POINCARE' NORMAL FORMS AND SIMPLE COMPACT LIE GROUPS

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Summary. We classify the possible behaviour of Poincaré-Dulac normal forms for dynamical systems in R^n with nonvanishing linear part and which are equivariant under (the fundamental representation of) all the simple Lie algebras and thus the corresponding simple Lie groups. We also briefly discuss the convergence of the normalizing (and where appropriate also renormalizing) transformations.

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1. Introduction.

In dealing with Poincaré-Dulac (and Birkhoff-Gustavson) normal forms [1, 2, 5, 24], the presence of symmetries introduces some peculiar features and even transforms into generic ones situations which are highly exceptional in the non-symmetric case.

Normal form in the presence of linear symmetries were studied by Belitsky [3] and, indipendently, by Elphick *et al.* [10] who completely clarified the issue in general terms; see also the very clear exposition given by Iooss and Adelmeyer [17]. The theory was subsequently extended to consider nonlinear group actions as well [7, 8].

In this note I will apply the general theory to obtain a classification of normal forms for systems equivariant under the fundamental (defining) representations of simple Lie algebras and thus in particular under the fundamental orthogonal representation of simple, including classical, Lie groups. I will also discuss the convergence of the transformation to normal form (normalizing transformation) for these systems, solely on the basis of the symmetry properties and of the linear part of the system.

The setting to be considered is that of dynamical systems (vector fields, formal power series) in $\mathbf{R}^n$ having a fixed point (a zero) in the origin; this of course is also representative of a more general situation, where we work in a local chart in the neighbourhood of a fixed point for a dynamical system (vector field) defined on a smooth n-dimensional manifold M^n. This dynamical system (DS) is written locally in terms of a (formal) power series expansion as

$$\dot{x} \;=\; f(x) \;=\; \sum_{k=0}^{\infty} f_k(x) \ , \tag{1}$$

where $x \in \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathrm{T}\mathbf{R}^n \approx \mathbf{R}^n$, and $f_k(x)$ is homogeneous of degree $k+1$ in x, i.e. $f_k(ax) \;=\; a^{(k+1)}\, f_k(x)$. Our results will apply to the case $f_0(x)$ is not identically zero. We can equivalently consider the vector field (VF) $X \equiv X_f$ given in the x coordinates by

$$X_f \;:=\; f^i(x)\, \frac{\partial}{\partial x^i} \ . \tag{2}$$

Remark 1. Notice that f_k is homogeneous of degree $k+1$ rather than k; this notation is justified on the one hand by the action of f_k under the fundamental Lie-Poisson bracket[1] $\{.,.\}$ defined by $\{\varphi,\psi\}^i := (\varphi^k \cdot \partial_k)\psi^i - (\psi^k \cdot \partial_k)\varphi^i$, so that $\{f_k,.\} : V_m \to V_{m+k}$, where V_m is the vector space of vector polynomial homogeneous of order $m+1$ in the x; and on the other hand by the fact that if we rescale coordinates by $x = \varepsilon\xi$ (so that ξ is of order one in the neighbourhood of the origin of order ε we will have to consider), then (1) will read as $\dot{\xi} \;=\; \sum_{k=0}^{\infty} \varepsilon^k\, f_k(\xi)$. $\odot$

2. Systems equivariant under simple Lie algebras and classical Lie groups.

We will then consider a simple Lie algebra $\mathcal{G}$ and the (simply connected, compact and actually orthogonal[2]) simple Lie group G obtained from this by exponentiation; we consider for these the fundamental (defining) matrix representation in $\mathbf{R}^n$; the representation of G in $\mathbf{R}^n$ will be given by $T = \{T_g, g \in G\}$, and the Lie algebra $\mathcal{G}$ will act via matrices S; we will denote a basis of $\mathcal{G}$ (i.e. of $\{S\}$, which is also the tangent space to T in $I = T_e$) by $\{H_{(1)}, ..., H_{(r)}\}$. Equivalently, $\mathcal{G}$ is realized by vector fields $\{Y_1, ..., Y_r\}$ given in x coordinates by $Y_h = \varphi^i_h(x)(\partial/\partial x^i)$ with $\varphi^i_h(x) = (H_{(h)})^i_k x^k$.

[1] The homological operator $\mathcal{L}$ is defined in terms of this as $\mathcal{L}(.) := \{f_0, .\}$.

[2] We recall that for each simple Lie algebra $\mathcal{G}$ there exists such a group admitting $\mathcal{G}$ as the group algebra (Ado's theorem).

A (possibly formal) dynamical system of the form (1) [vector field of the form (2), power series of the form in the r.h.s. of (1)] is G-equivariant if $\forall g \in G$ and $\forall x \in \mathbf{R}^n$, $f(T_g x) = T_g f(x)$ (we use here linearity of the action and also implicitely the standard euclidean connection on $\mathbf{R}^n$ to identify tangent spaces at x and at $T_g x$); this implies in particular that it is $\mathcal{G}$-symmetric, i.e. that

$$\{f(x), \varphi_h(x)\} \;=\; 0 \quad \forall h = 1, ..., r \ . \tag{3}$$

Remark 2. Notice that for groups obtained by direct exponentiation of $\mathcal{G}$ and thus simply connected – such as e.g. $SO(n)$ or $SU(n)$ – the G-equivariance is implied by $\mathcal{G}$-symmetry, while for a multiply connected group – such as e.g. $O(n)$ or $U(n)$ – we should still check that f satisfies the equivariance relation with respect to one element of each connected component of G (i.e. for G/G^0, with G^0 the connected component of the identity in G). Thus, for all the $\mathcal{G}$-symmetric NF given below, one should still add the restriction to terms allowed by the equivariance condition under such a discrete group. $\odot$

Remark 3. A situation which is also of physical interest is the one in which the system is Hamiltonian; or also it enjoys, besides the symmetry corresponding to the connected Lie group generated by $\mathcal{G}$, a *reversing symmetry* [4, 20, 21], or even more generally a k-symmetry [19]. It is well known that in these case the NF will also be, respectively, Hamiltonian [1, 2, 17] or reversible [4, 12, 20, 21], or k-symmetric [19]. $\odot$

We will consider the classical groups corresponding to the simple Lie algebras A_n, B_n, C_n, D_n, and also the exceptional groups corresponding to the algebras E_6, E_7, E_8, F_4, G_2, acting through their fundamental *real* representation (thus for $SU(m)$ groups this will be $2m$-dimensional).

All these group representations share a common property: all polynomial dynamical systems which are $\mathcal{G}$-equivariant (equivalently, G-equivariant) are necessarily written in the form

$$\dot{x} \;=\; \sum_{p=0}^{s} \alpha_p(r^2)\, K_p x \ , \tag{4}$$

where $r^2 = \sum_{i=1}^n (x^i)^2$, the α_p are power series in r^2, and the K_p are a basis of matrices commuting with T.

This result can be obtained in several ways exploiting their properties, see [16]. One of the ways of proving it uses simply the fact that all these group representations are orthogonal (isometric) and are transitive on the unit sphere $S^{n-1} \subset \mathbf{R}^n$ of their carrier space. Equivalently, all the H_i are orthogonal matrices, and the vectors $\eta^i = H_{(i)} x$ span the tangent space to the sphere of radius $|x|$ in x; see [16] for more detail.

Coming back to (4) this means, in other words, that the K_p span the centralizer $\mathcal{C}(T)$ of T in $GL(n, \mathbf{R})$; we will take $K_0 = I$, and $s + 1 \equiv c(T)$ is the dimension of $\mathcal{C}(T)$.

It happens that $\mathcal{C}(T)$ can assume only a very limited set of forms, and correspondingly $c(T) \in \mathbf{N}$ a very limited set of values; these are completely classified by the real version of Schur Lemma, which we quote here in full generality from Kirillov [18]:

Schur Lemma. *Suppose that the dimension of the irreducible representation T in a linear space over the field $\mathbf{K}$ is no larger than countable. Then, if $\mathbf{K} = \mathbf{C}$, $\mathcal{C}(T) \simeq \mathbf{C}$; if $\mathbf{K} = \mathbf{R}$, then $\mathcal{C}(T)$ is isomorphic to either $\mathbf{R}$ or $\mathbf{C}$ or $\mathbf{H}$.*

Corollary. *The intertwining number $c(T) = \dim[\mathcal{C}(T)]$ is equal to 1 for $\mathbf{K} = \mathbf{C}$ and can take the values 1 or 2 or 4 if $\mathbf{K} = \mathbf{R}$.*

A representation over $\mathbf{R}$ is called of real, complex or quaternionic type according to the form of $\mathcal{C}(T)$. We have [18]:

Proposition. *Let $T_{\mathbf{C}}$ be the complexification of the real irreducible representation T. If T is of real or complex or quaternionic type, then $T_{\mathbf{C}}$ is, respectively, irreducible or the sum of two inequivalent irreducible representations or the sum of two equivalent irreducible representations.*

Remark 4. As well known, and as readily seen from the above formulation, when $\mathbf{K} = \mathbf{C}$ – i.e. for complex representations – the Schur lemma tells $\mathcal{C}(T) \approx \mathbf{R}$, i.e. $\mathcal{C}(T)$ is trivial; the real case is thus more interesting. It is not to us to choose what kind of representation, over the reals or over the complex numbers should be considered: this depends on the problems we are considering, and first of all on the nature of the variables involved. However, any representation over $\mathbf{C}$ can be written as a representation over $\mathbf{R}$, so that we can in full generality just deal with the latter. $\odot$

Remark 5. The (polynomial) VFs of the form (4) are sometimes called *quasilinear*, corresponding to a module over the set $\mathcal{I}_{\mathcal{G}}$ of (polynomial) invariant functions for $\mathcal{G}$ [7], which in this case (due to transitivity over the unit sphere) is simply given by functions of $r = (x_1^2 + ... + x_n^2)^{1/2}$: we denote the set of quasilinear (polynomial) equivariant VFs as $\mathcal{Q}_{\mathcal{G}} := \mathcal{I}_{\mathcal{G}} \otimes \mathcal{M}_{\mathcal{G}}$, where $\mathcal{M}_{\mathcal{G}}$ are the linear equivariant VFs, $\mathcal{M}_{\mathcal{G}} = \{Mx \,:\, [M, H_i] = 0 \;\forall i = 1, ..., r\}$. Notice that dealing with normal forms we will have to consider polynomial vector fields, hence the dependence on r^2 instead than r in (4). The above mentioned results ensuring (4) applies can then be stated by saying that in the case of the defining representation of simple Lie algebras (and groups), and more generally of any linear representation which is transitive on the unit sphere of its carrier space (or equivalently satisfies the other conditions discussed in [16]), all the equivariant vector fields are quasilinear, and all the polynomial equivariant VFs are polynomial equivariant quasilinear VFs. We will actually use only the fact that (4) represents the most general equivariant VF in these cases. $\odot$

Remark 6. It should be mentioned that (4) ensures the DS (VF) can be written in terms of generalized gradients [16] in terms of $(s+1)$ scalar functions $\{\mathcal{H}_0(r^2), ..., \mathcal{H}_s(r^2)\}$ as $\dot{x}^i = \sum_{p=0}^{s} (K_{(p)})^{ij} \, \nabla_j \, \mathcal{H}_p$. As $s+1$ is never higher than n, it can be convenient to deal with these scalar functions rather than with the corresponding DS or VF, which are given by n-dimensional vectors. $\odot$

3. The Schur classification.

We do now focus on groups rather than algebras (for ease of language) and use the classification provided by Schur Lemma, considering separately the three possible cases. For ease of notation, we will write $\beta_p := \alpha_p(0)$, so that the linear part of (4) reads $f_0(x) \ = \ Ax \ = \ \sum_{p=0}^s \beta_p \, K_p \, x$.

3.1. Case A: $\mathcal{C}(T) \simeq \mathbf{R}$.

In this case $\mathcal{C}(T)$ corresponds to multiples of the identity, i.e. $s = 0$ and (4) reduces to

$$\dot{x} \ = \ \alpha_0(r^2) \, x \ ; \tag{5}$$

if $\beta_0 = 0$ (see above), the linear part of (5) vanishes, and we cannot do anything (at least with the standard Poincaré procedure; see below) about normalizing it; for $\beta_0 \neq 0$, all the eigenvalues $\{\lambda_1, ..., \lambda_n\}$ are equal to $\beta_0 \in \mathbf{R}$, and thus no resonance is present, and moreover the spectrum of A belongs to a Poincaré domain: thus we know [1, 2, 5] that the system can be brought to its Poincaré normal form, which coincides with its linear part $\dot{x} = Ax$, by a transformation which is convergent in a ball of radius ε around the origin, with $\varepsilon > 0$.

Notice that in this case (see remark 6) we have a gradient system, and we can equivalently Birkhoff normalize the function (potential) $\mathcal{H}_0(\rho)$, where $\rho = r^2$, reducing it (in normalized coordinates $\widehat{x}$) to its linear part in $\widehat{\rho}$; in terms of potentials, this means reducing $\mathcal{H}_0$ to its quadratic part (in normalized coordinates).

The description of nonzero solutions to (5) is immediate: $x(t) = \rho(t)x(0)$, where $\rho(t)$ is a scalar function solution to $\dot{\rho} = 2\rho\alpha_0(\rho)$ with $\rho(0) = |x(0)|^2$

An example of occurrence of case A is provided by $\mathcal{G} = so(3) \equiv B_1$.

3.2. Case B: $\mathcal{C}(T) \simeq \mathbf{C}$.

Now, according to Schur Lemma, $T = T_0 \oplus \widetilde{T}_0$ as a complex representation (notice this implies $n = 2m$); using coordinates corresponding to this decomposition of T, (4) reads

$$\dot{x} \ = \ \alpha_0(r^2) \, x \ + \ \alpha_1(r^2) \, Jx \ , \tag{6}$$

where J is the standard simplectic matrix in dimension $n = 2m$, written (with I the m-dimensional identity matrix) as

$$J \ = \ \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

The linear part is given by $A \ = \ \beta_0 I \ + \ \beta_1 J$; if $\beta_0 = \beta_1 = 0$, it vanishes and we cannot do anything to normalize (6); for $A \neq 0$, we should distinguish among various subcases.

Case B1. If $\beta_0 \neq 0$ and $\beta_1 = 0$, we are in the same situation as in case A considered above: i.e. all the eigenvalues of A are real and equal to β_0, so they belong to a Poincaré domain and no resonances are present; the system can be normalized by a convergent transformation (in a ball of radius $\varepsilon > 0$), and the normal form is linear, $\dot{w} = Aw$ in normalized coordinates.

Case B2. If $\beta_0 \neq 0$ and $\beta_1 \neq 0$ as well, the $n = 2m$ eigenvalues of A split in two groups: m eigenvalues are equal to $\lambda_j \ = \ \beta_0 + i\beta_1$, and m are equal to $\lambda_j = \beta_0 - i\beta_1$. In this case the eigenvalues do still belong to a Poincaré domain, so that the normalizing transformation is guaranteed to be convergent in a ball of radius $\varepsilon > 0$, and they cannot give origin to resonances [since any combination of the λ_i with integer coefficients m_i satisfying $\sum_i m_i = m > 1$, i.e. $\sum_i m_i \lambda_i$, has real part equal to $m\beta_0 \neq \mathrm{Re}(\lambda_j)$], and thus the NF is again linear, $\dot{w} = Aw$ in normalized coordinates.

Case B3. If $\beta_0 = 0$ but $\beta_1 \neq 0$, the eigenvalues of A do again split in two groups, with m eigenvalues $\lambda_j = i\beta_1 \equiv i\omega$, and m eigenvalues $\lambda_j = -i\omega$. Now the eigenvalues do *not* belong to a Poincaré domain, and moreover they give origin to an infinite number of resonances.

It is easy to check that in this case (6) is already in NF, and conversely the most general NF with linear part $A = \beta_1 J$ ($\beta_1 \neq 0$) is given exactly by (6). Thus, no (standard) Poincaré-Dulac normalization is possible here.

Notice that here the linear part $\dot{x} = Ax$ represents a Hamiltonian system, while the full system (6) is in general – i.e. unless $\alpha_0(r^2) \equiv 0$ – non Hamiltonian; the NF is thus also in general non-Hamiltonian.

In this case we can apply to the system the "further normalization" procedure [13]; the formal computations are analogous (and therefore are not repeated here) to those for $m = 1$, which in this context can be called the $SO(2)$ case [13], and lead to the same result: if $\alpha_0(r^2)$ is not identically zero, we write $\alpha_0(r^2) = \sum_k a_k r^{2k}$, $\alpha_1(r^2) = \sum_k b_k r^{2k}$ and denote by μ, ν the smallest nonzero k for which a_k and respectively b_k are nonzero (we assume both exist). Then [13],

Lemma. *The system can be reduced by a recursive sequence of (in general, only formal) changes of coordinates to the form*

$$\dot{x} \ = \ \beta_1 J x \ + \ \left[(c_1 r^{2\mu} + c_2 r^{4\mu}) I \ + \ c_3 r^{2\nu} J \right] \, x$$

with $c_i \in \mathbf{R}$ and $c_1 = a_\mu$; if $\nu > \mu$, then $c_3 = 0$.

It should be stressed that the convergence of the renormalizing transformation has not been studied yet in general (see however [14] for partial results), but from the explicit formulae giving the transformation in this case [13] (when both μ and ν exists, i.e. in particular for non-hamiltonian systems) it appears that no small denominator can however occurr in it.

Remark 7. In the hamiltonian case all the a_k are zero, and the renormalization procedure as described in [13] is ineffective, leaving the standard normal form unchanged [14]; however, the same result is known to hold from different considerations, and is actually a classical result for $n = 2$, i.e. one degree of freedom [22], recently generalized to higher dimension [11]. $\odot$

Remark 8. Notice that in all the B cases, $\mathcal{C}(T)$ is abelian; this is connected to the fact that no Poincaré-Dulac normalization is possible in the subcase B3, nor any "further normalization" for hamiltonian systems falling in this subcase. $\odot$

Remark 9. The interpretation in terms of "potentials" (or "Hamiltonians") $\mathcal{H}_i$ is immediate: the system (6) can be written in terms of two of them, $\dot{x} = \nabla \mathcal{H}_0(r^2) + J \nabla \mathcal{H}_1(r^2)$; in case B1 we can always normalize so that $\mathcal{H}_1 \equiv 0$

in normalized coordinates, in case B2 we can always obtain $\mathcal{H}_1 = c\mathcal{H}_0$ in normalized coordinates, and finally in case B3 we get no simplification. By further normalization, in the last case B3 we can (provided both μ and ν exist) reduce the Hamiltonians to the form $\mathcal{H}_0 = (c_1/(2\mu+2))(r^2)^{\mu+1} + (c_2/(4\mu+2))(r^2)^{2\mu+1}$ and $\mathcal{H}_1 = (c_3/(2\nu+2))(r^2)^{\nu+1}$. $\odot$

An example of case B is provided by $\mathcal{G} = so(2) \equiv D_1$.

3.3. Case C: $\mathcal{C}(T) \simeq \mathbf{H}$.

This case is actually quite rare; in particular, the only simple group for which the fundamental real representation is of this type, i.e. quaternionic, is $G = SU(2)$ (with $\mathcal{G} = su(2) \equiv A_1$): that is, the quaternion group itself. This special case is not only needed to complete our classification, but also quite important mathematically and physically.

Examples of higher representations of other classical groups which are also of quaternionic type exist [9], but are of quite high dimension and seem to have no special (including physical) relevance, beside being not relevant to our present classification. I will therefore directly discuss the $SU(2)$ case.

The basis matrices $\{H_1, H_2, H_3\}$ for $\mathcal{G} = su(2)$ will be taken to be

$$H_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \ H_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} ,$$

$$H_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} .$$

We have now four basic matrices K_p, with $K_0 = I$, spanning $\mathcal{C}(T)$. The three matrices K_α (greek indices run from 1 to 3, latin ones from 0 to 3), satisfy $K_\alpha^2 = -I$ and the quaternionic relations $K_\alpha K_\beta = \epsilon_{\alpha\beta\gamma} K_\gamma$, and therefore the $su(2)$ commutation and anticommutation relations $[K_\alpha, K_\beta] = 2\epsilon_{\alpha\beta\gamma} K_\gamma$ and $\{K_i, K_j\} = -2\delta_{ij}$. Thus, they span another $su(2)$ algebra, not equivalent to the one spanned by the H_α (they correspond to different orientations of space and are dual to each other [15]).

The matrices K_α can be taken to be

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \ K_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} ,$$

$$K_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} .$$

Thus, the most general $SU(2)$-equivariant DS (VF) expressed as a power series is

$$\dot{x} \ = \ \sum_{p=0}^{3} \alpha_p(r^2)\, K_p\, x \ ; \tag{7}$$

the linear part of this is given by $A = \sum_{p=0}^{3} \beta_p K_p$, and we will write $\omega = \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}$. It is easily checked, e.g. by explicit computation, that the eigenvalues of A are equal to $\beta_0 \pm i\omega$.

Remark 10. By a rotation in $\mathbf{R}^4$ – which can actually be realized using the matrices K_α themselves, see section 4 – changing the coordinates x into coordinates $\widetilde{x}$, this can be reduced to a combination of K_0 and one of the K_α, say $A = \beta_0 K_0 + \beta_1 K_1$, where $\widetilde{\beta}_0 = \beta_0$ and $\widetilde{\beta}_1 = \omega$; the general expression of (7) in the new coordinates will be of the same kind. $\odot$

As usual, if all the β_p are equal to zero, $A = 0$ and there is no Poincaré normalization to be considered; we will thus assume $A \neq 0$ and consider again three different subcases depending on the actual form of A.

Case C1. If $\beta_0 \neq 0$ and $\omega = 0$, we are in the same situation as in cases A and B1 considered above: i.e. all the eigenvalues of A are real and equal to β_0, so they belong to a Poincaré domain and no resonances are present; the system can be normalized by a convergent transformation (in a ball of radius $\varepsilon > 0$), and the normal form is linear, $\dot{w} = Aw$ in normalized coordinates.

Case C2. If $\beta_0 \neq 0$ and $\omega \neq 0$ as well, the $n = 2m$ eigenvalues of A split in two groups: m eigenvalues are equal to $\lambda_j = \beta_0 + i\omega$, and m are equal to $\lambda_j = \beta_0 - i\omega$. In this case the eigenvalues do still belong to a Poincaré domain, so that the normalizing transformation is guaranteed to be convergent in a ball of radius $\varepsilon > 0$, and they cannot give origin to resonances, and thus the NF is again linear, $\dot{w} = Aw$ in normalized coordinates.

Case C3. If $\beta_0 = 0$ but $\omega \neq 0$, the eigenvalues of A do again split in two groups, with m eigenvalues $\lambda_j = i\omega$, and m eigenvalues $\lambda_j = -i\omega$. Now the eigenvalues do not belong to a Poincaré domain, and moreover they give origin to an infinite number of resonances.

However, now $\mathcal{C}(T)$ is *not* abelian, so the system is in general *not* already in NF. To be more specific, it is better to use the possibility to assume $A = \beta_0 I + \omega J$ with $J = K_\alpha$ (e.g. $J = K_1$), see remark 10; equivalently, consider $A = \beta_0 I + \omega J$, where $J = \sum_\alpha (\beta_\alpha/\omega) K_\alpha$. In the subcase we are considering, $\beta_0 = 0$.

Now, $A^+ = -A$, and the most general matrix in $\mathcal{C}(T)$ which commutes with this A is a linear combination of I and J. From this it follows at once that the NF for (7) – with $\beta_0 = 0$ and $\omega \neq 0$ – is of the form

$$\dot{w} \ = \ \omega\, J w \ + \ \sum_{k=1}^{\infty} r^{2k}\, [a_k I + b_k J]\, w \ . \tag{8}$$

Notice that again here the linear part $\dot{x} = Ax$ represents a Hamiltonian system, while the full system (7) is in general – i.e. unless $\alpha_0(r^2) \equiv 0$ and $\alpha_\beta(r^2) = c_\beta P(r^2)$ for a single scalar function P – non Hamiltonian; the NF is also in general non-Hamiltonian, unless all the a_k vanish.

Again if the a_k are not all zero, we can "further normalize" this NF; once again the formal computations are exactly the same as in the $SO(2)$ case [13], and we arrive at the same result, i.e. the statement of the above Lemma. Notice that in this case the initial system has first to be put in standard NF, and that the convergence of the normalizing transformation for this first step is not guaranteed a priori (see section 4 below), so that the convergence of the renormalizing transformation does not in itself guarantee that the initial system is actually conjugated to the renormalized form in any neighbourhood of the origin.

Remark 11. The interpretation in terms of "potentials" (or "Hamiltonians") $\mathcal{H}_i$ is immediate: the system (8) can be written in terms of four of them, $\dot{x} = \sum_i K_i \nabla \mathcal{H}_i(r^2)$; in case C1 we can always normalize so that $\mathcal{H}_0 = (\beta_0/2)|w|^2$ and $\mathcal{H}_\alpha \equiv 0$, in case C2 we can always obtain $\mathcal{H}_0 = (\beta_0/2)|w|^2$, $\mathcal{H}_\alpha = c_\alpha \mathcal{H}_0$, while in case C3 we are guaranteed to get this situation only formally. $\odot$

4. Conclusions.

We have considered Poincaré-Dulac normal forms for dynamical systems (vector fields, formal power series) in $\mathbf{R}^n$ in the presence of a symmetry group G with Lie algebra $\mathcal{G}$ when G is a simple Lie groups (actually any group with transitive action on $S^{n-1} \subset \mathbf{R}^n$) and thus $\mathcal{G}$ a simple Lie algebra of the $A - G$ types, acting through its fundamental (defining) real representation. We have used the fact that in this case the equivariant dynamical systems (vector fields, formal power series) are of the form (4).

We have completely described the resulting normal form, which moreover are guaranteed to be obtained by a transformation which is convergent in some neighbourhood of the origin; in some cases the symmetry guarantees the system is necessarily in normal form without the need of any transformation. In the cases B3 and C3 the normal form contains an infinite number of terms, but it is possible to "further normalize" them (pass to "Poincaré renormalized forms" [13]) obtaining a much simpler expression, except when we are in case B3 and the system is Hamiltonian; this case is covered by other approaches, and the same result applies [11, 22].

We have also argued, on the basis of explicit computations presented elsewhere [13] that the "renormalizing" transformations needed in case B3 and C3 can be chosen to be convergent in a neighbourhood of the origin.

Some limitations of our discussion are as follows:

1) We are not able to give conditions guaranteeing the convergence (or divergence) of the standard normalizing transformation in the case C3. We have also not studied in detail the subtleties of the renormalizing procedure for Hamiltonian systems of type C3.

2) We have considered only systems with nonzero linear part A, as for the fully degenerate case $A = 0$ no Poincaré-Birkhoff normalization is possible; the technique of reduction to "renormalized forms" would also apply for $A = 0$, but

the convergence issue in this case is completely open and any result would be only utterly formal; thus we have not discussed this case.

3) We have only considered the simple (connected) compact Lie groups obtained by direct exponentiation of simple Lie algebras, leaving to the reader to apply the necessary restrictions on the normal forms we obtained for groups having several connected component.

4) Also, we have not discussed reducible representations, leaving again to the reader to combine several copies of the fundamental representation and to combine such a symmetry with discrete symmetries relating different irreducible blocks where appropriate.

5) Finally, as announced in the title we only considered compact simple Lie groups, i.e. we have not discussed non-compact simple Lie groups.

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