Results dealing with the behavior of the integrated density of states of random divergence operators

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Abstract

In this paper we generalize and improve results proven for acoustic operators in [9, 10]. It deals with the behavior of the integrated density of states of random divergence operators of the form $H_\omega = \sum_{i,j=1}^{d} \partial_{x_i} a_{i,j}(\omega, x)\partial_{x_j}$; on the internal band edges of the spectrum. We propose an application of such a result to get localization.

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1 Introduction

Let us consider the random divergence operator

$$H_\omega = -\nabla \rho_\omega \nabla = \sum_{i,j=1}^{d} \partial_{x_i} a_{i,j}(\omega, x)\partial_{x_j}; \tag{1.1}$$

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where $\rho_\omega = (a_{i,j}(\omega, x))_{1\le i,j\le d}$ is an elliptic, $d\times d$-matrix valued, $\mathbb{Z}^d$-ergodic random field. i.e there exists some constant $\rho_* > 1$, satisfying

$$\frac{1}{\rho_*}|\xi|^2 \le \langle \rho_\omega \xi, \xi\rangle \le \rho_*|\xi|^2, \quad \forall \xi \in \mathbb{C}^d. \tag{1.2}$$

This operator describes a vibrating membrane in the random medium as well as in the particular case when $\rho_\omega = \frac{1}{\varrho_\omega}\cdot I_d$, ($I_d$ is the identity matrix and ϱ_ω is a real function) we get the acoustic operator [1, 9, 10]. The great interest of this operator both from the physical and the mathematical point of view is quite obvious and known [14].
As this paper is devoted to the study of the behavior of the integrated density of states, we start by recalling that it is defined as follows: We note by $H_{\omega,\Lambda}$ the restriction of H_ω to Λ with self-adjoint boundary conditions. As H_ω is elliptic, the resolvent of $H_{\omega,\Lambda}$ is compact and, consequently, the spectrum of $H_{\omega,\Lambda}$ is discrete and is made of isolated eigenvalues of finite multiplicity [12]. We define

$$N_\Lambda(E) = \frac{1}{\mathrm{vol}(\Lambda)}\cdot \#\{\text{eigenvalues of } \mathrm{H}_{\omega,\Lambda} \le \mathrm{E}\}. \tag{1.3}$$

Here $\mathrm{vol}(\Lambda)$ is the volume of Λ in the Lebesgue sense and $\#E$ is the cardinal of E.
It is shown that the limit of $N_\Lambda(E)$ when Λ tends to $\mathbb{R}^d$ exists almost surely and is independent of the boundary conditions. It is called the **integrated density of states** of H_ω (IDS as an acronym). See [4].

1.1 The result

The essential goal of this work is to study internal Lifshitz tails for the operator defined by (1.1). We review some results proven previously and improve them. In [9, 10] we have studied the same question under a special regime of disorder, precisely under the assumption that $\lim\limits_{\varepsilon\to 0}\dfrac{\log|\log \mathbb{P}\{\omega_0 \le \varepsilon\}|}{\log \varepsilon} = 0$.
The main novelty here is that in the present work we omit this condition and we take a more general distribution of the random variables precisely we consider the case when $\log \mathbb{P}\{\omega_0 \le \varepsilon\} \sim -\varepsilon^{-\kappa}$ and we extend the result for another class of random Schrödinger operators.
As a possible application of our result we get localization near the band edges which is based on the fact that near those edges the integrated density of

states exhibits Lifshitz tails. The main interest of such a technique lies in a much weaker assumption on the probability distribution.

1.2 The model

Assume that ρ_ω is of Anderson type i.e. it has the form

$$\rho_\omega(x) = \rho^+(x) + \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma \rho^0(x-\gamma), \tag{1.4}$$

where
(**A.0**)

- $\rho^+ = (\rho^+_{i,j})_{1\le i,j\le d}$ is a, $\mathbb{Z}^d$-periodic and elliptic $d\times d$-matrix valued function.
- $\rho^0 = (\rho^0_{i,j})_{1\le i,j\le d}$ is a $d\times d$-matrix valued function such that for some elliptic matrix, ϱ^0_+, we have
$$0 \le \sum_{\gamma\in\mathbb{Z}^d} \rho^0(x-\gamma) \le \varrho^0_+.$$
- $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a family of non constant and positive, independent identically distributed random variables taking values in $[0,1]$.We note by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space and we suppose that
$$\lim_{\varepsilon\to 0^+} \frac{\log|\log \mathbb{P}(\{\omega_0 \le \varepsilon\})|}{\log \varepsilon} = -\kappa, \quad \kappa \in [0,+\infty[. \tag{1.5}$$

Let $h(\rho_\omega)$ be the quadratic form defined as follows: For $u \in H^1(\mathbb{R}^d) = \mathcal{D}(h(\rho_\omega))$

$$\begin{aligned} h(\rho_\omega)[u,u] &= \int_{\mathbb{R}^d} \rho_\omega(x)\nabla u(x)\cdot\overline{\nabla u(x)}dx \\ &= \sum_{1\le i,j\le d}\int_{\mathbb{R}^d} a_{i,j}(\omega,x)\partial_{x_i}u(x)\overline{\partial_{x_j}u(x)}dx. \end{aligned}$$

$h(\rho_\omega)$ is a positive and closed quadratic form. H_ω given by (1.1) is defined as the self adjoint operator associated to $h(\rho_\omega)$ [12]. By this, H_ω is a measurable family of self adjoint and ergodic operators[1, 14].

1.3 Reference operator

It is convenient to write H_ω as a perturbation of some background periodic operator H_0. More precisely we write:

$$H_\omega = H_0 + V_\omega,$$

with

$$H_0 = -\nabla \rho^+ \nabla$$

and

$$V_\omega(\cdot) = -\nabla(\sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma \rho^0(\cdot - \gamma))\nabla \geq 0.$$

1.3.1 Some facts from Floquet theory

Now we review some standard facts from the Floquet theory for periodic operators. Basic references of this material can be found in [12].
As ρ^+ is a $\mathbb{Z}^d$-periodic matrix, for any $\gamma \in \mathbb{Z}^d$, we have

$$\tau_\gamma H_0 \tau_\gamma^* = \tau_\gamma H_0 \tau_{-\gamma} = H_0.$$

Let $\mathbb{T}^* = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$. We define $\mathcal{H}$ by

$$\mathcal{H} = \{u(x,\theta) \in L^2_{loc}(\mathbb{R}^d) \otimes L^2(\mathbb{T}^*); \forall (x,\theta,\gamma) \in \mathbb{R}^d \times \mathbb{T}^* \times \mathbb{Z}^d;\ u(x+\gamma,\theta) = e^{i\gamma\theta}u(x,\theta)\}.$$

There exists U a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathcal{H}$ such that H_0 admits the Floquet decomposition [12]

$$UH_0U^* = \int_{\mathbb{T}^*}^{\oplus} H_0(\theta)d\theta.$$

Here $H_0(\theta)$ is the operator H_0 acting on $\mathcal{H}_\theta$, defined by

$$\mathcal{H}_\theta = \{u \in L^2_{loc}(\mathbb{R}^d); \forall \gamma \in \mathbb{Z}^d, u(x+\gamma) = e^{i\gamma\theta}u(x)\}.$$

As H_0 is elliptic, we know that, $H_0(\theta)$ has a compact resolvent; hence its spectrum is discrete [12]. We denote its eigenvalues, called Floquet eigenvalues of H_0, by

$$E_0(\theta) \leq E_1(\theta) \leq \cdots \leq E_n(\theta) \leq \cdots.$$

The functions $(\theta \to E_n(\theta))_{n\in\mathbb{N}}$ are Lipshitz-continuous, and we have

$$E_n(\theta) \to +\infty \text{ as } n \to +\infty \text{ uniformly in } \theta.$$

The spectrum $\sigma(H_0)$ of H_0 has a band structure. (i.e $\sigma(H_0) = \cup_{n\in\mathbb{N}} E_n(\mathbb{T}^*)$.) The periodic operator H_0 has an IDS which will be denoted by n. The behavior of n at a band edge E_+, is said to be **non-degenerate** if,

$$\lim_{\varepsilon\to 0^+} \frac{\log |n(E_+ + \varepsilon) - n(E_+)|}{\log \varepsilon} = \frac{d}{2}. \tag{1.6}$$

1.3.2 The main assumptions

As we study internal Lifshitz tails it is naturel to assume that H_0 has a spectral gap below E_+. More precisely we assume that:
(**A.1**)
There exists E_+ and $\delta > 0$ such that $\sigma(H_0) \cap [E_+, E_+ + \delta) = [E_+, E_+ + \delta)$ and $\sigma(H_0) \cap (E_+ - \delta, E_+] = \emptyset$.
As, $V_\omega \geq 0$, the spectrum Σ of H_ω contains an interval of the form $[E_+, E_+ + a]$ $(a > 0)$ [3]. As we are interested in the behavior of the IDS in the neighborhood of E_+, we require that E_+ remains always the edge of a gap for Σ, when the perturbation is turned on. More precisely, if for all $t \in [0, 1]$, we define $H_{\omega,t} = H_0 + tV_\omega$ and Σ_t is the almost sure spectrum of $H_{\omega,t}$, then one requires that the following assumption holds.
(**A.2**)
There exists $\delta' > 0$ such that for all $t \in [0, 1]$, $\Sigma_t \cap [E_+ - \delta', E_+) = \emptyset$.
We assume also the following:
(**A.3**)
We will state that the behavior of the IDS depends on the form of the perturbation. One distinguishes between two behaviors of ρ^0.
Let $C_0 = \{x \in \mathbb{R}^d; \forall 1 \leq j \leq d; -\frac{1}{2} < x_j \leq \frac{1}{2}\}$ and let $0 < g_- < g_+$ be two positives constants.
1. ρ^0 is of long range type.
There exists $\nu \in (d, d+2]$ such that for any $\gamma \in \mathbb{Z}^d$, $1 \leq i, j \leq d$ and almost every x in C_0 one has

$$g_- \leq \rho^0_{i,j}(x - \gamma) \cdot (1 + |\gamma|)^\nu \leq g_+; \tag{1.7}$$

and

$$g_- \leq |\partial x_i(\rho^0_{i,j})(x - \gamma)| \cdot (1 + |\gamma|)^\nu \leq g_+. \tag{1.8}$$

2. ρ^0 is of short range type.
There exists $\nu > d + 2$ such that for any $\gamma \in \mathbb{Z}^d$, $1 \leq i, j \leq d$ and almost

every x in C_0 one has

$$0 \leq \rho^0_{i,j}(x-\gamma) \cdot (1+|\gamma|)^\nu \leq g_+; \tag{1.9}$$

and

$$0 \leq |\partial x_i(\rho^0_{i,j})(x-\gamma)| \cdot (1+|\gamma|)^\nu \leq g_+. \tag{1.10}$$

2 Results and discussions

The main result of this paper is the following:

Theorem 2.1 *Let H_ω be the operator defined by (1.1). We assume that* **(A.1)**, **(A.2)** *hold. Then if*
1. ρ^0 is of long range type then,

$$\lim_{\varepsilon\to 0^+} \frac{\log\big(n(E_+ + \varepsilon) - n(E_+)\big)}{\log \varepsilon} = \frac{d}{2} \Rightarrow$$
$$\lim_{\varepsilon\to 0^+} \frac{\log|\log\big(N(E_+ + \varepsilon) - N(E_+)\big)|}{\log \varepsilon} = -\sup(\frac{d}{2}+\kappa, \frac{d}{\nu-d}),$$

if $\kappa + \frac{d}{2} < \frac{d}{\nu-d}$,

$$\lim_{\varepsilon\to 0^+} \frac{\log|\log\big(N(E_+ + \varepsilon) - N(E_+)\big)|}{\log \varepsilon} = -\frac{d}{\nu-d}. \tag{2.11}$$

2. ρ^0 is of short range type then,

$$\lim_{\varepsilon\to 0^+} \frac{\log|\log\big(N(E_+ + \varepsilon) - N(E_+)\big)|}{\log \varepsilon} = -(\frac{d}{2}+\kappa) \Leftrightarrow$$
$$\lim_{\varepsilon\to 0^+} \frac{\log\big(n(E_+ + \varepsilon) - n(E_+)\big)}{\log \varepsilon} = \frac{d}{2}.$$

Remark 2.2 *The result of Theorem 2.1 is stated for lower band edges. Under adequate assumptions the corresponding result is true for upper band edges.*

Now, let us comment the result. According to the Theorem 2.1 one notices that the behavior of the random variables is linked up to the lifshitz exponent, and determines if one is located in a classical regime or in a quantum one;

i.e if the kinetic energy intervenes or if it does not in the Lifshitz exponent. In the long range case one sees that it depends on the value of κ, the Lifshitz asymptotics are classical (if $\kappa < \frac{d}{\nu-d} - \frac{d}{2}$) or quantum (if $\kappa > \frac{d}{\nu-d} - \frac{d}{2}$). In other terms in the case of the long range potential, Lifshitz exponent depends on the uncertainty principle, i.e on the kinetic energy only in the case when $(\frac{d}{\nu-d} < \kappa + \frac{d}{2})$. In contrast, when $(\frac{d}{\nu-d} > \kappa + \frac{d}{2})$ then the Lifshitz asymptotics are not governed by the same considerations. This is due to the fact that in the long range case as the potential decreases slowly, locally the potential is an empirical average of random variables. This leads to the fact that its effect is more important and more influencing than the spatial extension of the considered state.
From what it has been said previously, one concludes that the value of κ is responsible for the transition between those two regimes.
The proof of the main result is now classic and based on the technic of periodic approximations which where originally stated by Klopp in [7]. It is quite close and follows the same steps used in [9, 10]. We omit details and we refer the reader to the above references.

2.1 Application

Now we state a useful result which can be related to the Theorem 2.1. Let

Theorem 2.3 *Let $\theta \in \mathbb{R}^d$ and $E_+ > 0$ a band edge of the spectrum of H_ω. Then for any $\alpha > 1$, integer $p > 0$, for $k \in \mathbb{N}$ sufficiently large, one has*

$$(\mathbf{P1})\ \mathbb{P}\Big(\big\{dist\big(\sigma(H^\theta_{\omega,\Lambda_{k^\alpha}}), E_+\big) \le \frac{1}{k}\big\}\Big) \le \frac{1}{k^p}.$$

Where Λ_k is the box centered in 0 of side length $2k+1$ and $A^\theta_{\omega,\Lambda_k}$ is the operator H_ω restricted to this box with θ-quasiperiodic boundary condition i.e with boundary condition $\varphi(x+\gamma) = e^{i\gamma\cdot\theta}\varphi(x)$ for any $\gamma \in (2k+1)\mathbb{Z}^d$.

To be able to apply the multiscale analysis [1, 14], we assume that ρ^0 is compactly supported. Indeed, when the single site is compactly supported H_ω satisfies a Wegner estimate [1] i.e for some $\alpha > 0$ and $n > 0$ for $E \in \mathbb{R}$ for $k \ge 1$ and $0 < \varepsilon < 1$, there exists $C(E) > 0$ such that one has

$$(\mathbf{P2})\ \mathbb{P}\Big(\big\{dist(\sigma(H^\theta_{\omega,\Lambda_k}), E) \le \varepsilon\big\}\Big) \le C(E) \cdot |\Lambda_k|^\alpha \cdot \varepsilon^n. \tag{2.12}$$

So, for a band edge energy E_+ using the Theorem 2.3 for $\theta = 0$, we obtain the initial estimate to start a multi-scale analysis. This proves that the spectrum of H_ω is exponentially localized in some interval around the energy E_+ i.e that in some neighborhood of E_+ eigenfunctions associated to energies in that interval are exponentially localized. More precisely we have

Theorem 2.4 *Let H_ω defined by (1.1). We assume that* **(A.1)** *and* **(A.2)** *hold and the single site is compactly supported. There exists $\varepsilon_0 > 0$ such that*
(i) $\Sigma \cap [E_+, E_+ + \varepsilon_0] = \Sigma_{pp} \cap [E_+, E_+ + \varepsilon_0]$.
(ii) an eigenfunction corresponding to an eigenvalue in $[E_+, E_+ + \varepsilon_0]$ decays exponentially.
(iii) for all $p > 0$,

$$\mathbb{E}\Big\{ \sup_{t>0} \Big|\Big| \Big|X\Big|^p e^{itH_\omega} P_{[E_+, E_+ + \varepsilon_0]}(H_\omega)\chi_K \Big|\Big| \Big\} < +\infty.$$

Here $P_I(H_\omega)$ is the spectral projection on the interval I, χ_K is the characteristic function of K, K is a compact of $\mathbb{R}^d$ and X is the position operator.

To comment upon Theorem 2.4, let us consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = H_\omega u. \tag{2.13}$$

The solution of (2.13) is given [14] by

$$u(t, \cdot) = \cos(t\sqrt{H_\omega})u_0 + \sin(t\sqrt{H_\omega})u_1,$$

where $u_0 = u(0, \cdot)$ and $\sqrt{H_\omega}u_1 = (\partial_t u)(0, \cdot)$ denote the initial data.
The result of Theorem 2.3 and the one of Theorem 2.4 can be related to the behavior of the integrated density of states in the neighborhood of the so-called fluctuation boundary E_+[6, 11]. This is done in the Schrödinger case in [15].

2.2 The periodic approximations

Let us consider the following periodic operator

$$H_{\omega,k} = -\nabla \cdot \rho_{\omega,k} \cdot \nabla,$$

where $\rho_{\omega,k}$ is the following matrix

$$\rho_{\omega,k} = \rho^+ + \sum_{\gamma \in C_k \cap \mathbb{Z}^d} \omega_\gamma \sum_{\beta \in (2k+1)\mathbb{Z}^d} \rho^0(\cdot - \gamma - \beta).$$

C_k is the cube

$$C_k = \{x \in \mathbb{R}^d; \forall 1 \le j \le d,\ -\frac{2k+1}{2} < x_j \le \frac{2k+1}{2}\}.$$

We set

$$V_{\omega,k} = -\nabla \cdot \Big(\sum_{\gamma \in C_k \cap \mathbb{Z}^d} \omega_\gamma \sum_{\beta \in (2k+1)\mathbb{Z}^d} \rho^0(\cdot - \gamma - \beta) \Big) \cdot \nabla$$

$H_{\omega,k}$ is $(2k+1)\mathbb{Z}^d$-periodic and essentially self adjoint operator. Let $\mathbb{T}_k^* = (\mathbb{R}^d)/2(2k+1)\pi\mathbb{Z}^d$. We define $N_{\omega,k}$ the IDS of $H_{\omega,k}$ by

$$N_{\omega,k}(E) = \frac{1}{(2\pi)^d} \sum_{n \in \mathbb{N}} \int_{\{\theta \in \mathbb{T}_k^*,\ E_{\omega,k,n}(\theta) \le E\}} d\theta. \tag{2.14}$$

Let $dN_{\omega,k}$ be the derivative of $N_{\omega,k}$ in the distribution sense. As $N_{\omega,k}$ is increasing, $dN_{\omega,k}$ is a positive measure; it is the density of states of $H_{\omega,k}$. We denote by dN the density of states of H_ω.

Theorem 2.5 *[5, 9] For any $\varphi \in \Lambda_0^\infty(\mathbb{R})$ and for almost all $\omega \in \Omega$ we have*

$$\lim_{k \to \infty} \langle \varphi, dN_{\omega,k} \rangle = \langle \varphi, dN \rangle.$$

In what follows we give a well-known [7, 9, 10] result stating that the IDS of H_ω is exponentially well approximated by the expectation of the IDS of the periodic operators $H_{\omega,k}$ when k is polynomial in ε^{-1}. More precisely

Lemma 2.6 *For any $\eta_0 > 1$ and $I \subset \mathbb{R}$ a compact interval, there exists $\nu_0 > 0$ and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0, E \in I$ and $k \ge k_1 = \varepsilon^{-\nu_0}$, we have*

$$\begin{aligned} &\mathbb{E}[N_{\omega,k}(E+\varepsilon/2) - N_{\omega,k}(E-\varepsilon/2)] - e^{\varepsilon^{-\eta_0}} \\ &\qquad \le N(E+\varepsilon) - N(E) \\ &\qquad \le \mathbb{E}[N_{\omega,k}(E+2\varepsilon) - N_{\omega,k}(E-2\varepsilon)] + e^{-\varepsilon^{-\eta_0}}. \end{aligned} \tag{2.15}$$

3 The proof of Theorem 2.1

To prove Theorem 2.1, we use periodic approximations. We prove a lower and an upper bounds on $N(E_+ + \varepsilon) - N(E_+)$. The upper and lower bounds are proven separately.

3.1 The lower bound

We postpone the proof of the lower bound. More details can be found in [9, 10]. It consists in proving the following theorem.

Theorem 3.1 *Let H_ω be the operator defined by (1.1). We assume that* **(A.1)**, **(A.2)** *hold. Then,*

- *if ρ^0 is of long range type, we have*

$$\liminf_{\varepsilon\to 0^+}\frac{\log\left|\log\Big(N(E_+ +\varepsilon)-N(E_+)\Big)\right|}{\log\varepsilon}\geq -\sup(\frac{d}{2}+\kappa,\frac{d}{\nu-2}). \tag{3.16}$$

- *if ρ^0 is of short range type, we have*

$$\liminf_{\varepsilon\to 0^+}\frac{\log\left|\log\Big(N(E_+ +\varepsilon)-N(E_+)\Big)\right|}{\log\varepsilon}\geq -(s\frac{d}{2}+\kappa). \tag{3.17}$$

Here $s<1$ if n is degenerate and $s=1$ if not.

Proof. By assumption, there is a spectral gap below E_+ of length at least $\delta'>0$. Thus, for $\varepsilon<\delta'$ we have

$$N(E_+ +\varepsilon)-N(E_+)=N(E_+ +\varepsilon)-N(E_+ -\varepsilon).$$

To prove Theorem 3.1, it suffice to lower bound $N(E_+ +\varepsilon)-N(E_+ -\varepsilon)$. Then, for N large, we will show that H_{ω,Λ_N} (H_{ω,Λ_N} is H_ω restricted to Λ_N with Dirichlet boundary conditions) has a large number of eigenvalues in $[E_+ -\varepsilon,E_+ +\varepsilon]$ with a large probability. For this we will construct a family of approximate eigenvectors associated to approximate eigenvalues of H_{ω,Λ_N} in $[E_+ -\varepsilon,E_+ +\varepsilon]$. These functions can be constructed from an eigenvector of H_0 associated with E_+. Locating this eigenvector in θ and imposing to ω_γ to be small for γ in some well chosen cube, one obtains an approximate eigenfunction of H_{ω,Λ_N}. Locating the eigenfunction in x in several disjointed places, we get several eigenfunctions two by two orthogonal.The subtlety is in the good choice of the size of the cube.
Using the same computation done in [9, 10] we get that we have to estimate the following two probabilities: For $1>\alpha>0$,

$$\mathbb{P}_{\varepsilon,\alpha,1}=\mathbb{P}\Big(\Big\{\omega;\ |\beta|\leq\varepsilon^{-(1+\alpha)/2};\ \sum_{\gamma\in\mathbb{Z}^d}\omega_\gamma(1+|\beta-\gamma|)^{-\nu}\leq\varepsilon^{1+\alpha}\Big\}\Big),$$

$$\mathbb{P}_{\varepsilon,\alpha,2} = \mathbb{P}\Big(\Big\{\omega;\ \sum_{\gamma\in\Lambda_\alpha(\varepsilon^s)} \omega_\gamma(1+|\gamma|)^{-\nu} \le \frac{\varepsilon^{1+\alpha}}{2}\Big\}\Big);$$

here $\Lambda_\alpha(\zeta) = \{\gamma \in \mathbb{Z}^d;\ \forall 1 \le j \le d;\ |\gamma_j| \le \zeta^{-(\frac{1}{2}+\alpha)}\}$.
Indeed we have the following relations[9, 10]:
• if ρ^0 is of long range type, we have

$$\liminf_{\varepsilon\to 0^+} \frac{\log\Big|\log\Big(N(E_+ + \varepsilon) - N(E_+)\Big)\Big|}{\log\varepsilon} \ge \liminf_{\varepsilon\to 0^+} \frac{\log\Big|\log(\mathbb{P}_{\varepsilon,\alpha,1})\Big|}{\log\varepsilon}. \quad (3.18)$$

• if ρ^0 is of short range type, we have

$$\liminf_{\varepsilon\to 0^+} \frac{\log\Big|\log\Big(N(E_+ + \varepsilon) - N(E_+)\Big)\Big|}{\log\varepsilon} \ge \liminf_{\varepsilon\to 0^+} \frac{\log\Big|\log(\mathbb{P}_{\varepsilon,\alpha,2})\Big|}{\log\varepsilon}. \quad (3.19)$$

Now one deals with the estimation of $\mathbb{P}_{\varepsilon,\alpha,1}$ and $\mathbb{P}_{\varepsilon,\alpha,2}$. We start by:
• The estimation of $\mathbb{P}_{\varepsilon,\alpha,1}$.
Let $\nu \in (d, d+2]$. First we notice that if

$$\omega_\gamma \le \varepsilon^{1+\alpha} \quad \text{for} \quad |\gamma| \le \varepsilon^{-(1-\alpha)/2}$$

and

$$\omega_\gamma \le \varepsilon^{1+\alpha}\Big(1 + dist(\gamma, C_{0,\varepsilon^{-(1-\alpha)/2}})\Big)^{(\nu-d)(1-\alpha)} \quad \text{for} \quad \varepsilon^{-(1-\alpha)/2} < |\gamma| \le \varepsilon^{-\frac{1+2\alpha}{\nu-d}},$$

then

$$\sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma(1+|\beta-\gamma|)^{-\nu} \le \varepsilon^{1+\alpha}.$$

So

$$\mathbb{P}_{\varepsilon,\alpha,1} \ge \mathbb{P}_2 \cdot \mathbb{P}_1. \quad (3.20)$$

Where

$$\mathbb{P}_1 = \mathbb{P}\{\omega;\ \forall\ \gamma \text{ such that } |\gamma| \le \varepsilon^{-(1-\alpha)/2},\ \omega_\gamma \le \varepsilon^{1+\alpha}\},$$

and

$$\mathbb{P}_2 = \mathbb{P}\Big\{\omega;\ \forall\ \ \gamma \text{ such that } \varepsilon^{-(1-\alpha)/2} < |\gamma| \le \varepsilon^{-\frac{1+2\alpha}{\nu-d}},$$
$$\omega_\gamma \le \varepsilon^{1+\alpha}\Big(1 + dist(\gamma, C_{0,\varepsilon^{-(1-\alpha)/2}})\Big)^{(\nu-d)(1-\alpha)}\Big\}.$$

As the random variables are i.i.d we get that

$$\mathbb{P}_1 = \Big(\mathbb{P}\{\omega_0 \le \varepsilon^{1+\alpha}\}\Big)^{\varepsilon^{-d(1-\alpha)/2}} \tag{3.21}$$

and

$$\mathbb{P}_2 = \prod_{\varepsilon^{(1-\alpha)/2} < |\gamma| \le \varepsilon^{-\frac{1+2\alpha}{\nu-d}}} \mathbb{P}\Big(\omega_0 \le \varepsilon^{1+\alpha}(1+dist(\gamma, C_{0,\varepsilon^{-(1-\alpha)/2}}))^{(\nu-d)(1-\alpha)}\Big). \tag{3.22}$$

Now by applaying the logarithm to (3.20) and taking into akount (3.21) and (3.22) wile using (1.5), for α and ε small enough we get that

$$\log \mathbb{P}_{\varepsilon,\alpha,1} \ge$$
$$-\varepsilon^{-(\kappa+d/2)(1+\alpha)} - \varepsilon^{-\kappa(1+\alpha)} \sum_{\varepsilon^{-(1-\alpha)/2} \le |\gamma| \le \varepsilon^{-\frac{1+2\alpha}{\nu-d}}} \Big(1+dist(\gamma, C_{0,\varepsilon^{-(1-\alpha)/2}})\Big)^{-\kappa(\nu-d)(1-\alpha)}. \tag{3.23}$$

As if $(\nu-d)\kappa > d$ the sum in (3.23) converges when α is chosen small enough such that $(1-\alpha)(\nu-d)\kappa > d$. So we get,

$$\liminf_{\varepsilon\to 0^+} \frac{\log\Big|\log(\mathbb{P}_{\varepsilon,\alpha,2})\Big|}{\log\varepsilon} \ge -(1+\alpha)\Big(\kappa + \frac{d}{2}\Big). \tag{3.24}$$

In the case when $(\nu-d)\kappa < d$, for ε small one computes the sum in (3.23) we get the following estimation

$$\sum_{\varepsilon^{-(1-\alpha)/2} \le |\gamma| \le \varepsilon^{-\frac{(1+2\alpha)}{(\nu-d)}}} \Big(1 + dist(\gamma, C_0, \varepsilon^{-(1-\alpha)/2})\Big)^{-\kappa(\nu-d)(1-\alpha)} \le$$
$$C \cdot \varepsilon^{\kappa(1-\alpha)} \cdot \varepsilon^{-d(1+\alpha)/(\nu-d)}. \tag{3.25}$$

Using equations (3.23) and (3.25) and the fact that $\frac{d}{\nu-d} - \kappa + \frac{d}{2} \ge 0$ we get

$$\log \mathbb{P}_{\varepsilon,\alpha,1} \ge -C\varepsilon^{-(\kappa+d/2)(1+\alpha)} \cdot \varepsilon^{-(d(1+\alpha)-2\alpha\kappa)/(\nu-d)}.$$

We apply the logarithm into the last equation taking into a count (1.5), (3.18) and (3.24) we get (3.16).

• The estimation of $\mathbb{P}_{\varepsilon,\alpha,2}$. Let us notice that there exists $C > 0$ such that we have

$$\mathbb{P}_{\varepsilon,\alpha,2} \geq \mathbb{P}\Big\{\omega; \forall \gamma \in \Lambda_\alpha(\varepsilon^s); \omega_\gamma \leq \frac{\varepsilon^{1+\alpha}}{C}\Big\}.$$

As the random variables are i.i.d we get that

$$\mathbb{P}_{\varepsilon,\alpha,2} \geq \Pi_{\gamma \in \Lambda_\alpha(\varepsilon^s)} \mathbb{P}\{\omega; \omega_\gamma \leq \frac{\varepsilon^{1+\alpha}}{C}\} = \Big(\mathbb{P}\{\omega_0 \leq \frac{\varepsilon^{1+\alpha}}{C}\}\Big)^{\sharp\Lambda_\alpha(\varepsilon^s)}.$$

Now taking into account (1.5), (3.19), the fact that $\sharp\Lambda_\alpha(\varepsilon) = \varepsilon^{-d(\frac{1}{2}+\alpha)}$, and $\alpha > 0$ small we end the proof of (3.17). So the proof of Theorem 3.1 is ended. □

3.2 The upper bound

To prove the upper bound, we compare $N(E_+ + \varepsilon) - N(E)$ to the IDS some reduced operators. More precisely, we prove that for an energy E close to E_+, $N(E) - N(E_+)$ can be upper bounded by the IDS of some random and bounded operator. Indeed we have,

Lemma 3.2 *[9] Let H_ω be the operator defined by (1.1). We assume that **(A.1)**, **(A.2)** and **(A.3)** hold. There exists $E_0 > E_+$ and $C > 1$ such that, for $E_+ \leq E \leq E_0$ we have*

$$0 \leq N(E) - N(E_+) \leq N_{\mathcal{E}_0}\Big(C \cdot (E - E_+) + E_+\Big) \tag{3.26}$$

where $N_{\mathcal{E}_0}$ is the IDS of $H^0_\omega = \Pi_0 H_\omega \Pi_0$ and Π_0, is the spectral projection for H_0 on the band starting at E_+.

Proof: The proof of Lemma 3.2 is given in the case of acoustic operators in [9]. It is still true in the divergence case. It is based on a localization in energy for the density of states. It goes as follows: We approach the density of states of H_ω by the density of states of periodic approximations, see section 2.2. In a neighborhood of E_+, we control the behavior of the density of states of periodic approximations via the density of states of periodic approximations of the reference operator i.e $H^0_\omega = H^0_0 + V^0_\omega$. We then compute the limit for the density of states of the reference operators and we obtain the sought for result. □

3.2.1 The short range case

We recall that in this case we assume that the IDS, n of the background operator H_0 is non-degenerate. We prove the following theorem:

Theorem 3.3 *Let H_ω be the operator defined by (1.1). We assume that **(A.1)** and **(A.2)** hold and n is non-degenerate at E_+, then*

$$\limsup_{\varepsilon \longrightarrow 0^+} \frac{\log|\log(N(E_+ + \varepsilon) - N(E_+))|}{\log \varepsilon} \leq -(\frac{d}{2} + \kappa).$$

Proof: Let us notice that by Lemma 3.2 to prove Theorem 3.3 it suffices to get the same upper bound for the reference operator. This represents several advantages: first, H_ω^0 it is a bounded random operator and equivalent to a random Jacobi matrix acting on $L^d(\mathbb{T}^*) \otimes \mathbb{C}^{n_0}$ (Here n_0 is the number of Floquet eigenvalues generating the band starting in E_+). The second advantage is that while, E_+ is an interior edge of a gap for H_ω, it becomes the bottom of the spectrum for H_ω^0.
The idea of the proof is based on the uncertainly principle. Indeed, as $V_\omega^0 \geq 0$, if a vector minimizes H_ω^0, it necessarily minimizes $H_0^0 = \Pi_0 H_0 \Pi_0$; hence, it has to be concentrated in the quasimomentum θ near the zeros of $(E_j(\theta) - E_+)_{1 \leq j \leq n_0}$. For this, we have to take into account all the points where the Floquet eigenvalues reach E_+. Let θ^0 be one of those points. As n is non-degenerate, for $c > 0$ small, in a neighborhood of θ^0 we have [9]

$$\mathbb{D} = c \sum_{j=1}^{d} (1 - \cos(\theta_j - \theta_j^0)) \leq H_0^0 - E_+ \cdot I_d. \qquad (3.27)$$

Here $\mathbb{D}$ is acting on $L^2(\mathbb{T}^*) \otimes \mathbb{C}^{n_0}$ and I_d is the identity matrix.
We recall that V_ω is the operator defined by

$$V_\omega = -\nabla \Big(\sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma \rho^0(\cdot - \gamma) \Big) \nabla. \qquad (3.28)$$

It is proved [9] that V_ω^0 can be lower bounded by

$$V_{2,\omega}^a = \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma \Pi_\gamma.$$

Here Π_γ is the orthogonal projection on the vector $\theta \mapsto e^{i\gamma\theta}$ in $L^2(\mathbb{T}^*) \otimes \mathbb{C}^{n_0}$. Now using the following (unitary operator) discrete Fourier transformation defined from $l^2(\mathbb{Z}^d)$ to $L^2([0,2\pi]^d)$ by

$$\mathcal{F}(u)(k) = \widehat{u}(k) = \sum_{n\in\mathbb{Z}} u(n)e^{-in.k},$$

we get that $\mathbb{D}$ is unitarily equivalent to the usual discrete Schrödinger operator. So H^0_ω is lower bounded by some opertaor which it self unitarly equivalent to the usual discrete random operator whose behavior of the IDS at the edges of the spectral gaps is already known [5, 9]. This lower bound on the operator immediately yields an upper bound on the density of states.

3.2.2 The long range case

In this section we shale prove:

Theorem 3.4 *Let H_ω be the operator defined by (1.1). We assume that* **(A.1)**, **(A.2)** *hold.*
If $\frac{d}{\nu-d} > \kappa + \frac{d}{2}$ then,

$$\limsup_{\varepsilon\to 0^+} \frac{\log|\log\left(N(E_+ + \varepsilon) - N(E_+)\right)|}{\log\varepsilon} \le -\frac{d}{\nu - d}. \tag{3.29}$$

If n, the IDS of H_0 is non-degenerate then,

$$\limsup_{\varepsilon\to 0^+} \frac{\log|\log\left(N(E_+ + \varepsilon) - N(E_+)\right)|}{\log\varepsilon} \le -\sup(\frac{d}{2} + \kappa, \frac{d}{\nu - d}). \tag{3.30}$$

Proof:
• If $\frac{d}{\nu-d} > \kappa + \frac{d}{2}$.
Notice that in this case we have no assumption made on the behavior of n, the IDS of the periodic operator. The proof goes exactly as the one given in [10], for this we omit details. From Lemma 2.6 and for $\eta_0 > 1/(\nu - d)$ and $k \sim \varepsilon^{-\delta}$ such that $\delta > \nu_0$ the proof of (3.29) is reduced to prove that

$$\limsup_{\varepsilon\to 0^+} \frac{\log\left|\log\left(\mathbb{E}(N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+))\right)\right|}{\log\varepsilon} \le -\frac{d}{\nu - d}. \tag{3.31}$$

Lemma 3.5 *[9]Let $k \sim \varepsilon^{-\rho}$ with $\rho > 1/(\nu - d)$. Define the event,*

$$\mathbf{E}_{\varepsilon,\omega} = \Big\{\omega; V_{\omega,k} \geq -\varepsilon\Delta = -\varepsilon \sum_{i=1}^{d} \partial_{x_i}^2 \Big\}.$$

Here we recall that

$$V_{\omega,k} = \nabla \cdot \Big(\sum_{\gamma \in C_k \cap \mathbb{Z}^d} \omega_\gamma \sum_{\beta \in (2k+1)\mathbb{Z}^d} \rho^0(\cdot - \gamma - \beta) \Big) \cdot \nabla.$$

Then $\mathbf{E}_{\varepsilon,\omega}$ has a probability at least $1 - \mathbb{P}_\varepsilon$ where $\mathbb{P}_\varepsilon$ satisfies

$$\limsup_{\varepsilon \to 0^+} \frac{\log|\log(\mathbb{P}_\varepsilon)|}{\log \varepsilon} \leq -\frac{d}{\nu - d}. \tag{3.32}$$

Using the fact that if for some $C > 0$ (depending only on δ and ρ^*), $V_{\omega,k} \geq -C\varepsilon\Delta$, then the spectrum of $H_{\omega,k}$ does not intersect $(E_+, E_+ + \varepsilon)$ for ε small. One computes

$$\begin{aligned}
\mathbb{E}\Big(N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)\Big) &= \mathbb{E}\Big([N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)]\mathbf{1}_{\{\omega; V_{\omega,k} \geq -C\varepsilon\Delta\}}\Big) \\
&+ \mathbb{E}\Big([N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)]\mathbf{1}_{\{\omega; V_{\omega,k} < -C\varepsilon\Delta\}}\Big) \\
&\leq C\mathbb{P}(\{\omega; V_{\omega,k} < -C\varepsilon\Delta\}) \\
&= C(1 - \mathbb{P}(\mathbf{E}_{C\cdot\varepsilon,\omega})) = C\mathbb{P}_{C\cdot\varepsilon}.
\end{aligned}$$

Here, we have used the fact that $N_{\omega,k}$ is bounded, locally uniformly in energy, uniformly in ω, k by C. Taking (3.32) into account, we end the proof of (3.31) and so (3.29) is proved. □

Now one deals with the proof of (3.30). We recall that here one supposes once more that n is non-degenerate. The idea is similar to the short range case (we will compare the IDS of our operator to the IDS of another one) and we will follow and use results given in [11, 13]. Let N^a be the IDS of the follwoing Anderson discrete operator acting on $l^2(\mathbb{Z}^d)$:

$$(H_\omega^a u)(\alpha) = E_+ \cdot u(\alpha) + \sum_{|\alpha - \beta| = 1} (u(\alpha) - u(\beta)) + (V_\omega^a u)(\alpha). \tag{3.33}$$

Here V_ω^a the diagonal infinite matrix with $v_\alpha(\omega) = \sum_{\beta \in \mathbb{Z}^d} \omega_\beta (1 + |\alpha - \beta|)^{-\nu}$ for the α^{th} diagonal coefficient.

For $k \in \mathbb{N}^*$ and $u \in l^2(\mathbb{Z}^d \cap C_k)$, let H_0^k, V_ω^k and H_ω^k be the following discrete operators

$$(H_0^k u)(\alpha) = E_+ \cdot u(\alpha) + \sum_{|\alpha-\beta|=1, \beta \in C_k} (u(\alpha) - u(\beta)), \quad (V_\omega^k u)(\alpha) = v_\alpha(\omega) u(\alpha)$$

and

$$H_\omega^k = H_0^k + V_\omega^k.$$

Let N_k^a be the IDS of H_ω^k defined by:

$$N_k^a(E) = \frac{1}{(2k+1)^d} \cdot \mathbb{E}\Big(\sharp\{\text{eigenvalues of } \ H_\omega^k \text{ less or equal to } \ E\}\Big).$$

From [11, 13], we know that for a good choice of k, the IDS at energy E is quite well approximated by the probability to find a state energy less than E. Precisely we have the following relation:

$$N^a(E) \leq N_k^a(E) \leq C \cdot \mathbb{P}_k(E). \tag{3.34}$$

Here

$$\mathbb{P}_k(E) = \mathbb{P}\Big(\Big\{H_\omega^k \text{ admits at least an eigenvalue less than } E\Big\}\Big).$$

To estimate this probability one proceeds as previously and lower bound H_ω^k by $H_{\widetilde{\omega}}^k$; obtained for $\delta > 0$ by changing ω_γ by $\widetilde{\omega}_\gamma = \omega_\gamma$ if $\omega_\gamma \leq \delta$ and $\widetilde{\omega}_\gamma = \delta$ if not. So if we denote the IDS of $H_{\widetilde{\omega}}^k$ by $\widetilde{N}_k^a$ then we have,

$$N_k^a(E) \leq \widetilde{N}_k^a(E).$$

One takes $k = c(E-E_+)^{-\frac{1}{2}}$ and $\delta = (E-E_+)/c$ positives. For a good choice of c and thus of δ, and E in a neighborhood of E_+, we get

$$\mathbb{P}_k(E) \leq \mathbb{P}\Big(\Big\{\widetilde{\omega};\ \frac{1}{(2k+1)^d} \sum_{|\gamma| \leq k} v_\gamma(\widetilde{\omega}) \leq \delta/K\Big\}\Big). \tag{3.35}$$

Now we have to estimate the last probability. Let $0 < \alpha < 1$, for some $C > 1$ we have:

$$\frac{1}{(2k+1)^d} \sum_{|\gamma| \leq k} v_\gamma(\widetilde{\omega}) = \frac{1}{(2k+1)^d} \sum_{\beta \in \mathbb{Z}^d} \widetilde{\omega}_\beta \Big(\sum_{|\gamma| \leq k} (1 + |\gamma - \beta|)^{-\nu} \Big).$$

$$\geq \frac{1}{C(2k+1)^d}\sum_{|\beta|\leq k}\widetilde{\omega}_\beta + \frac{1}{C}\sum_{k<|\beta|\leq \delta^{-(\nu-d)(1-\alpha)}}\widetilde{\omega}_\beta(1+|\beta|+k)^{-\nu}.$$

Thus

$$\mathbb{P}\Big(\Big\{\widetilde{\omega};\ \frac{1}{(2k+1)^d}\sum_{|\gamma|\leq k} v_\gamma(\widetilde{\omega}) \leq \delta/K\Big\}\Big) \leq$$

$$\mathbb{P}\Big(\Big\{\widetilde{\omega};\ \frac{1}{C(2k+1)^d}\sum_{|\beta|\leq k}\widetilde{\omega}_\beta \leq \delta/K \text{ and } \frac{1}{C}\sum_{k<|\beta|\leq \delta^{-(\nu-d)(1-\alpha)}}\widetilde{\omega}_\beta(1+|\beta|+k)^{-\nu} \leq \delta/K\Big\}\Big). \tag{3.36}$$

As the random variables are i.i.d we get for

$$\mathbb{P}_1 = \mathbb{P}\Big(\Big\{\widetilde{\omega};\ \frac{1}{C(2k+1)^d}\sum_{|\beta|\leq k}\widetilde{\omega}_\beta \leq \delta/K\ \Big\}\Big)$$

and

$$\mathbb{P}_2 = \mathbb{P}\Big(\Big\{\widetilde{\omega};\frac{1}{C}\sum_{k<|\beta|\leq \delta^{-(\nu-d)(1-\alpha)}}\widetilde{\omega}_\beta(1+|\beta|+k)^{-\nu} \leq \delta/K\Big\}\Big),$$

we have

$$\mathbb{P}\Big(\Big\{\widetilde{\omega};\ \frac{1}{(2k+1)^d}\sum_{|\gamma|\leq k} v_\gamma(\widetilde{\omega}) \leq \delta/K\Big\}\Big) \leq \mathbb{P}_1\cdot\mathbb{P}_2.$$

The estimation of $\mathbb{P}_1$ and $\mathbb{P}_2$ is based on large deviation results [2]. Briefly the idea is the following. Let $t>0$. Using the Markov inequality one estimates

$$\mathbb{P}_1 \leq \mathbb{E}\Big(e^{t(\delta/K-\frac{1}{(2k+1)^d}\sum_{|\beta|\leq k}\widetilde{\omega}_\gamma)}\Big). \tag{3.37}$$

As the random variables are i.i.d one gets

$$\begin{aligned}\mathbb{E}\Big(e^{t(\delta/K-\frac{1}{(2k+1)^d}\sum_{|\beta|\leq k}\widetilde{\omega}_\gamma)}\Big) &= e^{t\delta/K}\prod_{|\beta|\leq k}\mathbb{E}\Big(e^{-t\widetilde{\omega}_\beta/(2k+1)^d}\Big) &(3.38)\\ &= e^{t\delta/K}e^{(2k)^d\log\mathbb{E}(e^{-t\widetilde{\omega}_0/(2k+1)^d})} &(3.39)\end{aligned}$$

For k big enough using Taylor expansion of e^t, we get that

$$\mathbb{E}\Big(e^{-t\frac{\widetilde{\omega}_0}{(2k)^d}}\Big) = 1 - \frac{t\mathbb{E}(\widetilde{\omega}_0)}{(2k+1)^d} + o(\frac{t^2}{(2k+1)^{2d}}).$$

So we get

$$\mathbb{P}_1 \leq e^{t\delta/K} e^{(2k)^d \log\left(1 - \frac{t\mathbb{E}(\widetilde{\omega}_0=0)}{(2k+1)^d} + o(\frac{t^2}{(2k+1)^{2d}})\right)}.$$

As $\mathbb{E}(\widetilde{\omega}_0) > 0$ for k big enough and $t > 0$ well chosen we get that there exists $C > 0$ such that

$$\mathbb{P}_1 \leq e^{\left((2k)^d \log(\mathbb{P}\{\widetilde{\omega}_0=0\}\right)/C}.$$

The same computation as above gives

$$\mathbb{P}_2 \leq e^{-\left(k^d|\log(\mathbb{P}\{\widetilde{\omega}_0 \leq \delta\})| + \delta^{-d/(\nu-d)}\right)/C}.$$

So, we get that:

$$\log N^a(E) \leq -\Big(k^d|\log \mathbb{P}(\{\widetilde{\omega}_0 = 0\})| + |\log(\mathbb{P}(\widetilde{\omega}_0 = 0))| + \delta^{-d/(\nu-d)}\Big)/C.$$

Now, for $k = c(E - E_+)^{-\frac{1}{2}}$ and $\delta = (E - E_+)/c$ we get (3.30) for N^a.
As n is non-degenerate we get that (see the short range case and [9]) H^a_ω is unitarily equivalent by means of the Fourier transformation to some operator which lower bound H^0_ω. This yelds (3.30) for N. $\square$.

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