Lambda^{(j)}} \mathcal{L}K^{(j)}(Y)\right).
\end{aligned}$$

To use this theorem we pick $j_0 \gg 1$ so that $I^{(j)} \Longrightarrow I^{(j+1)}$ for all $j \geq j_0$ and then rename the initial $K(X)$ and initial Λ, K^{j_0} and Λ^{j_0} respectively. Then provided the norm of the initial K is small, as required by I^{j_0}, the theorem implies that we can repeat RG steps until $\Lambda^{(j)}$ becomes a single unit block Δ. In [18, 43, 41] this is used to calculate the logarithm of the partition function by reducing it to the logarithm of the partition function in a single block.

The conclusions follow from our discussion above except for $I^{(j)} \Longrightarrow I^{(j+1)}$, which will require the following Lemma 8.4 and Proposition 8.5.

Lemma 8.4 *Let $R = e^{-\mathcal{L}K} - 1$. There is C_2 such that for all $\epsilon, h, \kappa \geq 0$ with $\epsilon h^2 \leq 1$, and any Γ*

(i) $\|\mathcal{L}K\|_{\epsilon,\Gamma,h} \leq C_2 \frac{1}{\epsilon h^2} \|K\|_{\kappa,\Gamma,h}$.

If, in addition, $(\epsilon h^2)^{-1}\|K\|_{\kappa,\Gamma,h} \ll 1$, then

(ii) $\|R\|_{\epsilon,\Gamma,h} \leq C_2 \frac{1}{\epsilon h^2} \|K\|_{\kappa,\Gamma,h}$

(iii) $\|R + \mathcal{L}K\|_{\epsilon,\Gamma,h} \leq C_2 \left(\frac{1}{\epsilon h^2}\|K\|_{\kappa,\Gamma,h}\right)^2$.

For a detailed proof, see [18, Proposition 4.2]. However, notice that (i) follows easily from

$$|\phi(\xi)|^2 \leq O(\epsilon^{-1}) g_\epsilon(X, \phi)$$

if $\xi = (x, \mu), x \in X$. To understand (ii), notice that (i) says that for $\phi(\xi)$ large, $R \sim \exp\left(\frac{1}{2}\|K\|\frac{\phi^2}{h^2}\right)$, so we need $(\epsilon h^2)^{-1}\|K\| \leq 1$ to dominate R by g_ϵ. (iii) is expected since $R - \mathcal{L}K = O(\mathcal{L}K)^2$.

The next proposition is related to the fact that if a function $f(z)$ is analytic in a strip of width h then $f^{(n)}(z)$ is, in a strip of width αh, down in size by a factor α^n. The generalised norm $\|(\cdot)\|_{\kappa,\Gamma,\underline{h}}$ where each of the fields $\nabla\phi, \nabla\nabla\phi, \ldots$ is assigned a separate radius of analyticity makes an appearance here. $\underline{h} \equiv (h_{\nabla\phi}, h_{\nabla\nabla\phi}, \ldots) \equiv (h_{\partial^\beta\phi})_{\beta\in\mathrm{Range}(\beta)} \equiv (h_\beta)_{\beta\in\mathrm{Range}(\beta)}$. It is defined, by replacing in equation (7.11), $hd\xi$ by $h_\beta d\xi$. Recall that $\xi = (x, \beta)$, $\int d\xi$ integrates over x and β and that $\mathrm{Range}(\beta)$ was discussed when the localisation operator $\mathcal{L}$ was defined.

Proposition 8.5 *Let $[(1-\mathcal{L})K]_S$ denote the small set part of $(1-\mathcal{L})K$ as defined in Proposition 7.2.1. Let $\underline{\alpha h}$ be given by $(\underline{\alpha h})_\beta = \alpha L^{-|\beta|+1} h$. For each $p = 0, 1, \ldots$, there is $C_3(p)$ such that for all $\alpha, h, \kappa > 0$ with $((\underline{\alpha h})_\beta)^2\kappa \geq 1$ for all $\beta \in \mathrm{Range}(\beta)$, and $L^{-\frac{d}{2}} \leq \alpha \leq 1$, and any Γ,*

$$\|[(1-\mathcal{L})K]_S\|_{\kappa,\Gamma,\underline{\alpha h}} \leq C_3(p)\alpha^{p+1}\|K\|_{\kappa,\Gamma,h}\ .$$

In this section, $p = 3$, so I write $C_3 = C_3(p)$. Also I fix $\alpha = 2L^{-\frac{d}{2}}$, where L will be chosen below.

Sketch of proof. (A detailed proof of a very similar proposition is given in [18, Lemma 4.3].) First note that variational derivatives of order $n \geq p+1$ for K and $(1-\mathcal{L})K$

coincide. Because h is scaled by a factor of α in the left hand side the contribution of all derivatives of order $n \geq 4$ is down by a factor of $\alpha^n \leq \alpha^p$. This is part of the reason for α^p on the right hand side. The other part comes from the fact that we can write lower derivatives in terms of higher ones. For example consider the $n = 0$ contribution to the norm of $(1 - \mathcal{L})K$: we can express it in terms of the $p + 1$ variational derivative using (for $p = 3$),

$$(1 - \mathcal{L})K(\phi) = \int_0^1 dt\, \frac{(1-t)^3}{3!} \int \frac{\delta^4 K}{\delta \phi^4}(t\phi)\, \phi\phi\phi\phi + \ldots, \tag{8.33}$$

where $\ldots$ is the difference $M - [M]_{loc}$ discussed in (8.22), (8.23). I will drop this part because it is easy to estimate by $C\alpha^{p+1}\|K\|$ Therefore

$$\begin{aligned} |(1-\mathcal{L})K(\phi)g^{-1}(\phi)| \;&\leq\; \left| \int_0^1 dt\, \frac{(1-t)^3}{3!} \int \frac{\delta^4 K}{\delta\phi^4}(t\phi) g^{-1}(t\phi) \left\{ \phi\phi\phi\phi g^{-1}\left(\sqrt{1-t^2}\phi\right)\right\} \right| \\ &\leq\; \frac{4!}{h^4}\|K\|_{\kappa,\Gamma,h} \int_0^1 dt\, \frac{(1-t)^3}{3!} \frac{\text{Const.}}{(\kappa(1-t^2))^2} \\ &\leq\; \text{Const. } \alpha^4 \|K\|_{\kappa,\Gamma,h}, \end{aligned} \tag{8.34}$$

using convergence of the t integral and $(\alpha h)^2 \kappa \geq 1$. We can put together similar estimates on the $n = 1, 2, \ldots, p$ parts of the $\|[(1-\mathcal{L})K]_S\|$ norm to conclude as in Lemma 8.4. $\square$

Proof of Theorem 8.3. In particular this proves that $I^{(j)} \Longrightarrow I^{(j+1)}$ if $j \gg 1$.

(1) Renormalization. The task is to solve (8.28) for $\mathcal{E}K$ and estimate the norm of $\mathcal{E}K$. I begin by proving that

$$\mathcal{E}K = (1 - \mathcal{L})K + H, \tag{8.35}$$

($K = K_L^{(j)}$) where H, for higher order in K, is given by

$$H = (R + \mathcal{L}K) + R^+ + (R + R^+) \vee K, \tag{8.36}$$

$R \equiv e^{-\mathcal{L}K} - 1$, R^+ was defined in Definition 6.1.1, $A \vee B$ was defined in Proposition 7.3. To obtain this formula, first note that

$$\begin{aligned} e^{-\Omega(X)} \;&=\; \prod_{Y \subset X} \left(e^{-\mathcal{L}K(Y)} - 1 + 1 \right) \\ &=\; \prod_{Y \subset X} (1 + R(Y)) = \mathcal{E}\text{xp}[\square + R + R^+](X) \end{aligned} \tag{8.37}$$

by Proposition 7.2. Then, by definition of $\mathcal{E}K$,

$$\begin{aligned}\mathcal{E}\mathrm{xp}[\Box + \mathcal{E}K] &= e^{-\Omega}\mathcal{E}\mathrm{xp}[\Box + K] \\ &= \mathcal{E}\mathrm{xp}[\Box + R + R^{+}]\mathcal{E}\mathrm{xp}[\Box + K] \\ &= \mathcal{E}\mathrm{xp}[\Box + K + R + R^{+} + K \vee (R + R^{+})]\end{aligned}$$

(by Proposition 7.3)

$$= \mathcal{E}\mathrm{xp}[\Box + K - \mathcal{L}K + H] \tag{8.38}$$

which proves the formula for $\mathcal{E}K$.

H looks complicated and it is, but all we need is that there is a constant C such that

$$\|H\|_{\kappa_{j+1},\eta^{-1}\Gamma,h} \leq C\left(\frac{1}{\epsilon_j h^2}\|K\|_{\kappa_j,\Gamma,h}\right)^2, \tag{8.39}$$

where $\eta > 1$ was fixed once and for all just below Proposition 8.2.

I shall write $\|K\|$ instead of $\|K\|_{\kappa_j,\Gamma,h}$ in this proof. H consists of three pieces, $R + \mathcal{L}K, R^{+}$ and $(R + R^{+}) \vee K$. It is enough to prove (8.39) for each piece.

Piece 1: This is small because $R \sim -\mathcal{L}K$. Thus

$$\|R + \mathcal{L}K\|_{\kappa_{j+1},\eta^{-1}\Gamma,h} \leq \|R + \mathcal{L}K\|_{\epsilon_j,\eta^{-1}\Gamma,h}, \tag{8.40}$$

because $\kappa_{j+1} \geq \epsilon_j$

$$\leq \|R + \mathcal{L}K\|_{\epsilon_j,\Gamma,h} \tag{8.41}$$

because $\eta^{-1}\Gamma \leq \Gamma$. Now apply Lemma 8.4 and conclude the bound on piece 1.

Piece 2: R^{+} is small for the same reason that all $(\cdot)^{+}$ quantities are small — it is of order K^2. In Proposition 7.2, replace Γ by $\gamma^{-1}\Gamma$, set $\gamma = \sqrt{\eta}$, $\epsilon = \tau\epsilon_j$ and $K = R$. The result is a bound for R^{+} in terms of R which we estimate using Lemma 8.4 in terms of K obtaining: there is C such that for $j \gg 1$ (to make $\|K\|$ small)

$$\|R^{+}\|_{\tau\epsilon_j,\eta^{-\frac{1}{2}}\Gamma,h} \leq C\left(\frac{1}{\epsilon_j h^2}\|K\|\right)^2, \tag{8.42}$$

and since $\eta^{-\frac{1}{2}}\Gamma \geq \eta^{-1}\Gamma$ this is also a bound for the norm $\|R^{+}\|_{\tau\epsilon_j,\eta^{-1}\Gamma,h}$. Finally, for $j \gg 1$ we have $\tau\epsilon_j \leq \kappa_{j+1}$, so this is a bound for $\|R^{+}\|_{\kappa_{j+1},\eta^{-1}\Gamma,h}$ as required.

Piece 3: This is small because it also is of order K^2 since it is formed by the overlap of two or more polymers. First we combine (8.42) and Lemma 8.4 to get: there is C such that

$$\|R + R^{+}\|_{\tau\epsilon_j,\eta^{-\frac{1}{2}}\Gamma,h} \leq C\left(\frac{1}{\epsilon_j h^2}\|K\|\right). \tag{8.43}$$

Now we estimate $(R+R^+)\vee K$ by taking $A=R+R^+, B=K, g_A=g_{\tau\epsilon_j}, g_B=g_{\kappa_j}$ in Proposition 7.34, with $\gamma=\sqrt{\eta}$ and Γ replaced by $\eta^{-1}\Gamma$. The result is

$$\begin{aligned}&\|(R+R^+)\vee K\|_{\kappa_{j+1},\eta^{-1}\Gamma,h}\\ &\quad\leq \sum_{N,M\geq(1,1)} C_\gamma^{N+M}\|R+R^+\|^N_{\tau\epsilon_j,\eta^{-\frac{1}{2}}\Gamma,h}\|K\|^M,\end{aligned} \tag{8.44}$$

where again I used $\eta^{-\frac{1}{2}}\Gamma\leq\Gamma$ in the K norm. We conclude the bound (8.39) on piece 3 using (8.43) and $j\gg 1$. This concludes the proof of H bound (8.39).

(2) Reblocking. $\mathcal{E}K=(1-\mathcal{L})K+H$ as in (8.35). From Proposition 8.5 we estimate the small set part of $(1-\mathcal{L})K$

$$\begin{aligned}\|[(1-\mathcal{L})K]_S\|_{\kappa_{j+1},\eta^{-1}\Gamma,\underline{\alpha h}} &\leq \|[(1-\mathcal{L})K]_S\|_{\kappa_j,\Gamma,\underline{\alpha h}}\\ &\leq 16C_3L^{-2d}\|K\|,\end{aligned} \tag{8.45}$$

where $\alpha=2L^{-d/2}$. Using $H=O(K^2)$, (8.39), we can take $j\gg 1$ so that

$$\|[\mathcal{E}K]_S\|_{\kappa_{j+1},\eta^{-1}\Gamma,\underline{\alpha h}}\leq 17C_3L^{-2d}\|K\| \tag{8.46}$$

(write $[\mathcal{E}K]_S=[(1-\mathcal{L})K]_S+[H]_S$ and estimate each small set part separately by the triangle inequality). Now we apply the reblocking Proposition 8.2 to conclude, again using $H=O(K^2)$, that for $j\gg 1$,

$$\|\mathcal{B}\mathcal{E}K\|_{\kappa_{j+1},\eta\Gamma,\underline{\alpha h}}\leq(34C_3+2)L^{-d}\|K\| \tag{8.47}$$

which by the inductive hypothesis is less than

$$\leq(C_1L^{-d})^{(j+1)} \tag{8.48}$$

provided

$$C_1=34C_3+2. \tag{8.49}$$

Thus, the C_1 in the inductive hypothesis is determined by C_3 in Proposition 8.5.

(3) Rescaling. Letting $x=Lx', \phi(x)=L^{1-d/2}\phi'(x'), (\nabla\phi)(x)=L^{-d/2}\nabla'\phi'(x')$, which implies $h_{\nabla\phi}=L^{d/2}h'_{\nabla\phi}$, $h_{\nabla\nabla\phi}=L^{d/2+1}h'_{\nabla\nabla\phi}$, ... and, referring to the definition of g in (8.6)

$$g_{\kappa_{j+1},L}(X,\phi)=g_{\kappa_{j+1},1}(L^{-1}X,\phi'). \tag{8.50}$$

Thus, recalling $\alpha=2L^{-d/2}$ and (8.48),

$$\begin{aligned}\|K_1^{(j+1)}\|_{\kappa_{j+1},\eta\Gamma,2h} &\equiv \|\mathcal{R}\mathcal{B}\mathcal{E}K\|_{\kappa_{j+1},\eta\Gamma,2h}\\ &= \|\mathcal{B}\mathcal{E}K\|_{\kappa_{j+1},\eta\Gamma,\underline{\alpha h}}\leq(C_1L^{-d})^{(j+1)}.\end{aligned} \tag{8.51}$$

(4) Integrating out. First we choose the deformation of covariance that we want to use in Theorem 7.1. We want to deform the covariance of $d\mu_{1,\mathcal{R}\Omega}$ into the covariance of $d\mu_{L,\sigma^{(j+1)}}$ which is

$$\hat{C}_L = \frac{F(L^2k^2)}{\sigma^{(j+1)}k^2} \tag{8.52}$$

Ω has the form (up to a ϕ independent part)

$$\Omega(\Lambda^{(j)}) = \sum_{Y \subset \Lambda^{(j)}} c_{\underline{\alpha}}(Y) \int_Y dz \partial^{\alpha_1}\phi(z)\partial^{\alpha_2}\phi(z), \tag{8.53}$$

which is a translation invariant functional of ϕ, so passing to the Fourier transform and remembering the parity assumptions on K there are $\delta\sigma$ and ω such that

$$\mathcal{R}\Omega(\Lambda^{(j)}) = -\frac{1}{2}\sum_k \{\delta\sigma k^2 + \omega(k)\}|\hat{\phi}(k)|^2, \tag{8.54}$$

where the polynomial $\omega(k) = O((k^2)^2)$ as $k \to 0$. The quadratic parts of Ω and consequently ω are formed from the second derivatives of K. Therefore by their construction, c.f. (8.22), there is C such that

$$|\partial_k^\alpha \omega(k=0)| \leq \frac{C}{h^2}\, \|K\| \tag{8.55}$$

and

$$|\delta\sigma| \leq \frac{C}{h^2}\, \|K\|. \tag{8.56}$$

By definition the covariance of $d\mu_{1,\mathcal{R}\Omega}$ is

$$\left(\frac{\sigma^{(j)}k^2}{F(k^2)} + \delta\sigma k^2 + \omega(k)\right)^{-1} = \frac{F(k^2)}{\sigma^{(j+1)}k^2[1+G(k)]} \tag{8.57}$$

where

$$\begin{aligned} \sigma^{(j+1)} &= \sigma^{(j)} + \delta\sigma \\ G(k) &= \delta\sigma(F(k^2)-1) + \frac{\omega(k)}{k^2}F(k^2). \end{aligned} \tag{8.58}$$

Define the deformation to be, for $1 \leq \ell \leq L$,

$$\hat{C}'_\ell = \frac{F(\ell^2k^2)}{\sigma^{(j+1)}k^2[1+a(\ell)G(k)]}, \tag{8.59}$$

where $a(\ell)$ is a linear function such that $a(1) = 1, a(L) = 0$, so that $\hat{C}'_\ell$ interpolates between $\hat{C}_L$ and the covariance of $d\mu_{1,\mathcal{R}\Omega}$. The *crucial point* is that the coefficient of k^2 in the denominator, i.e., the strength of the singularity, does not change because $G(0) = 0$. Provided $\sigma^{(j+1)}$ remains bounded away from zero uniformly in $j \gg 1$ this deformation has the following properties: for $j \gg 1$,

(i) The constants A and B in (8.12), the hypothesis for the Lemma 8.1 on g_κ, will be bounded by a constant $C(L) = C$ since L is chosen;

(ii) $\|\dot{C}'_\ell\|_{\eta\Gamma}$ is bounded by a constant $C(L) = C$ since L is chosen.

From (8.56) we see that if $\sigma^{(j)} \geq 1 - \frac{1}{8}\sum_0^{(j)} 2^{-n}$, then $\sigma^{(j+1)} \geq 1 - \frac{1}{8}\sum_0^{(j+1)} 2^{-n}$ *which is part (ii) of the inductive hypothesis* $I^{(j+1)}$. In particular, $\sigma^{(j)} \geq \frac{3}{4}$ for any $j \gg 1$.

Now we apply Theorem 7.1 with the choices: (h in Theorem 7.1) $= 2h$, $h' = h$, (Γ in Theorem (6.1) $= \eta\Gamma$, (g_ℓ in Theorem 7.1) $= \bar{g}_\ell$, where (c.f., (8.10),

$$\bar{g}_\ell(X,\phi) = g_{\kappa_{j+1},\ell}(X,\phi)\ell^{v(\kappa_{j+1}|X|}. \tag{8.60}$$

By Lemma 8.1, this satisfies $\mu_{\ell'\ell} * \bar{g}_\ell \leq g_{\ell'}$ which is one hypothesis of Theorem 7.1.

Choose $j \gg 1$, so that by the bound (8.51) on $K_1^{(j+1)}$

$$\|K_1^{(j+1)}\|_{\kappa_{j+1},\eta\Gamma,2h} \leq \frac{h^2}{16\int_1^L \|\dot{C}'_\ell\|_{\eta\Gamma} d\ell} \tag{8.61}$$

which is the other hypothesis of Theorem 7.1. Then by the choice of κ_0 in Parameters part (B),

$$\|K_L^{(j+1)}\|_{\kappa_{j+1},\Gamma,h} \leq \|K_L^{(j+1)}\|_{\bar{g}_L,\eta\Gamma,h} \tag{8.62}$$

because the factor $L^{v(\kappa)|X|}$ in $\bar{g}$ is cancelled by the factor $\eta^{|X|}$ in $\eta\Gamma$. By the conclusion of Theorem 7.1,

$$\leq \|K_1|_{\kappa_{j+1},\eta\Gamma,2h} \tag{8.63}$$

which by the bound (8.51) is

$$\leq \left(\frac{C_1}{L^d}\right)^{(j+1)}. \tag{8.64}$$

Thus, we have proved the Inductive Assumption $I^{(j+1)}$ and the proof of Theorem 8.3 is complete. □

References

[1] A. Alastuey and Ph. A. Martin. Absence of exponential clustering for the static quantum correlations and the time-displaced correlations in charged fluids. *Europhys. Lett.*, 6:385–390, 1988.

[2] A. Alastuey and Ph. A. Martin. Absence of exponential clustering in quantum Coulomb fluids. *Phys. Rev. A*, 40:6485–6520, 1989.

[3] T. Balaban. Large field renormalization, II. Localization, exponentiation and bounds for the R operation. *Comm. in Math. Physics*, 122:355–392, 1989. See also previous papers.

[4] G. Battle and P. Federbush. A phase cell cluster expansion for Euclidean field theories. *Annals of Physics*, 142:95–139, 1982.

[5] G. Battle and P. Federbush. A note on cluster expansions, tree graph identities, extra $\frac{1}{N!}$ factors!!! *Letters in Mathematical Physics*, 8:55–57, 1984.

[6] G. Benfatto. An iterated Mayer expansion for the Yukawa gas. *J. Statistical Physics*, 41:671–684, 1985.

[7] G. Benfatto, N. Cassandro, G. Gallavotti, F. Nicolo, E. Olivieri, E. Presutti, and E. Scacciatelli. Some probabilistic techniques in field theory. *Comm. in Math. Physics*, 59:143–166, 1978.

[8] G. Benfatto, G. Gallavotti, and F. Nicolo. On the massive Sine-Gordon theory in the first few regions of collapse. *Comm. in Math. Physics*, 83:387–410, 1982.

[9] D. C. Brydges. A short course on cluster expansions. In K. Osterwalder and R. Stora, editors, *Critical Phenomena, Random Systems, Gauge Theories*, pages 131–183, Amsterdam, New York, 1984. Les Houches Summer School, North Holland.

[10] D. C. Brydges and Seiler E. Absence of screening in certain lattice gauge and plasma models. *J. Statistical Physics*, 42:405–424, 1986.

[11] D. C. Brydges and P. Federbush. A new form of the Mayer expansion in classical statistical mechanics. *J. Math. Phys.*, 19:2064–2067, 1978.

[12] D. C. Brydges and P. Federbush. Debye screening. *Comm. in Math. Physics*, 73:197–246, 1980.

[13] D. C. Brydges and P. Federbush. Debye screening in classical Coulomb systems. In G. Velo and A.S. Wightman, editors, *Rigorous Atomic and Molecular Physics*, pages 371–439, Oxford, 1981. Erice Summer School, Plenum Press.

[14] D. C. Brydges, J. Fröhlich, and T. Spencer. The random walk representation of classical spin systems and correlation inequalities. *Comm. in Math. Physics*, 83:123–150, 1982.

[15] D. C. Brydges and G. Keller. Debye screening in quantum Coulomb systems. Work in progress.

[16] D. C. Brydges and T. Kennedy. Convergence of multiscale Mayer series. In M. Mebkhout and R. Sénéor, editors, *VIII International Congress on Mathematical Physics*, pages 311–317, Singapore, 1986. World Scientific Publishing Co.

[17] D. C. Brydges and T. Kennedy. Mayer expansions and the Hamilton-Jacobi equation. *J. Statistical Physics*, 48:19–49, 1987.

[18] D. C. Brydges and H.T. Yau. Grad ϕ perturbations of massless Gaussian fields. *Comm. in Math. Physics*, 129:351–392, 1990.

[19] P.A.F da Veiga. *Construction de modeles non-renormalisables en théorie quantique des champs.* PhD thesis, Université de Paris-Sud Centre Orsay, 1991.

[20] P. G. De Gennes. Exponents for the excluded volume problem as derived by the Wilson method. *Physics Lett.*, 38A:339–340, 1972.

[21] E.B. Dynkin. Markov processes as a tool in field theory. *J. Funct. Analysis*, 50:167–187, 1983.

[22] E.B. Dynkin. Polynomials of the occupation field and related random fields. *J. Funct. Analysis*, 58:20–52, 1984.

[23] P. Federbush. A note on the ultimate tree graph inequality. University of Michigan preprint, 1990.

[24] P. Federbush. A phase cell approach to Yang Mills theory V. *Comm. in Math. Physics*, 127:433–457, 1990. See also previous papers.

[25] J. Feldman, V. Magnen, J.and Rivasseau, and R. Seneor. A renormalizable field theory: the massive Gross Neveu model in two dimensions. *Comm. in Math. Physics*, 103:67–103, 1986.

[26] J. Feldman, V. Magnen, J.and Rivasseau, and R. Seneor. Construction and Borel summability of infrared ϕ_4^4 by a phase space expansion. *Comm. in Math. Physics*, 109:437–480, 1987.

[27] R. Fernández, J. Fröhlich, and A. Sokal. *Random Walks, Critical Phenomena and Triviality in Quantum Field Theory.* Springer-Verlag, New York, 1992.

[28] J. Fröhlich and Y.M. Park. Correlation inequalities and the thermodynamic limit for classical and quantum continuous systems. *Comm. in Math. Physics*, 59:235–266, 1978.

[29] G. Gallavotti. Renormalization group. Technical report, Troisième Cycle de la Physique en Suisse Romande, 1990.

[30] K. Gawedzki and A. Kupiainen. Block spin renormalization group for dipole gas and $(\bigtriangledown\phi)^4$. *Ann. Phys.*, 147:198, 1983.

[31] K. Gawedzki and A. Kupiainen. Massless lattice ϕ_4^4 theory: Rigorous control of a renormalizable asymptotically free model. *Comm. in Math. Physics*, 99:197–252, 1985.

[32] K. Gawedzki and A. Kupiainen. Asymptotic freedom beyond perturbation theory. In K. Osterwalder and R. Stora, editors, *Critical Phenomena, Random Systems, Gauge Theories.* Les Houches, North Holland, 1986.

[33] I.M. Gel'fand, G.E. Shilov, N. Ya. Vilenkin, and M.I. Graev. *Generalised Functions*, volume 4. Academic Press, New York, 1964–1968.

[34] J. Ginibre. Some applications of functional integration in statistical mechanics. In C. DeWitt and R. Stora, editors, *Statistical Mechanics and Quantum Field Theory.* Les Houches, Gordon and Breach, 1971.

[35] J. Glimm, A. Jaffe, and T. Spencer. The Wightman axioms and particle structure in the $P(\phi)_2$ quantum field model. *Ann. Math.*, 100:585, 1974.

[36] M. Göpfert and G. Mack. Iterated Mayer expansions for classical gases. *Comm. in Math. Physics*, 81:97–126, 1981.

[37] C. Gruber and H. Kunz. General properties of polymer systems. *Comm. in Math. Physics*, 22:133–161, 1971.

[38] J. Imbrie. Debye screening for Jellium and other Coulomb systems. *Comm. in Math. Physics*, 87:515–565, 1983.

[39] J. Imbrie. Iterated Mayer expansions and their application to Coulomb systems. In J. Fröhlich, editor, *Scaling and Self Similarity in Physics. Renormalization in Statistical Mechanics and Dynamics*. Birkhäuser, Boston, 1983.

[40] J. Imbrie. Renormalization group methods in gauge theories. In K. Osterwalder and R. Stora, editors, *Critical Phenomena, Random Systems, Gauge Theories*, pages 297–358, Amsterdam, New York, 1984. Les Houches Summer School, North Holland.

[41] Dimock J. and T. R. Hurd. A renormalisation group analysis of correlation functions for the Dipole gas. McMaster University preprint.

[42] Dimock J. and T. R. Hurd. A renormalisation group analysis of the Kosterlitz-Thouless phase. McMaster University preprint.

[43] Dimock J. and T. R. Hurd. Construction of the two dimensional Sine-Gordon model for $\beta < 8\pi$. McMaster University Preprint, 1992.

[44] M. Kac. On the partition function of one dimensional gas. *Phys. Fluids*, 2:8–12, 1959.

[45] G. Mack and A. Pordt. Convergent perturbation expansions in Euclidean quantum field theory. *Comm. in Math. Physics*, 97:267–298, 1985.

[46] G. Mack and A. Pordt. Convergent weak coupling expansions for lattice field theories that look like perturbation series. *Rev. in Math. Phys.*, 1:47–87, 1989.

[47] A.C. Maggs and N.W. Ashcroft. Electronic fluctuation and cohesion in metals. *Phys. Rev. Lett.*, 59:113–116, 1987.

[48] V.A. Malyshev. Uniform cluster estimates for lattice models. *Comm. in Math. Physics*, 64:131–157, 1979.

[49] A.J. McKane. Reformulation of $n \to 0$ models using anticommuting scalar fields. *Physics Lett. A*, 76:22, 1980.

[50] F. Nicolo, J. Renn, and A. Steinman. On the massive Sine-Gordon equation in all regions of collapse. *Comm. in Math. Physics*, 105:291–326, 1986.

[51] G. Parisi and N. Sourlas. Self-avoiding walk and supersymmetry. *J. Phys. Lett.*, 41:1403–1406, 1980.

[52] O. Penrose. Convergence of fugacity expansions for classical systems. In T. A. Bak, editor, *Statistical Mechanics, Foundations and Applications*, pages 101–109, New York, 1967. I.U.P.A.P. Meeting (Copenhagen 1966), Benjamin Inc.

[53] J. Polchinski. Renormalization and effective Lagrangians. *Nuclear Physics B*, 231:269, 1984.

[54] A. Pordt. Convergent multigrid polymer expansions and renormalisation for Euclidean field theory. DESY preprint, 1992.

[55] G. C. Rota. On the foundations of combinatorial theory I. Theory of Mobius functions. *Wahrsch. Z. Verw. Geb.*, 2:340–368, 1964.

[56] D. Ruelle. *Statistical Mechanics.* W.A. Benjamin Inc., London, 1969.

[57] A.J.F. Siegert. Partition functions as averages of functionals of Gaussian random functions. *Physica*, 26:530–535, 1960.

[58] K. Symanzik. Euclidean quantum theory. In R. Jost, editor, *Local Quantum Theory.* Academic Press, New York, London, 1969.

[59] S.R.S. Varadhan. *Stochastic Processes.* Courant Institute Lecture Notes, New York, 1968.

[60] K. G. Wilson and J. Kogut. The renormalization group and the ϵ expansion. *Phys. Rep. (Sect C of Phys Lett.)*, 12:75–200, 1974.