On the Periodic Magnetic Schrödinger Operator in R^d. Eigenvalues and Model Functions.

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September 7, 2001

ABSTRACT An asymptotic formula for a rich family of eigenvalues of the periodic magnetic Schrödinger operator is constructed in the high-energy region. Model functions, which solve the equation for eigenfunctions approximately are described explicitly. It is shown that the system of model functions is almost orthogonal and form a rich subspace in $L_2(R^d)$.

1 Introduction.

We consider the operator

$$H=\sum_{j=1}^{d}\left(\frac{1}{i}\frac{\partial}{\partial x_j}+a_j(x)\right)^2+\alpha a_0(x) \tag{1}$$

in $L_2(R^d)$, where $a_0(x)$, $a_1(x), ..., a_d(x)$ are real periodic potentials, α is a real parameter. We assume that the potentials have the same periods $\vec{b}_1, ..., \vec{b}_d$. For the sake of simplicity, we consider the periods to be orthogonal, however all the results are also valid for nonorthogonal periods. Without loss of generality we assume that the periods are directed along the coordinate axes and

$$\int_Q a_s(x)dx=0, \quad s=0,1,...,d, \quad Q=[0,b_1]\times...\times[0,b_d].$$

We use the Fourier representation of the potentials:

$$a_s(x)=\sum_{m\in Z^d,\ m\neq 0} a_{sm}\exp i(\vec{p}_m(0),x), \quad s=0,1,...,d,$$

*Research partially supported by USNSF Grant DMS-9803498.

where $(\cdot,\cdot)$ is the inner product in R^d and $\vec{p}_m(0)$ is a vector of the dual lattice:

$$\vec{p}_m(0) = 2\pi(m_1 b_1^{-1}, ..., m_d b_d^{-1}). \tag{2}$$

We use the notation $\vec{A}(x) = (a_0(x), a_1(x), ..., a_d(x))$. It is known (see e.g. [ReSi]) that spectral analysis of H can be reduced to analysis of a family of operators $H(t)$, $t \in K$, where K is the elementary cell of the dual lattice,

$$K = [0, 2\pi b_1^{-1}) \times ... \times [0, 2\pi b_d^{-1}).$$

The vector t is called the *quasimomentum*. The operator $H(t)$, $t \in K$, acts in $L_2(Q)$. It is described by the formula (1) and the quasiperiodic conditions:

$$u(b_1, x_2, ..., x_d) = \exp(it_1 b_1) u(0, x_2, ... x_d),$$

$$.......$$

$$u(x_1, .., x_{d-1}, b_d) = \exp(it_d b_d) u(x_1, .., x_{d-1}, 0). \tag{3}$$

The first derivatives with respect to x_1,... x_d have to satisfy the analogous conditions. Each operator $H(t)$, $t \in K$, has a discrete semibounded below spectrum $\Lambda(t)$:

$$\Lambda(t) = \cup_{n=1}^{\infty} \lambda_n(t), \ \ \lambda_n(t) \to_{n\to\infty} \infty.$$

The spectrum Λ of the operator H is the union of the spectra of the operators $H(t)$: $\Lambda = \cup_{t\in K} \Lambda(t) = \cup_{n\in N, t\in K} \lambda_n(t)$. The functions $\lambda_n(t)$ are continuous, so Λ has a band structure:

$$\Lambda = \cup_{n=1}^{\infty} [q_n, Q_n], \ \ q_n = \min_{t\in K} \lambda_n(t), \ \ Q_n = \max_{t\in K} \lambda_n(t).$$

It is proven in [BiSu, Sob] that the spectrum of operator H is absolutely continuous, i.e. $[q_n, Q_n]$ can not degenerate to a point.

The eigenfunctions of $H(t)$ and H are simply related. Extending all the eigenfunctions of the operators $H(t)$ quasiperiodically (see (3)) to R^d, we obtain a complete system of generalized eigenfunctions of H.

In the case of $\vec{A} = 0$, the eigenvalues and eigenfunctions of the corresponding operators $H_0(t)$, $t \in K$, are naturally indexed by the points of Z^d: $\Psi_m^0(t, x) = \exp i(\vec{p}_m(t), x)$, $\lambda_m^0(t) = p_m^2(t)$, $m \in Z^d$, here and below $\vec{p}_m(t) = \vec{p}_m(0) + t$, $\vec{p}_m(0)$ being given by (2), and $p_m^2(t) = |\vec{p}_m(t)|^2$.

The goal of this paper is to construct an asymptotic formula in the high energy region for a rich set of eigenvalues of the operators $H(t)$, $t \in K$, and to construct explicitly model functions, which solve the equation for eigenfunctions approximately. We will show that the system of model functions is almost orthogonal and form a rich subspace in $L_2(R^d)$.

We consider the case of $\vec{A}=(a_0(x),a_1(x),...,a_d(x))$ being a trigonometric polynomial of the "length" R_0:

$$a_s(x)=\sum_{m\in Z^d,0<|m|\leq R_0} a_{sm}\exp i(\vec{p}_m(0),x), \quad s=0,1,...,d. \tag{4}$$

It turns out that the model functions of $H(t)$, $t\in K$, are not close to the unperturbed eigenfunctions $\exp i(\vec{p}_m(t),x)$. This makes the magnetic case different from the Schrödinger operator with only an electric potential ($a_s(x)=0, s\geq 1$), where perturbed eigenfunctions are mostly close to unperturbed ones (see [K]). The new phenomena is due to the fact that the perturbation by a magnetic field is much stronger than the perturbation by an electric field only and, therefore, leads to a more complicated diffraction picture. The construction of the model functions includes the construction of approximate solutions of the Eikonal equation by perturbative methods.

Let us introduce some notations. First,

$$\vec{a}(x)=(a_1(x),...,a_d(x)).$$

Let $C^\infty(Q)$ be the class of infinitely differentiable functions periodic on Q (all the derivatives are also periodic.) Let $\|\cdot\|_\nu$, $\nu=0,1,...$ be the family of norms in $C_\infty(Q)$ defined as follows. For any $f\in C^\infty(Q)$,

$$\|f\|_\nu=|f_0|+\sum_{r\in Z^d\setminus\{0\}}|f_r||r|^\nu,$$

f_r being the Fourier coefficients of $f(x)$. If $\vec{f}(x)$ is a vector function: $f:Q\rightarrow \mathbf{C^N}$, $\vec{f}(x)=(f_1(x),...,f_N(x))$, then

$$\|\vec{f}\|_\nu=\sum_{s=1}^N\|f_s\|_\nu.$$

Obviously, $\|D^{\vec{\nu}}\vec{f}\|_0\leq\|\vec{f}\|_\nu$, $\vec{\nu}=(\nu_1,...,\nu_d)$, $\nu=\nu_1+...+\nu_d$.

We denote by $\|L\|_*$ the norm of a matrix L:

$$2\|L\|_*=\sup_i\sum_j|L_{ij}|+\sup_j\sum_i|L_{ij}|. \tag{5}$$

Last, let $|Q|=b_1\cdot ...\cdot b_d$, $b_0=\max\{2\pi/b_1,...,2\pi/b_d,1\}$. Further, $(\cdot,\cdot)$ is the scalar product in R^d, $<\cdot,\cdot>$ is the scalar product in $L_2(Q)$.

We denote by C an absolute constant independent of the parameters of H, by C_b a constant defined only by $b_1...b_d$, by $C_{b,\vec{A}}$ a constant defined only by $b_1,..b_d,\vec{A}$, etc.

2 Construction of Model Functions and an Asymptotic Formula for Eigenvalues.

In this section we will prove that the model functions are almost orthogonal (Theorem 2) and form a rich subspace in $L_2(Q)$ (Theorem 3).

Let k be a large parameter and $Z_0(k,\beta,\varepsilon)$ be the set of $j \in Z^d$ satisfying the conditions:

1.
$$k/4 < p_j(t) < 7k/4. \tag{6}$$

2. For any $q \in Z^d : 0 < |q| \leq k^{\varepsilon}$:
$$|(\vec{p}_j(t), \vec{p}_q(0))| > k^{1/2+\beta}, \tag{7}$$

   ε, β, being parameters, $0 < \beta < 1/2$, $0 < \varepsilon d < 1/2 - \beta$.

Clearly, $Z_0(k,\beta,\varepsilon)$ is a rich set in the sense that

$$\lim \frac{\# Z_0(k,\beta,\varepsilon)}{\# S(k)}_{k\to\infty} = 1, \tag{8}$$

here $\# Z_0(k,\beta,\varepsilon)$ and $\# S(k)$ are the number of points j in $Z_0(k,\beta,\varepsilon)$ and in the set $S(k) = \{j : k/4 < p_j(t) < 7k/4\}$, respectively. The goal of this section is to construct the model functions and formula for perturbed eigenvalues when $j \in Z_0(k,\beta,\varepsilon)$. The construction of model functions follows immediately (see (9) – (17)). The formula for eigenvalues is given in Theorem 1 at the end of this section.

If $j \in Z_0(k,\beta,\varepsilon)$, we define the model function Ψ_j by the formula

$$\Psi_j(x) = \frac{1}{|Q|^{1/2}} e^{i(\vec{p}_j(t),x)+S_j(x)}. \tag{9}$$

with some periodic function $S_j(x)$. We want Ψ_j to satisfy the equation $H(t)\Psi_j = \lambda_j \Psi_j$ as precise as possible, λ_j being a constant. This consideration gives us an asymptotic equation to define the periodic function $S_j(x)$:

$$2i\,(\vec{p}_j(t), \nabla S_j(x)) + \Delta S_j(x) + (\nabla S_j(x), \nabla S_j(x)) + 2i\,(\nabla S_j(x), \vec{a}(x)) =$$

$$2\,(\vec{p}_j(t), \vec{a}(x)) + (\vec{a}(x), \vec{a}(x)) + \alpha a_0(x) - i\mathrm{div}\vec{a}(x) + C_{j,t,N} + \epsilon_N(x), \tag{10}$$

$$\|\epsilon_N(x)\|_0 < ck^{1-2(N+2)\beta}, \tag{11}$$

where $C_{j,t,N}$ is a constant, N is a fixed natural number. The choice of N is defined by the accuracy required from the approximation: $N = 0$ corresponds to a simpler formula for

$S_j(x)$ (see (13)), while a larger N requires additional terms in the formula for $S_j(x)$ (see (12)), which provide a higher accuracy of the approximation.

To satisfy this equation we define $S_j(x)$ by the formula:

$$S_j(x)=\sum_{n=0}^{N+1} S_j^{(n)}(x), \tag{12}$$

where

$$S_j^{(0)}(x)=-\sum_{r\in Z^d\setminus\{0\}}\frac{2(\vec{p}_j(t),\vec{a}_r)+(\alpha a_0-i\mathrm{div}\vec{a})_r}{p_{j+r}^2(t)-p_j^2(t)}e^{i(\vec{p}_r(0),x)}, \tag{13}$$

$\vec{a}_r$, a_{0r} being the Fourier coefficients of $\vec{a}(x)$ and $a_0(x)$, respectively. Next,

$$S_j^{(1)}(x)=\sum_{r\in Z^d\setminus\{0\}}\frac{\tilde{a}_{jr}^{(1)}}{p_{j+r}^2(t)-p_j^2(t)}e^{i(\vec{p}_r(0),x)}, \tag{14}$$

where $\tilde{a}_{jr}^{(1)}$ are the Fourier coefficients of the function

$$\tilde{a}_j^{(1)}(x)=2i\left(\nabla S_j^{(0)}(x),\vec{a}(x)\right)+\left(\nabla S_j^{(0)}(x),\nabla S_j^{(0)}(x)\right)-(\vec{a}(x),\vec{a}(x)). \tag{15}$$

The functions $S_j^{(n)}$, $n\geq 2$, are defined by the recurrent formula:

$$S_j^{(n)}(x)=\sum_{r\in Z^d\setminus\{0\}}\frac{\tilde{a}_{jr}^{(n)}}{p_{j+r}^2(t)-p_j^2(t)}, \tag{16}$$

where $a_{jr}^{(n)}$ are the Fourier coefficients of the function

$$\tilde{a}_j^{(n)}=2i\left(\nabla S_j^{(n-1)},\vec{a}\right)+2\left(\nabla S_j^{(n-1)},\sum_{s=0}^{n-2}\nabla S_j^{(s)}\right)$$
$$+(\nabla S_j^{(n-1)},\nabla S_j^{(n-1)}). \tag{17}$$

Lemma 1 *Suppose $0<\beta<1/2$, $0<\varepsilon d<1/2-\beta$, $j\in Z_0(k,\beta,\varepsilon)$, the parameter α in the definition of H satisfies the estimate $|\alpha|\leq k$, $\vec{A}$ being a trigonometric polynomial of the length R_0 (see (4)) and $N\in\mathbf{N}$. Then, the functions $S_j^{(n)}(x)$, n=0,1,...,N+1, and $\tilde{a}_j^{(n)}(x)$, $n=1,...,N+1$, have the following properties:*

$$S_{jr}^{(n)}=\tilde{a}_{jr}^{(n)}=0 \quad \text{if } |r|>2^{n+1}R_0, \tag{18}$$

$$\left\|D^{\vec{\nu}}S_j^{(n)}\right\|_0\leq c_{n,\nu}\left\|\vec{A}\right\|_{\nu+n}^{n+1}k^{1/2-(1+2n)\beta}, \tag{19}$$

$$\left\|D^{\vec{\nu}}\Re S_j^{(n)}\right\|_0\leq c_{n,\nu}\left\|\vec{A}\right\|_{\nu+n+2}^{n+1}k^{-2\beta(n+1)}, \tag{20}$$

$$\left\|D^{\vec{\nu}}\left(S_j^{(n)}-S_m^{(n)}\right)\right\|_0 \le c_{n,\nu}\left\|\vec{A}\right\|_{\nu+n}^{n+1}|m-j|k^{-2(n+1)\beta}, \tag{21}$$

if $m\in Z_0(k,\beta,\varepsilon)$,

$$\left\|D^{\vec{\nu}}\tilde{a}_j^{(n)}\right\|_0 \le c_{n,\nu}\left\|\vec{A}\right\|_{\nu+n}^{n+1}k^{1-2n\beta}, \tag{22}$$

$$\left\|D^{\vec{\nu}}\Im a_j^{(n)}\right\|_0 \le c_{n,\nu}\left\|\vec{A}\right\|_{\nu+n}^{n+1}k^{1/2-2n\beta}, \tag{23}$$

$$\left\|D^{\vec{\nu}}\left(\tilde{a}_j^{(n)}-\tilde{a}_m^{(n)}\right)\right\|_0 \le c_{n,\nu}\left\|\vec{A}\right\|_{\nu+n}^{n+1}|m-j|k^{1/2-(2n+1)\beta}, \tag{24}$$

if $m\in Z_0(k,\beta,\varepsilon)$,

where $c_{n,\nu}=c^{(n+1)(n+1+\nu)}$. *The estimates (19) – (24) hold for for any order* ν *of derivatives and* k *large enough, namely* $k: k^{\varepsilon}>2^{N+2}R_0b_0^2$, $k^{2\beta}>c_{n,\nu}\|\vec{A}\|_{\nu+N}$.

Corollary 1

$$\left\|D^{\vec{\nu}}S_j\right\|_0 \le c\left\|\vec{A}\right\|_{\nu}k^{1/2-\beta}, \tag{25}$$

$$\left\|D^{\vec{\nu}}\Re S_j\right\|_0 \le c\left\|\vec{A}\right\|_{\nu+2}k^{-2\beta}, \tag{26}$$

$$\left\|e^{S_j}\right\|_0 < 1+\left\|\vec{A}\right\|_2 k^{-2\beta}, \tag{27}$$

$$\left\|D^{\vec{\nu}}\left(S_j-S_m\right)\right\|_0 \le c\left\|\vec{A}\right\|_{\nu}|m-j|k^{-2\beta}. \tag{28}$$

Proof. By (4), $\vec{A}_r=0$ when $|r|>R_0$. From the definition of $S_j^{(0)}$ we get $S_{jr}^{(0)}=0$ when $|r|>R_0$. It follows now from (15) that $\tilde{a}_{jr}^{(1)}=0$ when $|r|>2R_0$. Using (16) we immediately obtain $S_{jr}^{(1)}=0$ when $|r|>2R_0$. Further, using (17) and (16), we obtain (18) for all n by mathematical induction.

Next we prove the pair of estimates (19), (22) by mathematical induction. Using (7) and the inequality $2^{N+2}R_0<k^{\varepsilon}$, we get

$$\left|p_{j+r}^2(t)-p_j^2(t)\right|>k^{1/2+\beta} \text{ for all } r\in Z^d, 0<|r|\le 2^{N+2}R_0. \tag{29}$$

Using the definition (13) of $S_j^{(0)}(x)$, inequality (29) and the estimates $|\alpha|\le k$, $\|\mathrm{div}\ \vec{a}\|_0\le\|\vec{A}\|_1$, we easily get (19) for $n=0$. Next, we get (22) for $\tilde{a}_j^{(1)}$. In fact, it is easy to see that for any $\vec{f},\vec{g}:Q\to R^d$ the following definition holds:

$$\left\|D^{\vec{\nu}}\left(\vec{f}(x),\vec{g}(x)\right)\right\|_0 \le 2^{\nu}\|\vec{f}\|_{\nu}\|\vec{g}\|_{\nu}. \tag{30}$$

Applying (30) in (15) we obtain:

$$\|D^{\vec{\nu}}\tilde{a}_j^1(x)\|_0 \le 2^{\nu}\left(2\|\nabla S_j^0(x)\|_{\nu}\|\vec{a}(x)\|_{\nu}+\|\nabla S_j^{(0)}(x)\|_{\nu}^2+\|\vec{a}(x)\|_{\nu}^2\right).$$

Using already proven estimate (19) for $S_j^{(0)}$, we get (22) for $\tilde a_j^1$. It follows from (16) and (29) that for all $\nu\geq 0$ and $n=1,....,N+1$,

$$\|S_j^{(n)}(x)\|_\nu \leq \|\tilde a_j^{(n)}(x)\|_\nu k^{-1/2-\beta}. \tag{31}$$

Formulae (17) and (30) yield:

$$\|D^{\vec\nu}\tilde a_j^{(n)}(x)\|_0\leq \|\tilde a_j^{(n)}(x)\|_\nu \leq$$

$$2^\nu\left(\|\nabla S_j^{(n-1)}(x)\|_\nu\|\vec a(x)\|_\nu+2\|\nabla S_j^{(n-1)}(x)\|_\nu\sum_{s=0}^{n-1}\|\nabla S_j^{(s)}(x)\|_\nu\right).$$

Using (31), we obtain

$$\|\tilde a_j^{(n)}(x)\|_\nu\leq 2^\nu k^{-1/2-\beta}\|\tilde a_j^{(n-1)}(x)\|_{\nu+1}\times$$

$$\left(\|\vec a(x)\|_\nu+2\|\nabla S_j^{(0)}(x)\|_\nu+2k^{-1/2-\beta}\sum_{s=1}^{n-1}\|\tilde a_j^{(s)}(x)\|_{\nu+1}\right).$$

Suppose (22) holds for $s=1,...,n-1$. Then, taking into account the estimate $2^{N+\nu}\|\vec A\|_{N+\nu}k^{-2\beta}<1$ and inequality (19) for $n=0$, we obtain

$$\|\vec a\|_\nu+2\|\nabla S_j^{(0)}(x)\|_\nu+2k^{-1/2-\beta}\sum_{s=1}^{n-1}\|\tilde a_j^{(s)}(x)\|_{\nu+1}\leq ck^{1/2-\beta}\|\vec A\|_{\nu+1}.$$

Thus, $\|\tilde a_j^{(n)}(x)\|_\nu<c2^\nu k^{-2\beta}\|\vec A\|_{\nu+1}\|\tilde a_j^{(n-1)}(x)\|_{\nu+1}$. Using (22) for $\tilde a_j^{(n-1)}(x)$, we get (22) for $\tilde a_j^{(n)}(x)$. Estimate (19) follows from (22) and (31).

Let us prove the pair of estimates (20), (23). By elementary calculation we obtain from (13):

$$\Re S_j^{(0)}=-\sum_{r\in Z^2\setminus\{0\}}\frac{(2(\vec p_j(t),\vec a_r)+(\alpha a_0-i\mathrm{div}\vec a)_r)\,p_r^2(0)}{(p_{j+r}^2(t)-p_j^2(t))(p_{j-r}^2(t)-p_j^2(t))}e^{i(\vec p_r(0),x)}.$$

The estimate (20) for $n=0$ easily follows from the last formula and (29). Using this estimate and formula (15) for $\tilde a_j^{(1)}$, and taking into account (20) and (19) for $n=0$, it is easy to show that $\|\Im\tilde a_j^{(1)}\|_0\leq c2^\nu k^{1/2-3\beta}$, i.e., (23) holds for $n=1$. Formula (16) for $S_j(x)$ yields

$$\|\Re S_j^{(n)}\|_\nu<c\|\tilde a_j^{(n)}\|_{\nu+2}k^{-1-2\beta}+c\|\Im\tilde a_j^{(n)}\|_\nu k^{-1/2-\beta}.$$

Using the estimate (22), we obtain:

$$\|\Re S_j^{(n)}\|_\nu<c_{n,\nu}\|A\|_{\nu+n+2}^{n+1}k^{-2\beta(n+1)}+c\|\Im\tilde a_j^{(n)}\|_\nu k^{-1/2-\beta}. \tag{32}$$

Formulae (17) and (30) yield:

$$\|D^{\vec\nu}\Im\tilde a_j^{(n)}(x)\|_0\leq\|\Im\tilde a_j^{(n)}(x)\|_\nu\leq$$

$$2^\nu \left(\|\Re\nabla S_j^{(n-1)}(x)\|_\nu \|\vec{a}\|_\nu + 2\|\Re\nabla S_j^{(n-1)}(x)\|_\nu \sum_{s=0}^{n-1} \|\nabla S_j^{(s)}(x)\|_\nu + \right. \tag{33}$$

$$\left. 2\|\nabla S_j^{(n-1)}(x)\|_\nu \sum_{s=0}^{n-2} \|\Re\nabla S_j^{(s)}(x)\|_\nu \right).$$

Suppose (20) holds for $n = s - 1$, and (23) for $n = s$, $s \geq 1$. Then, using (32), we conclude that (20) holds for $n = s$, and using (33), we get (23) for $n = s + 1$.

Let us prove (21) and (24). Using formula (13) for $S_j^{(0)}$ it is easy to check (21) when $n = 0$. Considering (15) we readily check (24) for $n = 1$. It is easy to show that

$$\|S_j^{(n)} - S_m^{(n)}\|_\nu < c|j-m|\|\tilde{a}_j^{(n)}\|_\nu k^{-1-2\beta} + c\|\tilde{a}_j^{(n)} - \tilde{a}_j^{(m)}\|_\nu k^{-1/2-\beta}.$$

Using the estimate (22), we obtain:

$$\|S_j^{(n)} - S_m^{(n)}\|_\nu < c_{n,\nu}|j-m|\|A\|_{\nu+n}^{n+1} k^{-2\beta(n+1)} + c\|\tilde{a}_j^{(n)} - \tilde{a}_j^{(m)}\|_\nu k^{-1/2-\beta}. \tag{34}$$

Formulae (17) and (30) yield:

$$\|D^{\vec{\nu}}\left(\tilde{a}_j^{(n)}(x) - \tilde{a}_m^{(n)}(x)\right)\|_0 \leq \|\tilde{a}_j^{(n)}(x) - \tilde{a}_m^{(n)}(x)\|_\nu \leq$$

$$2^\nu \left(\|\nabla S_j^{(n-1)}(x) - \nabla S_m^{(n-1)}(x)\|_\nu \|\vec{a}\|_\nu + \right.$$

$$2\|\nabla S_j^{(n-1)}(x) - \nabla S_m^{(n-1)}(x)\|_\nu \sum_{s=0}^{n-1} \|\nabla S_j^{(s)}(x)\|_\nu +$$

$$\left. 2\|\nabla S_j^{(n-1)}(x)\|_\nu \sum_{s=0}^{n-2} \|\nabla S_j^{(s)}(x) - \nabla S_m^{(n-1)}(x)\|_\nu \right). \tag{35}$$

Suppose (21) holds for $n = s - 1$, and (24) for $n = s$, $s \geq 1$. Then, using (34), we conclude that (21) holds for n=s, and using (35), we get (24) for $n = s + 1$. $\square$

Lemma 2 *Under the conditions of Lemma 1 the function $S_j(x)$ defined by (12) – (17) satisfies the relations (10), (11), where*

$$C_{j,t,N} = \sum_{n=1}^{N+1} \tilde{a}_{j0}^{(n)}. \tag{36}$$

Corollary 2

$$C_{j,t,N} < c\|\vec{A}\|_1^2 k^{1-2\beta}. \tag{37}$$

Estimate (37) follows from (36) and (22).

Corollary 3 *If $j, m \in Z_0(k, \beta, \varepsilon)$, then*

$$|C_{j,t,N} - C_{m,t,N}| < c \left\| \vec{A} \right\|_1^2 |m - j| k^{1/2 - 3\beta}. \tag{38}$$

This corollary follows from formulas (36) and (24).

<u>Proof.</u> Formula (13) and the straightforward calculation imply:

$$2i \left(\vec{p}_j(t), \nabla S_j^{(0)} \right) + \Delta S_j^{(0)} - 2(\vec{p}_j(t), \vec{a}) - \alpha a_0 + i \mathrm{div} \vec{a} = 0. \tag{39}$$

Using the definition of $S_j^{(1)}$ (see (14) and (15)) it is easy to check that

$$2i(\vec{p}_j(t), \nabla S_j^{(1)}) + \Delta S_j^{(1)} + 2i(\nabla S_j^{(0)}, \vec{a}) + \left(\nabla S_j^{(0)}, \nabla S_j^{(0)} \right) - (\vec{a}, \vec{a}) = \tilde{a}_{j0}^1,$$

we've got $\tilde{a}_{j0}^1$ because the term corresponding to $r = 0$ is omitted in formula (14). Formulae (16) and (17) easily lead to the equation for $S_j^{(n)}$:

$$2i(\vec{p}_j(t), \nabla S_j^{(n)}) + \Delta S_j^{(n)} + 2i(\nabla S_j^{(n-1)}, \vec{a})$$

$$+ \left(\sum_{s=0}^{n-1} \nabla S_j^{(s)}, \sum_{s=0}^{n-1} \nabla S_j^{(s)} \right) - \left(\sum_{s=0}^{n-2} \nabla S_j^{(s)}, \sum_{s=0}^{n-2} \nabla S_j^{(s)} \right) = \tilde{a}_{j0}^{(n)}. \tag{40}$$

Adding formulae (39) – (40) for $n = 0, 1, 2, ...N + 1$ and using (19) for $n = N, N + 1$, we get (10), (11) and (36). □

Lemma 3 *The function $\Psi_j(t, x)$ satisfies the equation:*

$$H\Psi_j = \left(p_j^2(t) - C_{j,t,N} \right) \Psi_j + \delta\Psi_j, \tag{41}$$

$\delta\Psi_j$ obeying the estimate

$$\|\delta\Psi_j\|_0 \le c(\vec{A}) k^{1 - 2(N+2)\beta}. \tag{42}$$

We prove this formula by direct substitution of (9) into equation and using (10). To get (42) from (11) we use (27). From the last lemma the next theorem follows immediately:

Theorem 1 *There is a set of eigenvalues λ_j of the operator $H(t)$, which can be parameterized by $j \in Z_0(k, \beta, \delta)$ and obey the asymptotic formula:*

$$\lambda_j(t) = p_j^2(t) - C_{j,t,N}, \tag{43}$$

$C_{j,t,N}$ being given by (36).

3 Properties of the Model Functions.

Let G be the Gram matrix of the set of functions $\Psi_j(x)$, $j \in Z_0(k,\beta,\varepsilon)$ and I_0 be the identity matrix for $j \in Z_0(k,\beta,\varepsilon)$:

$$G_{lj} =< \Psi_j, \Psi_l >, \quad (I_0)_{jm} = \delta_{jm}, \ j,m \in Z_0(k,\beta,\varepsilon).$$

We estimate $\|G - I_0\|_*$, $\|\cdot\|_*$ being defined by (5).

Theorem 2 *Let $0 < \beta < 1/2$. Then,*

$$\|G - I_0\|_* =_{k\to\infty} O(k^{-2\beta}). \tag{44}$$

First, let us prove two lemmas.

Lemma 4 *Let $0 < \gamma_1 < \gamma_2$, $M \in \mathbf{N}$, $k > 1$. Suppose functions Ψ_j, Ψ_m, $j,m \in Z^d$, can be represented in the form $\Psi_j = f_j(x)\exp i(\vec{p}_j(t), x)$, $\Psi_m = f_m(x)\exp i(\vec{p}_m(t), x)$, where $|\vec{p}_j(t) - \vec{p}_l(t)| > k^{\gamma_2}$ and*

$$\int_Q |D^{\vec{\nu}} f_j(x) dx| < (ck^{\gamma_1})^{\nu}, \int_Q |D^{\vec{\nu}} f_m(x)| dx < (ck^{\gamma_1})^{\nu}$$

for any $\nu \le M$. Then,

$$< \Psi_j, \Psi_m >= O\left(c^M k^{-M(\gamma_2-\gamma_1)}\right). \tag{45}$$

<u>Proof</u>. Let ν_{jm} be the unit vector in the direction of $\vec{p}_j(t) - \vec{p}_m(t)$. Noting that

$$\left(\nu_{jm}, \nabla e^{i(\vec{p}_j(t)-\vec{p}_m(t),x)}\right) = |\vec{p}_j(t) - \vec{p}_m(t)| e^{i(\vec{p}_j(t)-\vec{p}_m(t),x)},$$

and integrating M times by parts in the direction of ν_{jm}, we arrive at (45). □

Lemma 5 *Any model function Ψ_j, $j \in Z_0(k,\beta,\varepsilon)$, can be represented in the form:*

$$\Psi_j = f_j(x)\exp i(\vec{p}_j(t), x), \tag{46}$$

where

$$\int_Q |D^{\vec{\nu}} f_j(x)| dx < \left(C_{\|\vec{A}\|_0,\beta,\varepsilon} k^{1/2+\beta}\right)^{\nu}, \ \textit{if } \nu < k^{1/2+\beta-\varepsilon}. \tag{47}$$

<u>Proof.</u> If $j \in Z_0(k,\beta,\varepsilon)$, then the estimate (47) easily follows from formula (9) and estimates (25) and (27). □

The proof of Theorem 2. Suppose $|\vec{p}_j(t)-\vec{p}_m(t)|>k^{1/2+\beta+\delta}$, δ being a positive auxiliary parameter, $\delta\leq\varepsilon$. Then using Lemmas 4 and 5, we get

$$<\Psi_m,\Psi_j>=O\left(k^{(1/2+\beta)M}|\vec{p}_m(t)-\vec{p}_j(t)|^{-M}\right)=$$

$$O(k^{2+4\beta-\delta(M-4)}|\vec{p}_m(t)-\vec{p}_j(t)|^{-4})$$

for any $M<k^{1/2+\beta-\varepsilon}$. Choosing M large enough, say $M>10d/\delta$, we obtain:

$$\sigma_1=\sup_j\sum_{m:|\vec{p}_m(t)-\vec{p}_j(t)|>k^{1/2+\beta+\delta}}|<\Psi_m,\Psi_j>|<C_{\|\vec{A}\|_0,\beta,\varepsilon,\delta}k^{-2}.$$

Let

$$|\vec{p}_m(t)-\vec{p}_j(t)|\leq k^{1/2+\beta+\delta}.$$

Then,

$$<\Psi_m,\Psi_j>=\frac{1}{|Q|}\int_Q\exp\left(i(\vec{p}_m(t)-\vec{p}_j(t),x)+\left(S_m+\bar{S}_j\right)(x)\right)dx. \tag{48}$$

For $m=j$, obviously,

$$<\Psi_j,\Psi_j>=\frac{1}{|Q|}\int_Q e^{2\Re S_j(x)}dx=1+O(k^{-2\beta}),$$

the inequality (27) has been used to obtain the last estimate. Suppose $m\neq j$. Let us consider $S_m(x)+\bar{S}_j(x)$. Using formulae (26) and (28) we obtain:

$$\left\|D^{\vec{\nu}}\left(S_m(x)+\bar{S}_j(x)\right)\right\|_0<C_{\nu,N}\|\vec{A}\|_{\nu+2}|m-j|k^{-2\beta},\quad m\neq j.$$

Integrating (48) by parts in the direction $\vec{p}_m(t)-\vec{p}_j(t)$, we get:

$$<\Psi_m,\Psi_j>=O(k^{-2\beta}|j-m|^{-M})\ \text{ for any } M.$$

Therefore,

$$\sigma_2=\sup_j\sum_{m\neq j}|<\Psi_m,\Psi_j>|=O(k^{-2\beta}).$$

□

Let us denote by $\hat{Z}_0(k,\beta,\varepsilon)$ a smaller version of $Z_0(k,\beta,\varepsilon)$, namely, let $\hat{Z}_0(k,\beta,\varepsilon)$ be a set of $j\in Z^d$ satisfying the conditions:

1. $$k/2<p_j(t)<3k/2.$$

2. For any $q\in Z^d: 0<|q|\leq k^{\varepsilon}$:

$$|(\vec{p}_j(t),\vec{p}_q(0))|>2k^{1/2+\beta}.$$

Let L be the linear span of $\{\Psi_j\}_{j\in Z_0(k,\beta,\varepsilon)}$, and L_0 be the linear span of $\{\Psi_j^0\}_{j\in \hat{Z}_0(k,\beta,\varepsilon)}$, $\Psi_j^0(x) = |Q|^{-1/2}e^{(\vec{p}_j(t),x)}$. Obviously, both L and L_0 are finite dimensional subspaces of $L_2(Q)$. We denote by P and P_0 the projections on L and L_0 correspondingly. Let us consider the operator P_0P.

Theorem 3 *If* $\frac{3d+2}{2(3d+6)} < \beta < 1/2$, *then*

$$P_0P = P_0 + \delta P,$$

where δP *is an operator satisfying the asymptotic estimate:*

$$\|\delta P\| =_{k\to\infty} O(k^{-k^{\mu}}),\ \ \mu < 2\beta. \tag{49}$$

<u>Proof.</u> It suffice to prove that each $\Psi_j^0(x)$, $j \in \hat{Z}_0(k,\beta,\varepsilon)$, can be represented as a linear combination of Ψ_m with a high accuracy:

$$\Psi_j^0(x) = \sum_{|m-j|<k^{1/2-\beta+\mu'}} c_{jm}\Psi_m(x) + \epsilon_j(x), \quad \|\epsilon_j\|_0 = O(k^{-k^{\mu}}), \tag{50}$$

where $\mu < \mu' < 2\beta$.

Step 1. First we check that each $\Psi_j^0(x)$, $j \in \hat{Z}_0(k,\beta,\varepsilon)$, can be represented as a linear combination of Ψ_m with a good accuracy:

$$\Psi_j^0(x) = \sum_{m:|m-j|<k^{1/2-\beta+\delta}} c_{jm}\Psi_m(x) + \epsilon_j(x), \tag{51}$$

$$\|\epsilon_j\|_{L_2(Q)} = O(k^{-\gamma_0/2+\delta(d+2)/2}),$$

here $\gamma_0 = \min\{2\beta, \frac{\beta(d+6)}{2} - \frac{d+2}{4}\} > 0$ and $0 < \delta < \gamma_0/(d+2)$. To prove (51) it is enough to check that the projection of $\Psi_j^0(x)$ on L has the L_2-norm greater than $1 - k^{-\gamma_0+\delta(d+2)}$. This readily follows from the inequality

$$\sum_{m:|m-j|<k^{1/2-\beta+\delta}} |<\Psi_m, \Psi_j^0>|^2 > 1 - k^{-\gamma_0+\delta(d+2)}, \tag{52}$$

since the system of functions Ψ_m is almost orthogonal (Theorem 2). Let us prove (52). If $j \in \hat{Z}_0(k,\beta,\varepsilon)$ and $|m-j| < k^{1/2-\beta+\delta}$, then, obviously, $m \in Z_q(k,\beta,\varepsilon)$ and

$$<\Psi_m(x), \Psi_j^0(x)>= |Q|^{-1}\int_Q e^{i(\vec{p}_m(t)-\vec{p}_j(t),x)+S_m(x)}dx.$$

Using (26), we easily replace $S_m(x)$ by $-\bar{S}_j(x)$, with a small error following :

$$<\Psi_m(x), \Psi_j^0(x)>= |Q|^{-1}\int_Q e^{-\bar{S}_j(x)+i(\vec{p}_m(t)-\vec{p}_j(t),x)}dx+$$

$$O\left(|m-j|k^{-2\beta}+k^{-2\beta}\right). \tag{53}$$

Considering that $|j-m|\leq k^{1/2-\beta+\delta}$ we obtain $O\left(|m-j|k^{-2\beta}+k^{-2\beta}\right) = O(k^{1/2-3\beta+\delta})$. Note that the above integral is the $(j-m)$-th Fourier coefficient of the function $e^{-\bar{S}_j(x)}$. We will denote it by $(e^{-\bar{S}_j(x)})_{j-m}$. Thus,

$$\sum_{m:|m-j|<k^{1/2-\beta+\delta}} |<\Psi_m(x),\Psi_j^0(x)>|^2>$$

$$\sum_{m:|m-j|<k^{1/2-\beta+\delta}} \left|(e^{-\bar{S}_j(x)})_{j-m}+O(k^{1/2-3\beta+\delta})\right|^2.$$

Using the Cauchy-Schwarz inequality and considering that the sum contains about $ck^{(1/2-\beta+\delta)d}$ terms, we obtain:

$$\sum_{m:|m-j|<k^{1/2-\beta+\delta}} |<\Psi_m(x),\Psi_j^0(x)>|^2 > \sum_{m:|m-j|<k^{1/2-\beta+\delta}} |(e^{-\bar{S}_j(x)})_{j-m}|^2 +$$

$$O(\|e^{-\bar{S}_j}\|_{L_2(Q)}k^{1/2-3\beta+\delta+(1/2-\beta+\delta)d/2})+O(k^{1-6\beta+(1/2-\beta+\delta)d}). \tag{54}$$

Now we estimate an incomplete sum of the Fourier coefficients on the right-hand side of the last formula. We have the following obvious formula for the complete sum:

$$\sum_{m\in Z^d}|(e^{-\bar{S}_j(x)})_{j-m}|^2=\|e^{-\bar{S}_j}\|^2_{L_2(Q)}=1+O(k^{-2\beta}). \tag{55}$$

Thus, it remains to estimate the tail of the series. Let us consider $(e^{-\bar{S}_j(x)})_{j-m}$ for $|j-m|\geq k^{1/2-\beta+\delta}$. Integrating $\tilde{M}$ times by parts the corresponding integral (see the right-hand side of (53)) in the direction of $\vec{p}_m(t)-\vec{p}_j(t)$ and using that $\|D^lS_j\|=O(k^{1/2-\beta})$ for all l, we get

$$|(e^{-\bar{S}_j(x)})_{j-m}|<C_{\tilde{M}}k^{(1/2-\beta)\tilde{M}}|m-j|^{-\tilde{M}})<c_{\tilde{M}}k^{-\delta\tilde{M}}$$

for any positive $\tilde{M}$. Choosing $\tilde{M}>6d/\delta$, we obtain:

$$\sum_{m:|j-m|\geq k^{1/2-\beta+\delta}} |<\Psi_m(x),\Psi_j^0(x)>|^2<k^{-2}.$$

Taking into account that β satisfies the estimate $\beta>\frac{d+2}{2(d+6)}$, we replace the $O(\cdot)$ in (54), (55) by $O(k^{-\gamma_0+\delta(d+2)/2})$.

Step 2. We prove that the function ε_j in (50) satisfies the estimate

$$\|\epsilon_j\|_0=O(k^{-\gamma'}),\quad \gamma'=\gamma_0/2-\delta(d+2)/2-(1/2-\beta+\delta)d/2. \tag{56}$$

Let us consider the set W_1: $W_1:\{m:|m-j|<2k^{1/2-\beta+\delta}\}$ and let P_1^0 be the projection on the linear span of Ψ_m^0, $m\in W_1$. Estimate (56) easily follows from the two estimates:

$$\|P_1^0\epsilon_j\|_0=O(k^{-\gamma'}), \tag{57}$$

$$\|(I-P_1^0)\epsilon_j\|_0 < C_{\tilde{M}} k^{-\delta \tilde{M}}, \tilde{M} \in \mathbf{N}. \tag{58}$$

First we consider $P_1^0\epsilon_j$. The estimate for ϵ_j in (51) yields $\|P_1^0\epsilon_j\|_{L_2(Q)} < k^{-\gamma_0/2+\delta(d+2)/2}$. Considering that W_1 contains about $k^{(1/2-\beta+\delta)d}$ elements and using the Cauchy-Schwarz inequality, we get (57). To prove (58), we apply projection $I-P_1^0$ to both sides of the (51). Taking into account that $(I-P_1^0)\Psi_j^0 = 0$, $|(\Psi_m)_r| < C_M k^{1/2-\beta}|m-r|)^{-M}$ for any $r \notin W_1$ and $|c_{mj}| < 1 + o(1)$, we easily obtain (58). Thus, (56) holds.

Step 3. Let us consider the sets W_n: $W_n : \{m : |m-j| < (n+1)k^{1/2-\beta+\delta}\}$ and let P_n^0 be the corresponding projections on the linear spans of Ψ_m^0, $m \in W_n$. Clearly, $\cup_{n=1}^{N_1} W_n \in \hat{Z}_q$, $N_1 = [k^{\mu'}]$. Let us consider the reminder $\varepsilon_j(x)$ in (51). Using formula (51) for each $m \in W_1$, we get

$$P_1^0\varepsilon_j = \sum_{m\in W_1}(\epsilon_j)_m\Psi_m^0(x) = \sum_{r\in W_2} c_{jr}\Psi_r(x) + \sum_{m\in W_1}(\epsilon_j)_m\epsilon_m(x),$$

here, obviously, $c_{jr} = \sum_m(\epsilon_j)_m c_{mr}$. Clearly,

$$\left\|\sum_{m\in W_1}(\epsilon_j)_m\epsilon_m(x)\right\|_0 \le \|\epsilon_j\|_0 \sup_m \|\epsilon_m\|_0 \le k^{-2\gamma'}.$$

Using (58) and the last estimate, we obtain:

$$\Psi_j^0(x) = \sum_{m\in W_2} c_{jr}\Psi_r(x) + \epsilon_j(x), \quad \|\epsilon_j\|_0 = O(k^{-2\gamma'}).$$

Repeating this procedure N_1 times, we get (50). □

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