REPRESENTATIONS OF NILPOTENT LIE GROUPS VIA MEASURABLE DYNAMICAL SYSTEMS

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ABSTRACT. We study unitary representations associated to cocycles of measurable dynamical systems. Our main result establishes conditions on a cocycle, ensuring that ergodicity of the dynamical system under consideration is equivalent to irreducibility of its corresponding unitary representation. This general result is applied to some representations of finite-dimensional nilpotent Lie groups and to some representations of infinite-dimensional Heisenberg groups.

1. Introduction

A measurable dynamical system is a measure space (X,μ) endowed with a group action on the right $X\times S\to X$, $(x,s)\mapsto x.s$, for which the measure μ is quasi-invariant, hence $\mathrm{d}\mu(x.s)=j(x,s)\mathrm{d}\mu(x)$ for a suitable a.e. defined positive measurable function $j(\cdot,s)$ on X. A scalar cocycle of this measurable dynamical system is a family $\{a(\cdot,s)\}_{s\in S}$ of a.e. defined measurable functions on X with values in the unit circle $\mathbb{T}$, for which the map $\pi_a\colon S\to\mathcal{B}(L^2(X,\mu))$ is a unitary representation, where

$$\pi_a(s)\colon L^2(X,\mu)\to L^2(X,\mu),\quad (\pi_a(s)\varphi)(x)=j(x,s)^{1/2}a(x,s)\varphi(x.s).$$

Our main abstract result is Theorem 2.4 which establishes conditions on the cocycle a, ensuring that ergodicity of the above dynamical system is equivalent to irreducibility of the unitary representation π_a. The unifying force of this result is then illustrated by a variety of applications, including unitary irreducible representations of finite-dimensional nilpotent Lie groups and some representations of infinite-dimensional Heisenberg groups.

Some preliminaries on measure theory.

Lemma 1.1. *Let (X,μ) be any measure space and $\mathcal{H}:=L^2(X,\mu)$. For any $\psi\in L^\infty(X,\mu)$ let $M_\psi\in\mathcal{B}(\mathcal{H})$ be the multiplication-by-ψ operator, and define $\mathcal{A}:=\{M_\psi\mid\psi\in L^\infty(X,\mu)\}$. If at least one of the following conditions is satisfied:*

1. *X is a locally compact space and μ is a Radon measure;*
2. *one has $\mu(X)<\infty$ and $\mathcal{H}$ is separable;*

then $\mathcal{A}$ is a maximal abelian self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$.

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Proof. If the first condition is satisfied then the assertion follows by [Di69, Ch. I, §7, no. 3, Th. 2]. If the second condition is satisfied, then the constant function $1 \in L^\infty(X,\mu) \subseteq L^2(X,\mu)$ is a cyclic vector for $\mathcal{A}$, hence the conclusion follows by [SS08, Th. 2.3.4]. $\square$

Definition 1.2. Let $\alpha\colon G\times X\to X$ be any group action by measurable transformations of a measure space (X,μ). The action α is called *ergodic* if for every measurable set $A\subseteq X$ with $\mu((A\triangle\alpha_g(A))=0$ for all $g\in G$, one has either $\mu(A)=0$ or $\mu(X\setminus A)=0$. (Here, for two sets X and Y, $X\triangle Y$ denotes the symmetric difference $X\triangle Y=(X\setminus Y)\cup(Y\setminus X)$.)

Remark 1.3. In the framework of Definition 1.2 it is straightforward to check that the group action α is ergodic if and only if the equivalence classes of a.e. constant functions in $L^\infty(X,\mu)$ are the only elements $\varphi\in L^\infty(X,\mu)$ with $\varphi\circ\alpha_g=\varphi$ for all $g\in G$. This implies that if α is a transitive action (or more generally, if for every $x\in X$ with its orbit $G.x:=\{\alpha_g(x)\mid g\in G\}$ one has $\mu(X\setminus G.x)=0$), then α is ergodic.

We refer to [Ta03] for the role of ergodic actions in the theory of operator algebras.

2. General results

To begin with, we recall some ideas from [Is96, Ch. I, Subsect. 1.4].

Definition 2.1. A *measurable dynamical system* consists of a measure space (X,μ) endowed with a group action on the right

$$\beta\colon X\times S\to X,\quad (x,s)\mapsto\beta_s(x)=:x.s,$$

for which the measure μ is quasi-invariant. Then for every $s\in S$ there is an a.e. defined positive function $j(\cdot,s)$ on X for which $(\beta_s)_*(\mu)=j(\cdot,s)\mu$, where $(\beta_s)_*(\mu)$ denotes the pushforward of the measure μ through the map β_s. Hence for every measurable set $E\subseteq X$ one has

$$\mu(\beta_s(E))=\int\limits_E j(x,s)\mathrm{d}\mu(x).$$

A *scalar cocycle* of this measurable dynamical system is a family $\{a(\cdot,s)\}_{s\in S}$ consisting of a.e. defined measurable functions on X with values in the unit circle $\mathbb{T}$, satisfying the conditions

$$a(x,s_1s_2)=a(x,s_1)a(x.s_1,s_2)\text{ and }a(x,\mathbf{1})=x$$

for a.e. $x\in X$ and all $s_1,s_2\in S$.

In the above setting we also define $\mathcal{H}:=L^2(X,\mu)$ and for every $s\in S$,

$$\pi_a(s)\colon\mathcal{H}\to\mathcal{H},\quad \pi_a(s)\varphi=j(\cdot,s)^{1/2}a(\cdot,s)(\varphi\circ\beta_s)(\cdot).$$

Remark 2.2. In Definition 2.1, since $(\beta_{s_1s_2})_*(\mu)=(\beta_{s_1})_*((\beta_{s_2})_*(\mu))$ for all $s_1,s_2\in S$, it is easily checked that the family $\{a(\cdot,s)\}_{s\in S}$ satisfies the conditions of a scalar cocycle, except that the functions from this family take values in the multiplicative group $(0,\infty)$ instead of the unit circle $\mathbb{T}$.

The following result is a special case of [Is96, Ch. I, Props. 1.1–1.2] whose proof was not included therein, so we give the sketch of a proof here, for the sake of completeness.

Proposition 2.3. *Assume the setting of Definition 2.1. Then the following assertions hold:*

(i) The map $\pi_a\colon S \to \mathcal{B}(\mathcal{H})$ is a unitary representation.

(ii) If the representation π_a is irreducible, then the action of S on X is ergodic.

(iii) If S is a topological group and one has

$$\lim_{s\to\mathbf{1}} \mu(E \bigtriangleup (E.s)) = \lim_{s\to\mathbf{1}} \int\limits_{E \bigtriangleup (E.s)} |j(\cdot,s)^{1/2} - 1|^2 \mathrm{d}\mu = \lim_{s\to\mathbf{1}} \int\limits_{E} (a(s,\cdot) - 1)\mathrm{d}\mu = 0$$

for every measurable set $E \subseteq X$ with $\mu(E) < \infty$, then the representation π_a is continuous.

Proof. Assertion (i) is based on a straightforward computation.

For Assertion (ii) note that for every measurable set $E \subseteq X$ which is G-invariant, the multiplication operator $M_{\chi_E} \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection whose image is invariant under $\pi_a(s)$ for all $s \in S$.

For Assertion (iii) we use that the values of π_a are unitary operators on $\mathcal{H}$, hence an $(\varepsilon/3)$-argument shows that it suffices to check that $\lim\limits_{s\to\mathbf{1}} \|\pi_a(s)\varphi - \varphi\| = 0$ for φ in some subset of $\mathcal{H}$ that spans a dense linear subspace. Using the assumptions, one can check that $\lim\limits_{s\to\mathbf{1}} \|\pi_a(s)\chi_E - \chi_E\| = 0$ for every measurable set $E \subseteq X$ with $\mu(E) < \infty$, and this completes the proof. □

For the following theorem we recall that a *multiplicity-free representation* is a unitary representation whose commutant is commutative.

Theorem 2.4. *Assume the setting of Definition 2.1, where (X,μ) satisfies either of the conditions in Lemma 1.1, and let*

$$S_0 := \{s \in S \mid x.s = x \text{ for a.e. } x \in X\}.$$

If the set $\{a(\cdot,s) \mid s \in S_0\}$ spans a w^*-dense linear subspace of $L^\infty(X,\mu)$, then $\pi_a\colon S \to \mathcal{B}(\mathcal{H})$ is a multiplicity-free representation and moreover the following assertions are equivalent:

(i) The action of S on (X,μ) is ergodic.

(ii) The representation π_a is irreducible.

Proof. Recall that $\mathcal{H} = L^2(X,\mu)$ and for any $\psi \in L^\infty(X,\mu)$ we denote by $M_\psi \in \mathcal{B}(\mathcal{H})$ be the operator of multiplication by ψ. By Lemma 1.1, the operator algebra

$$\mathcal{A} := \{M_\psi \mid \psi \in L^\infty(X,\mu)\} \subseteq \mathcal{B}(\mathcal{H})$$

is a maximal self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. To prove that π_a is a multiplicity-free representation, we will show that $\pi_a(S)' \subseteq \mathcal{A}$. Hence we must prove that if $T \in \mathcal{B}(\mathcal{H})$ and $T\pi(s) = \pi(s)T$ for all $s \in S$, then $T \in \mathcal{A}$. In fact we will prove a stronger fact, namely if $T \in \mathcal{B}(\mathcal{H})$ and $T\pi(s) = \pi(s)T$ for all $s \in S_0$, then $T \in \mathcal{A}$.

If $s \in S_0$, then it is clear that $j(x,s) = 1$ and $\varphi(x.s) = \varphi(x)$ for a.e. $x \in X$, where $\varphi \in L^2(X,\mu)$ is arbitrary, and it then follows by the definition of π_a that $\pi_a(s)$ is the operator of multiplication by $a(\cdot,s) \in L^\infty(X,\mu)$. Since the set $\{a(\cdot,s) \mid s \in S_0\}$ spans a w^*-dense linear subspace of $L^\infty(X,\mu)$ by hypothesis, it then follows that if $T \in \mathcal{B}(\mathcal{H})$ and $T\pi(s) = \pi(s)T$ for all $s \in S_0$, then $T \in \mathcal{A}'$. We have seen above that $\mathcal{A}$ is a maximal self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, hence $\mathcal{A}' = \mathcal{A}$, and then $T \in \mathcal{A}$, as claimed above. This completes the proof of the fact that π_a is a multiplicity-free representation.

Moreover, if the representation π_a is irreducible, then the action of S on (X,μ) is ergodic by Proposition 2.3(ii). Conversely, let us assume that the action of S on (X,μ) is ergodic. In order to prove that the representation π_a is irreducible, we must show that if $T\in\mathcal{B}(\mathcal{H})$ satisfies $T\pi(s)=\pi(s)T$ for all $s\in S$, then T is a scalar multiple of the identity operator on $\mathcal{H}$. In fact, using the condition $T\pi(s)=\pi(s)T$ for all $s\in S_0$, we obtain by the above reasoning that $T=M_\psi$ for some $\psi\in L^\infty(X,\mu)$. Then for all $s\in S$ and $\varphi\in L^\infty(X,\mu)$ one has

$$\begin{aligned}\psi(x)j(x,s)^{1/2}\varphi(x.s)&=(M_\psi\pi_a(s)\varphi)(x)\\&=(T\pi_a(s)\varphi)(x)\\&=(\pi_a(s)T\varphi)(x)\\&=(\pi_a(s)M_\psi\varphi)(x)\\&=j(x,s)^{1/2}\psi(x.s)\varphi(x.s)\end{aligned}$$

for a.e. $x\in X$. This implies that for all $s\in S$ one has $\psi(x)=\psi(x.s)$ for a.e. $x\in X$. Since $\psi\in L^\infty(X,\mu)$ and the action of S on (X,μ) is ergodic, it then follows that ψ is constant a.e. on X, hence the multiplication operator $T=M_\psi$ is a scalar multiplication of the identity operator on $\mathcal{H}$, and this completes the proof. □

Remark 2.5. As we will see in the examples presented in the following sections of this paper, the group S_0 from Theorem 2.4 is an abstract version of the Lie subgroup that corresponds to a polarization of a nilpotent Lie algebra. More precisely, one can interpret the representation $\pi_a\colon S\to\mathcal{B}(L^2(X,\mu))$ as the representation induced from the character $\chi_0\colon S_0\to\mathbb{T}$, $\chi_0(s):=a(x_0,s)$, for some fixed $x_0\in X$ (if any) with $x_0.s=x_0$ for all $s\in S_0$. It is worth noting that if there exists such a point $x_0\in X$, then the above χ_0 is a group homomorphism because of the cocycle properties of a.

3. Applications to group actions on locally compact spaces

The following proposition establishes irreducibility of some unitary representations that play a very significant role in [BB09] and [BB10a]. See Examples 3.4–3.5 below for more specific information in this connection.

Proposition 3.1. *Let G be a group and (X,μ) be any locally compact space endowed with a Radon measure. Assume that $\alpha\colon G\times X\to X$, $(g,x)\mapsto\alpha_g(x)$, is an action of G on X by measure-preserving transformations. Let $\mathcal{F}$ be any G-invariant vector space of real measurable functions on X, with the corresponding representation $\lambda\colon G\to\operatorname{End}(\mathcal{F})$, $\lambda_g(f):=f\circ\alpha_{g^{-1}}$. Assume in addition that the linear span of the set $\{\exp(\mathrm{i}f)\mid f\in\mathcal{F}\}$ is w^*-dense in $L^\infty(X,\mu)$. Then the following conditions are equivalent:*

1. *The action α is ergodic.*
2. *The unitary representation*

$$\pi\colon\mathcal{F}\rtimes_\lambda G\to\mathcal{B}(L^2(X,\mu)),\quad(\pi(f,g)\varphi)(x)=\mathrm{e}^{\mathrm{i}f(x)}\varphi(\alpha_{g^{-1}}(x))$$

is irreducible.

Proof. This is just a special case of Theorem 2.4, with $S=\mathcal{F}\rtimes_\lambda G$. Indeed in this case $S_0=(\mathcal{F},+)$, and it acts trivially on X. The fact that group action of $\mathcal{F}\rtimes_\lambda G$ on (X,μ) is ergodic is equivalent to ergodicity of the action of G on (X,μ), since the action of $\mathcal{F}$ on (X,μ) is trivial. □

Remark 3.2. In Proposition 3.1 the ergodicity hypothesis is necessary for the representation π to be irreducible, without imposing any condition on the linear span of the set $\{\exp(\mathrm{i}f) \mid f \in \mathcal{F}\}$. This follows by Proposition 2.3(ii).

For the transitive group action of a connected simply connected nilpotent Lie group on itself by left translations, the following corollary implies that the unitary representations constructed in [BB09, Subsect. 2.4] are irreducible. Using suitable global coordinates on coadjoint orbits of nilpotent Lie groups and the transitivity of coadjoint action on its orbits, this corollary also recovers the result of [BB10a, Prop. 5.1(2)]. See Examples 3.4–3.5 below for more details in this connection.

Corollary 3.3. *Let G be a group and X be a finite-dimensional real vector space with a Lebesgue measure μ. Assume that $\alpha\colon G \times X \to X$, $(g,x) \mapsto \alpha_g(x)$, is an action of G on X by measure-preserving transformations. Let $\mathcal{F}$ be any G-invariant vector space of real measurable functions on X, with the corresponding representation $\lambda\colon G \to \mathrm{End}\,(\mathcal{F})$, $\lambda_g(f) := f \circ \alpha_{g^{-1}}$, and define the unitary representation*

$$\pi\colon \mathcal{F} \rtimes_\lambda G \to \mathcal{B}(L^2(X,\mu)), \quad (\pi(f,g)\varphi)(x) = \mathrm{e}^{\mathrm{i}f(x)}\varphi(\alpha_{g^{-1}}(x)).$$

If the linear dual space of X satisfies $X^* \subseteq \mathcal{F}$, then the following conditions are equivalent:

(i) The action α is ergodic.
(ii) The representation π is irreducible.

Proof. If the representation π is irreducible, then α is ergodic by Remark 3.2.

Conversely, the result will follow by Proposition 3.1 as soon as we will have proved that the linear span of the set $\{\exp(\mathrm{i}\xi) \mid \xi \in X^*\}$ is w^*-dense in $L^\infty(X,\mu)$. To check this, recall that the predual of the von Neumann algebra $L^\infty(X,\mu)$ is $L^1(X,\mu)$ and the corresponding duality pairing is

$$L^\infty(X,\mu) \times L^1(X,\mu) \to \mathbb{C}, \quad (\varphi,\psi) \mapsto \langle\varphi,\psi\rangle := \int\limits_X \varphi\psi \mathrm{d}\mu.$$

On the other hand, if $\psi \in L^1(X,\mu)$ and $0 = \langle \exp(\mathrm{i}\xi), \psi\rangle = \int\limits_X \exp(\mathrm{i}\xi)\psi\mathrm{d}\mu$ for all $\xi \in X^*$, then $\psi = 0$ by the injectivity property of the Fourier transform. It then follows by the Hahn-Banach theorem that indeed the linear span of the set $\{\exp(\mathrm{i}\xi) \mid \xi \in X^*\}$ is w^*-dense in $L^\infty(X,\mu)$, and this completes the proof. $\square$

Example 3.4 ([BB09, Subsect. 2.4]). Let G be any connected, simply connected, nilpotent Lie group with some fixed left invariant Haar measure, and $\mathcal{F} \subseteq \mathcal{C}^\infty(G)$ be a linear subspace of the space of smooth functions on G. satisfying the following conditions:

(1) The linear space $\mathcal{F}$ is invariant under the representation of G by left translations, $\lambda\colon G \to \mathrm{End}\,(\mathcal{C}^\infty(G))$, $(\lambda_g\phi)(x) = \phi(g^{-1}x)$. We denote again by $\lambda\colon G \to \mathrm{End}\,(\mathcal{F})$ the restriction to $\mathcal{F}$ of the above representation λ of G.
(2) The mapping $G \times \mathcal{F} \to \mathcal{F}$, $(g,\phi) \mapsto \lambda_g\phi$ is continuous.
(3) We have $\mathfrak{g}^* \subseteq \{\phi \circ \exp_G \mid \phi \in \mathcal{F}\}$.

We define $\pi\colon \mathcal{F} \rtimes G \to \mathcal{B}(L^2(G))$ by

$$(\pi(\phi,g)f)(x) = \mathrm{e}^{\mathrm{i}\phi(x)}f(g^{-1}x)$$

for all $\phi \in \mathcal{F}$, $g \in G$, and $f \in L^2(G)$, and almost all $x \in G$.

Hence π is as in Proposition 3.1. In order to apply that proposition we must check that the linear span of the set $\{\exp(\mathrm{i}\phi) \mid \phi\in\mathcal{F}\}$ is w^*-dense in $L^\infty(G)$, hence that if $\psi\in L^1(G)$ and $\int\limits_G \psi\exp(\mathrm{i}\phi)=0$ for all $\phi\in\mathcal{F}$, then necessarily $\psi=0$. To this end, using the above condition for $\phi=\xi\circ\log_G$ with arbitrary $\xi\in\mathfrak{g}^*$ (note that $\phi\in\mathcal{F}$ by the hypothesis 3), we obtain that the Fourier transform of ψ is zero, hence $\psi=0$. Finally, the right action of $\mathcal{F}\rtimes G$ on G given by

$$G\times(\mathcal{F}\rtimes G)\to G,\quad (x,(g,\phi))\mapsto g^{-1}x$$

is transitive, hence ergodic (see Remark 1.3), and then by Proposition 3.1 the representation π is irreducible.

Let us also note that the above hypotheses on $\mathcal{F}$ ensure that $\mathcal{F}$ is a admissible function space in the sense of [BB09, Def. 2.8].

Example 3.5 ([BB10a, Prop. 5.1(2)]). Let G be any connected, simply connected, nilpotent Lie group, with its center Z and the corresponding Lie algebras $\mathfrak{z}\subseteq\mathfrak{g}$. Endow the coadjoint orbit $\mathcal{O}$ with its Liouville measure and define

$$\widetilde{\pi}\colon G\ltimes_{\mathrm{Ad}}\mathfrak{g}\to\mathcal{B}(L^2(\mathcal{O})),\quad (\widetilde{\pi}(g,Y)f)(\xi)=\mathrm{e}^{\mathrm{i}\langle\xi,Y\rangle}f(\mathrm{Ad}^*_G(g^{-1})\xi).$$

Then the following assertions hold:

(i) The group $\widetilde{G}:=G\ltimes_{\mathrm{Ad}}\mathfrak{g}$ is nilpotent and its center is $Z\times\mathfrak{z}$.
(ii) $\widetilde{\pi}$ is a unitary irreducible representation of $\widetilde{G}$.

We recall that the multiplication in the semi-direct product group $\widetilde{G}$ is given by

$$(g_1,Y_1)\cdot(g_2,Y_2)=(g_1g_2,Y_1+\mathrm{Ad}_G(g_1)Y_2) \tag{3.1}$$

and the bracket in the corresponding Lie algebra $\widetilde{\mathfrak{g}}=\mathfrak{g}\ltimes_{\mathrm{ad}}\mathfrak{g}$ is defined by

$$[(X_1,Y_1),(X_2,Y_2)]=([X_1,X_2],[X_1,Y_2]-[X_2,Y_1]).$$

An inspection of these equations shows that $\widetilde{\mathfrak{g}}$ is a nilpotent Lie algebra with its center $\mathfrak{z}\times\mathfrak{z}$.

To see that $\widetilde{\pi}$ is a representation we need to check that the function

$$a\colon\mathcal{O}\times\widetilde{G}\to\mathbb{T},\quad a(\xi,(g,Y)):=\mathrm{e}^{\mathrm{i}\langle\xi,Y\rangle}$$

is a cocycle in the sense of Definition 2.1. In fact, using the right action of $\widetilde{G}$ on $\mathcal{O}$ given by

$$\mathcal{O}\times\widetilde{G}\to\mathcal{O},\quad (\xi,(g,Y))\mapsto\xi\circ\mathrm{Ad}_G(g)=\mathrm{Ad}^*_G(g^{-1})\xi, \tag{3.2}$$

it follows by (3.1) and the above definition of a that

$$\begin{aligned}a(\xi,(g_1,Y_1)(g_2,Y_2))&=a(\xi,(g_1g_2,Y_1+\mathrm{Ad}_G(g_1)Y_2))\\&=\mathrm{e}^{\mathrm{i}\langle\xi,Y_1+\mathrm{Ad}_G(g_1)Y_2\rangle}\\&=\mathrm{e}^{\mathrm{i}\langle\xi,Y_1\rangle}\mathrm{e}^{\mathrm{i}\langle\xi\circ\mathrm{Ad}_G(g_1),Y_2\rangle}\\&=a(\xi,(g_1,Y_1))a(\xi.(g_1,Y_1),(g_2,Y_2)).\end{aligned}$$

The property $a(\xi,\mathbf{1})=\xi$ for al $\xi\in\mathcal{O}$ is clear from the definition of a. Also note that the Liouville measure on $\mathcal{O}$ is invariant under the group action (3.2). It then follows by Proposition 2.3 that $\widetilde{\pi}$ is a continuous unitary representation.

Moreover, to see that $\widetilde{\pi}$ is irreducible we will use Corollary 3.3. To this end, first note that the group action (3.2) is transitive, hence ergodic (see Remark 1.3). Furthermore, recall that the mapping

$$\mathcal{O}\to\mathfrak{g}_e^*,\quad \xi\to\xi|_{\mathfrak{g}_e}$$

is a global chart which takes the Liouville measure of $\mathcal{O}$ to a Lebesgue measure on $\mathfrak{g}_e^*$, where e is the jump index set of $\mathcal{O}$ with respect to some Jordan-Hölder basis in $\mathfrak{g}$ (see for instance [BB10a]). Then we can use the Fourier transform to see that the linear subspace generated by $\{\mathrm{e}^{\mathrm{i}\langle Y,\cdot\rangle} \mid Y \in \mathfrak{g}\}$ is weak*-dense in $L^\infty(\mathcal{O})$ $(\simeq L^1(\mathcal{O})^*)$. Therefore we can use Corollary 3.3 to obtain that $\widetilde{\pi}$ is irreducible.

In addition to the above properties of $\widetilde{\pi}$ we also recall some additional information on the irreducible representation $\widetilde{\pi}$ that was obtained in [BB10a, Prop. 5.1(2)]. Firstly, the space of smooth vectors for the representation $\widetilde{\pi}$ is $\mathcal{S}(\mathcal{O})$. Moreover, select any Jordan-Hölder basis $X_1, \dots, X_n$ in $\mathfrak{g}$ and define

$$\widetilde{X}_j = \begin{cases} (0, X_j) & \text{for } j = 1, \dots, n, \\ (X_{j-n}, 0) & \text{for } j = n+1, \dots, 2n. \end{cases}$$

Then $\widetilde{X}_1, \dots, \widetilde{X}_{2n}$ is a Jordan-Hölder basis in $\widetilde{\mathfrak{g}}$ and the corresponding predual for the coadjoint orbit $\widetilde{\mathcal{O}} \subseteq \widetilde{\mathfrak{g}}^*$ associated with the representation $\widetilde{\pi}$ is

$$\widetilde{\mathfrak{g}}_{\widetilde{e}} = \mathfrak{g}_e \times \mathfrak{g}_e \subseteq \widetilde{\mathfrak{g}},$$

where $\widetilde{e}$ is the set of jump indices for $\widetilde{\mathcal{O}}$.

4. Application to Gaussian measures on Hilbert spaces

We first recall here a few facts from [BB10b] and [BBM15].

Definition 4.1. If $\mathcal{V}$ is a real Hilbert space, $A \in \mathcal{B}(\mathcal{V})$ with $(Ax \mid y) = (x \mid Ay)$ for all $x, y \in \mathcal{V}$, and moreover $\operatorname{Ker} A = \{0\}$, then the *Heisenberg algebra* associated with the pair $(\mathcal{V}, A)$ is the real Hilbert space $\mathfrak{h}(\mathcal{V}, A) = \mathcal{V} \dotplus \mathcal{V} \dotplus \mathbb{R}$ endowed with the Lie bracket defined by $[(x_1, y_1, t_1), (x_2, y_2, t_2)] = (0, 0, (Ax_1 \mid y_2) - (Ax_2 \mid y_1))$. The corresponding *Heisenberg group* $\mathbb{H}(\mathcal{V}, A) = (\mathfrak{h}(\mathcal{V}, A), *)$ is the Lie group whose underlying manifold is $\mathfrak{h}(\mathcal{V}, A)$ and whose multiplication is defined by

$$(x_1, y_1, t_1) * (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + ((Ax_1 \mid y_2) - (Ax_2 \mid y_1))/2)$$

for $(x_1, y_1, t_1), (x_2, y_2, t_2) \in \mathbb{H}(\mathcal{V}, A)$.

Let $\mathcal{V}_-$ be a real Hilbert space and $(\cdot \mid \cdot)_-$ be its scalar product. For every $a \in \mathcal{V}_-$ and every symmetric, nonnegative, injective, *trace-class* operator K on $\mathcal{V}_-$ there is a unique probability Borel measure γ on $\mathcal{V}_-$ with

$$(\forall x \in \mathcal{V}_-) \qquad \int_{\mathcal{V}_-} \mathrm{e}^{\mathrm{i}(x|y)_-} \mathrm{d}\gamma(y) = \mathrm{e}^{\mathrm{i}(a|x)_- - \frac{1}{2}(Kx|x)_-}$$

and γ is called the *Gaussian measure with the mean a and the variance K.*

Now assume that $a = 0$ and let $\mathcal{V}_+ := \operatorname{Ran} K$ and $\mathcal{V}_0 := \operatorname{Ran} K^{1/2}$ be endowed with the scalar products given by $(Kx \mid Ky)_+ := (x \mid y)_-$ and $(K^{1/2}x \mid K^{1/2}y)_0 := (x \mid y)_-$, respectively, for all $x, y \in \mathcal{V}_-$, which turn the linear bijections $K \colon \mathcal{V}_- \to \mathcal{V}_+$ and $K^{1/2} \colon \mathcal{V}_- \to \mathcal{V}_0$ into isometries. We thus obtain the real Hilbert spaces

$$\mathcal{V}_+ \hookrightarrow \mathcal{V}_0 \hookrightarrow \mathcal{V}_-$$

where the inclusion maps are Hilbert-Schmidt operators, since $K^{1/2} \in \mathcal{B}(\mathcal{V}_-)$ is a Hilbert-Schmidt operator. Also, the scalar product of $\mathcal{V}_0$ extends to a duality pairing $(\cdot \mid \cdot)_0 \colon \mathcal{V}_- \times \mathcal{V}_+ \to \mathbb{R}$.

We also recall that for every $x \in \mathcal{V}_+$ the translated measure $\mathrm{d}\gamma(-x+\cdot)$ is absolutely continuous with respect to $\mathrm{d}\gamma(\cdot)$ and we have the Cameron-Martin formula

$$\mathrm{d}\gamma(-x+\cdot) = \rho_x(\cdot)\mathrm{d}\gamma(\cdot) \quad \text{with } \rho_x(\cdot) = \mathrm{e}^{(\cdot|x)_0 - \frac{1}{2}(x|x)_0}.$$

Definition 4.2. Let $\mathcal{V}_+$ be a real Hilbert space with the scalar product denoted by $(x,y) \mapsto (x \mid y)_+$. Also let $A\colon \mathcal{V}_+ \to \mathcal{V}_+$ be a nonnegative, symmetric, injective, trace-class operator. Let $\mathcal{V}_0$ and $\mathcal{V}_-$ be the completions of $\mathcal{V}_+$ with respect to the scalar products

$$(x,y) \mapsto (x \mid y)_0 := (A^{1/2}x \mid A^{1/2}y)_+$$

and

$$(x,y) \mapsto (x \mid y)_- := (Ax \mid Ay)_+,$$

respectively. Then the operator A uniquely extends to a nonnegative, symmetric, injective, trace-class operator $K \in \mathcal{B}(\mathcal{V}_-)$, hence by the above observations one obtains the Gaussian measure γ on $\mathcal{V}_-$ with variance K and mean 0.

One can also construct the Heisenberg group $\mathbb{H}(\mathcal{V}_+, A)$. The *Schrödinger representation* $\pi\colon \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{B}(L^2(\mathcal{V}_-, \gamma))$ is defined by

$$\pi(x,y,t)\phi = \rho_x(\cdot)^{1/2}\mathrm{e}^{\mathrm{i}(t+(\cdot|y)_0+\frac{1}{2}(x|y)_0)}\phi(-x+\cdot)$$

for $(x,y,t) \in \mathbb{H}(\mathcal{V}_+, A)$ and $\phi \in L^2(\mathcal{V}_-, \gamma)$.

Proposition 4.3. *The representation $\pi\colon \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{B}(L^2(\mathcal{V}_-, \gamma))$ from Definition 4.2 is irreducible.*

Proof. See for instance from [BB10b, Rem. 3.6] or [BBM15]. □

Corollary 4.4. *In the above setting, the action by translations of $\mathcal{V}_+$ on $(\mathcal{V}_-, \gamma)$ is ergodic.*

Proof. In the present framework, the representation π is the unitary representation associated to the measure space $(\mathcal{V}_-, \gamma)$ acted on by the additive group $(\mathcal{V}_+, +)$ by translations. The cocycle of that measurable dynamical system which gives rise to the representation π is given by

$$a(\cdot, (x,y,t)) = \mathrm{e}^{\mathrm{i}(t+(\cdot|y)_0+\frac{1}{2}(x|y)_0)}$$

for all $(x,y,t) \in \mathbb{H}(\mathcal{V}_+, A)$. The conclusion follows by Propositions 4.3 and 2.3(ii) for the right group action

$$\mathcal{V}_- \times \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{V}_-, \quad (v, (x,y,t)) \mapsto -x+v$$

and we are done. □

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