ANDERSON TRANSITIONS FOR A FAMILY OF ALMOST PERIODIC SCHRÖDINGER EQUATIONS IN THE ADIABATIC CASE

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Abstract. This work is devoted to the study of a family of almost periodic one-dimensional Schrödinger equations. We define a monodromy matrix for this family. We study the asymptotic behavior of this matrix in the adiabatic case. Therefore, we use a complex WKB method for adiabatic perturbations of periodic Schrödinger equations. The study of the monodromy matrix enables us to get spectral results for the initial family of almost periodic equations.

Résumé. Ce travail est consacré à l'étude d'une famille d'équations de Schrödinger quasi-périodiques en dimension 1. Nous définissons une matrice de monodromie pour cette famille. Nous étudions le comportement asymptotique de cette matrice dans le cas adiabatique. A cette fin, nous utilisons une version de la méthode WKB complexe pour l'équation de Schrödinger périodique avec une perturbation adiabatique. L'étude de la matrice de monodromie nous permet d'obtenir des résultats spectraux pour la famille d'équations quasi-périodiques initiale.

1. Introduction

In this paper, we study spectral properties of the ergodic family of Schrödinger equations

$$H_{\phi,\varepsilon}\psi = -\frac{d^2}{dx^2}\psi(x) + (V(x-\phi) + \alpha\cdot\cos(\varepsilon x))\psi(x) = E\psi(x), \quad x \in \mathbb{R}. \tag{1.1}$$

We assume that

- V is a locally square integrable, real valued, periodic function,

$$V(x+1) = V(x), \quad x \in \mathbb{R}, \tag{1.2}$$

- $\alpha > 0$ is a coupling constant,
- ε is a positive number such that $2\pi/\varepsilon \notin \mathbb{Q}$.
- $\phi \in \mathbb{R}$ is a parameter numbering the equations,

We note that, under our assumptions, the potential $V(x-\phi) + \alpha\cos(\varepsilon x)$ is quasi-periodic.

The goal of this paper is to study the family (1.1) at "low" energies. Some of these results were announced in [21, 22].

During the last 20 years, quasi-periodic Schrödinger operators have been the object of many studies both rigorous and numerical (see, e.g. [25, 29, 33, 28, 14] for references). Motivated by the physics of condensed matter, among the main goals was the understanding of their spectra.

One of the question that has drawn a lot of interest is the nature of the spectrum; this is one of the main issues dealt with in this paper. One of the most remarkable phenomena exhibited by the spectrum of these operators is the so-called Anderson transition or "metal-insulator" transition. It was observed numerically (e.g. [27, 28]) that, for certain quasi-periodic Schrödinger operators, the spectrum is split into energy intervals where it is purely absolutely continuous and other energy intervals where it is singular. Roughly speaking, from the physicist's point of view, the singular spectrum corresponds to a phase where the material is an insulator, and the absolutely continuous spectrum to a phase where the material is a conductor. The transitions from one of these phases to the other one are expected to be sharp and to happen at discrete energies called *mobility edges*. Such transitions do not always exist; e.g. for highly symmetrical models like the Almost Mathieu equation or for small potentials, it is believed and proved in many cases that no such transitions happen, the spectral type being determined by parameters of this equation ([13, 30]).

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As for the existence of transitions, in [23], for a model similar to ours, it was proved that the bottom of the spectrum is pure point in the large coupling constant limit. On the other hand, it is known that, for one dimensional quasi-periodic Schrödinger operators with analytic potentials, the spectrum is absolutely continuous for sufficiently high energies ([13]). So, different types of spectra may coexist.

In the present paper, we define the *monodromy matrix* for the family of equations (1.1), compute the asymptotics of a monodromy matrix in the adiabatic limit i.e. as $\varepsilon \to 0$, and use these results to study spectral properties of equation (1.1). More precisely, we show that, at low energies, the spectrum of the family (1.1) is located in exponentially small intervals and prove the coexistence of zones where the spectrum is singular and others where most of the spectrum is absolutely continuous. We also prove that the transitions between these regions are asymptotically sharp as $\varepsilon \to 0$. The energies where these transitions occur are called *asymptotic mobility edges.* We give a simple description of these energies in terms of analytic objects naturally associated to equation (1.1). To the best of our knowledge, this is the first instance of quasi-periodic Schrödinger operators for which such a coexistence has been proved outside of the large coupling regime and sharp transitions have been found. Also, in the present paper, we handle locally square integrable potentials, whereas in previous works analyticity of the potential has always been a crucial assumption.

Let us now describe our model more precisely. Consider the periodic Schrödinger operator

$$(H_0\psi)(x) = -\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x). \tag{1.3}$$

We assume that

- the first gap in the spectrum of H_0 is open; denote it by (E_2, E_3).
- let E_1 be the bottom of the spectrum of H_0 (see Fig. 5). We restrict our study to the bottom of the spectrum, more precisely, to energies satisfying

$$E - \alpha < E_1 \text{ and } E_1 < E + \alpha < E_2. \tag{1.4}$$

The techniques that we present here can also be used to study more general adiabatic quasi-periodic perturbations of H_0 i.e. we can replace the cosine by other real analytic potentials. It can also be used to study the situation at higher energies (see [17, 20, 16]).

Let us now briefly describe our results, the next section being devoted to the precise mathematical statements. Let $\mathbf{E}(\kappa)$ be the dispersion relation associated to H_0 and consider the *real* and the *complex iso-energy curves* $\Gamma_{\mathbb{R}}$ and Γ defined by

$$\Gamma_{\mathbb{R}}: \quad \mathbf{E}(\kappa) + \alpha \cdot \cos(\varphi) = E, \quad \kappa, \varphi \in \mathbb{R}, \tag{1.5}$$

$$\Gamma: \quad \mathbf{E}(\kappa) + \alpha \cdot \cos(\varphi) = E, \quad \kappa, \varphi \in \mathbb{C}. \tag{1.6}$$

The connected components of $\Gamma_{\mathbb{R}}$ are called the *real branches* of the iso-energy curve (1.5). The real iso-energy curve is 2π-periodic in the κ- and φ- (vertical and horizontal) directions. It is symmetric with respect to the φ-axis. The role of the real iso-energy curve for adiabatic problems is well known (see, for example, [5]).
Under assumption (1.4), the curve Γ has the topology corresponding to Fig. 1. In this figure, the real branches of $\Gamma_{\mathbb{R}}$ are represented by full lines, and the dashed lines are complex loops in Γ. Define the actions S_h, S_v and Φ by

$$S_h = -i\oint_{\gamma_2} \kappa d\varphi, \; S_v = i\oint_{\gamma_3} \kappa d\varphi \text{ and } \Phi = \oint_{\gamma_1} \kappa d\varphi. \tag{1.7}$$

Here, γ_1 is a connected component of $\Gamma_{\mathbb{R}}$, γ_2 a loop in Γ connecting γ_1 and $\gamma_1 - (2\pi, 0)$, and, γ_3 a loop in Γ connecting γ_1 and $\gamma_1 + (0, 2\pi)$. We discuss these loops in section 2.6. These loops are oriented so that the integrals in (1.7) be positive.

We prove that, on the interval (1.4), the spectrum is situated in exponentially small (in $1/\varepsilon$) intervals such that the distances between any two neighboring such intervals are of order ε; the "centers" of these intervals are given by the quantization condition $\cos(\Phi(E)/\varepsilon) = 0$. Moreover, if we set $\Delta S(E) = S_h(E) - S_v(E)$, then, the subintervals where $\Delta S(E) < 0$ contain only singular spectrum, and the subintervals where $\Delta S(E) > 0$ contain absolutely continuous spectrum; actually, in the latter subintervals, most of the spectrum is absolutely continuous. Finally, we study general properties of the function $\Delta S(E)$. For small α, ΔS takes positive values; hence, the bottom of the spectrum of the operator $H_{\phi,\varepsilon}$ still contains absolutely continuous

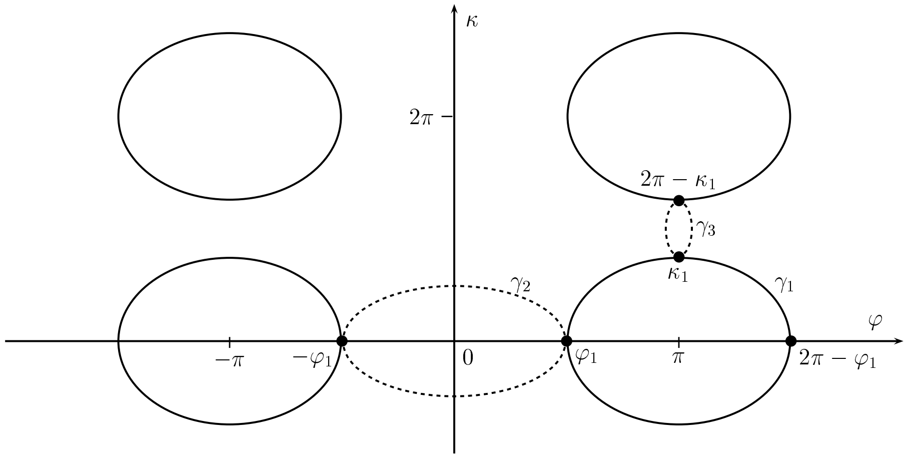

FIGURE 1. The contours on Γ

spectrum for α small. On contrary, for large α, the function ΔS becomes negative, and thus, the bottom of the spectrum of the unperturbed operator becomes singular, the absolutely continuous spectrum disappears. For intermediate values of α, there are always one or several asymptotic mobility edges i.e. points where the function ΔS vanishes. These are the points where the transitions from absolutely continuous to singular spectrum (or vice versa) occur.
One can define two tunneling coefficients, a vertical one associated to $S_v : t_v = \exp(-\frac{1}{\varepsilon}S_v)$, and, a horizontal one associated to $S_h : t_h = \exp(-\frac{1}{\varepsilon}S_h)$. We see that, roughly, if the vertical tunneling is larger than the horizontal one, the spectrum is singular; and if the horizontal is the larger one, the spectrum is absolutely continuous. For Harper's equation, an heuristic leading to a similar interpretation has been developed in [42].

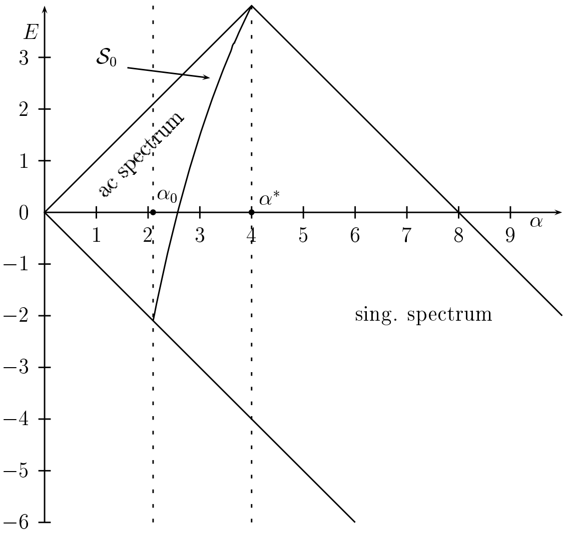

FIGURE 2. The phase diagram

To illustrate our results, we describe an example where we have computed the asymptotic mobility edges numerically i.e. we have solved $\Delta S = 0$ numerically. We take the simplest example of V i.e we take V to be a one-gap potential ([36]) associated to the periodic spectrum $[E_1, E_2] \cup [E_3, +\infty)$. In our case $E_1 = 0$, $E_2 = 8$. The value of E_3 is computed so that the potential V do have period 1.
In Fig. 2, we represented the *phase diagram* i.e. the curve $\mathcal{S}_0 = \{(\alpha, E) : \Delta S(E, \alpha) = 0\}$ of the asymptotic mobility edges (for the potential $V(\cdot - \phi) + \alpha\cos(\varepsilon\cdot)$). The spectrum is represented vertically and the parameter α horizontally. The oblique rectangle delimits the region (α, E) satisfying (1.4). It comprises the bottom of the spectrum of $H_{\phi,\varepsilon}$. For α fixed, the domain marked by "ac spectrum" (resp. "sing. spectrum") in Fig. 2 contains absolutely continuous (resp. singular) spectrum. α_0 is the value of α at which mobility

edges start to exist in our energy interval. When one turns the coupling constant α on, the lowest edge of the spectrum starts to "localize" first, though not immediately. When $\alpha > \alpha^*$, all the spectrum in the window we are considering is singular. In a more general case i.e. if, in (1.1), we replace the cosine by a more general potential, the phase diagram may be quite different (see [17]). Such phase diagrams were already obtained for quasi-periodic finite difference equations using numerical simulations ([27]).

Remark 1.1. In the description given above, we used the word "localized" very loosely to denote singular spectrum.

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2. The main results

In this section, we introduce the central object of our study, the monodromy matrix, and present our main results.

2.1. Monodromy equation.

We define *the monodromy matrix* and introduce *the monodromy equation.*

2.1.1. *Monodromy matrix.* For any ϕ fixed, let $\psi_{1,2}(x,\phi)$ be two linearly independent solutions of equation (1.1). We say that they form a *consistent basis* if their Wronskian is independent of ϕ and if these solutions are 1-periodic in ϕ i.e.

$$\psi_{1,2}(x,\phi+1)=\psi_{1,2}(x,\phi),\quad \forall x,\phi, \tag{2.1}$$

The existence of a consistent basis is clear: it suffices to take the solutions with canonical Cauchy data at $x=0$ i.e. $\psi_1(0,\phi)=1$ and $\psi_1'(0,\phi)=0$, and $\psi_2(0,\phi)=0$ and $\psi_2'(0,\phi)=1$.

Consider a consistent basis $(\psi_{1,2})$. As cosine is 2π-periodic, the functions $\psi_{1,2}(x+2\pi/\varepsilon,\phi+2\pi/\varepsilon)$ are solutions of equation (1.1). Therefore, one can write

$$\begin{pmatrix}\psi_1(x+2\pi/\varepsilon,\phi+2\pi/\varepsilon)\\ \psi_2(x+2\pi/\varepsilon,\phi+2\pi/\varepsilon)\end{pmatrix}=M(E,\phi)\begin{pmatrix}\psi_1(x,\phi)\\ \psi_2(x,\phi)\end{pmatrix}, \tag{2.2}$$

where $M(E,\phi)$ is a 2×2 matrix with coefficients independent of x.

The matrix M is called *the monodromy matrix* corresponding to the consistent basis $(\psi_{1,2})$. Note that if the potential in (1.1) and the solutions $\psi_{1,2}$ are independent of ϕ, then our definition becomes the standard definition of the monodromy matrix for the periodic Schrödinger equation (see e.g. [12, 41]).

One immediately checks that, for any consistent basis, the monodromy matrix satisfies

$$\det M(E,\phi)\equiv 1,\quad M(E,\phi+1)=M(E,\phi),\quad \forall\phi. \tag{2.3}$$

2.1.2. *Monodromy equation.* Set

$$h=\frac{2\pi}{\varepsilon}\mathrm{mod}1. \tag{2.4}$$

Let M be a monodromy matrix corresponding to a consistent basis $(\psi_{1,2})$. The spectral analysis of (1.1) can be reduced to the investigation of the solutions of the *monodromy equation*

$$F_{n+1}=M(E,\phi+nh)F_n,\quad F_n\in\mathbb{C}^2,\quad \forall n\in\mathbb{Z}. \tag{2.5}$$

Going from equation (1.1) to the monodromy equation is close to the monodromization transformation introduced in [7] to construct Bloch solutions of difference equations. In our case, it appears that the behavior of solutions of (1.1) for $x\to\pm\infty$ repeats the behavior of solutions of the monodromy equation for $n\to\mp\infty$. And, it is well known that the spectral properties of one dimensional Schrödinger equations can be described in terms of the behavior of its solutions as $x\to\pm\infty$ (see [24, 31]). Let us formulate the precise statement. In the sequel, identifying the vector solutions of the monodromy equation (2.5) with functions $F:\mathbb{Z}\to\mathbb{C}^2$, we denote the values of F by F_n. One has

Theorem 2.1. *Let $\psi_j(x,\phi)$, $j=1,2$, be a consistent basis solutions of (1.1), bounded in (x,ϕ) together with their derivatives in x. Fix $\phi=\phi_0\in\mathbb{R}$ and consider solutions of the monodromy equation. Then, there exists a positive constant C such that, for any vector solution F of the equation (2.5), there exists a unique solution f of (1.1) satisfying the estimates*

$$\frac{1}{C}\|F_{-n}\|_{\mathbb{C}^2}\leq\left\|\begin{pmatrix} f(x+2\pi n/\varepsilon,\phi)\\ f'(x+2\pi n/\varepsilon,\phi)\end{pmatrix}\right\|_{\mathbb{C}^2}\leq C\|F_{-n}\|_{\mathbb{C}^2},\quad \forall x\in[0,2\pi/\varepsilon), n\in\mathbb{Z}. \tag{2.6}$$

and reciprocally.

Both, for equations (1.1) and (2.5), one can define the Lyapunov exponent (see e.g. [4, 8, 10, 37]). It is defined for almost every ϕ and independent of ϕ. Let $\Theta(E,\phi)$ be the Lyapunov exponent at energy E for (1.1) and $\theta(E,\phi)$ be the one for (2.5). Theorem 2.1 then immediately implies

Corollary 2.1. *The Lyapunov exponents $\Theta(E,\phi)$ and $\theta(E,\phi)$ satisfy the relation*

$$\Theta(E,\phi)=\frac{\varepsilon}{2\pi}\theta(E,\phi). \tag{2.7}$$

2.2. **The adiabatic limit.** We see that the spectral analysis of (1.1) reduces to the analysis of solutions of the monodromy equation (2.5). To compute a monodromy matrix, we use the adiabatic limit $\varepsilon\to 0$.

In the adiabatic case, the asymptotics of the monodromy matrices take up a very simple form so that one can speak of reducing of (1.1) to some simple model difference equation (i.e. the respective monodromy equation).

To describe the reduced model, we first recall some basic facts about one dimensional periodic Schrödinger operators and define some related objects.

2.3. **Periodic Schrödinger operator.** Consider the periodic Schrödinger operator (1.3). Its spectrum on $L^2(\mathbb{R})$, denoted by $\sigma(H_0)$, is absolutely continuous and consists of intervals $[E_1,E_2]$, $[E_3,E_4]$, …, $[E_{2n+1},E_{2n+2}]$, …, of the real axis such that

$$E_1<E_2\leq E_3<E_4\dots E_{2n}\leq E_{2n+1}<E_{2n+2}\leq\dots,$$
$$E_n\to+\infty,\quad n\to+\infty.$$

The above intervals are called the *spectral bands*, and the open intervals (E_2,E_3), (E_4,E_5), …, (E_{2n},E_{2n+1}), …, are called the *spectral gaps*.

Denote the *dispersion law* associated to (1.3) by $\mathbf{E}(\kappa)$. The inverse function $k(E)$ is called *the Bloch quasi-momentum*. It is a multi-valued analytic function; its branch points are the edges of the gaps and are of square root type. It is real on the spectral bands; on the spectral gaps, its real part is constant and its imaginary part does not vanish. For more details on this function, we refer to section 5.

2.4. **The complex momentum.** Consider the function $\kappa(\varphi)$ defined by the relation (1.6) i.e.

$$\kappa(\varphi)=k(E-\alpha\cos(\varphi)). \tag{2.8}$$

We call it *the complex momentum*. This function is also multi-valued and analytic. The function κ will play a crucial role in the determination of the asymptotics of the monodromy matrix.

2.5. **The branch points of κ and the set $\arccos(\mathbb{R})$.** As the branch points of the Bloch quasi-momentum, all the branch points of κ are of square root type. They are given by the equation

$$E-\alpha\cos(\varphi)=E_n,\ n\in\mathbb{N}. \tag{2.9}$$

The set of the branch points is 2π-periodic in φ; if $E\in\mathbb{R}$, it is symmetric with respect to the real line. Moreover, all of them are on $\arccos(\mathbb{R})=\mathbb{R}\cup\cup_{n\in\mathbb{Z}}(i\mathbb{R}+n\pi)$, the pre-image of $\mathbb{R}$ by the cosine.
Under condition (1.4), the complex momentum has a single branch point in $(0,\pi)$. Denote it by φ_1; it is defined by $E_1=E-\alpha\cos(\varphi_1)$ (see Fig. 3).
The complex momentum also has non-real branch points. Let φ_2 and φ_3 be the branch points lying on the line $\pi+i\mathbb{R}_+$, closest to π, indexed so that $\operatorname{Im}\varphi_2<\operatorname{Im}\varphi_3$; they satisfy (2.9) for $n=2,3$ (see Fig. 3).

Consider the map $\mathcal{E}:\varphi\mapsto E-\alpha\cos(\varphi)$. We denote by Z the pre-image of $\sigma(H_0)$, the spectrum of H_0 and by G the pre-image of the spectral gaps of H_0 i.e. $\mathbb{R}\setminus\sigma(H_0)$. Clearly, $Z,G\subset\arccos(\mathbb{R})$. The connected components of Z and G are separated by branch points of the complex momentum. All the branches of the complex momentum take real values on Z. And, the imaginary part of any branch of the complex momentum is non-zero on G.

2.6. **The phases and tunneling coefficients.** We now describe the main analytical objects needed to write out the asymptotics of the coefficients of the monodromy matrix: the phase Φ and the tunneling coefficients t_h and t_v, in terms of contours in the complex plane of φ.
There is a branch of the complex momentum taking values in $i\mathbb{R}_+$ on the interval $(-\varphi_1, \varphi_1)$, and in $[0, \pi]$ on the "cross" $[\varphi_1, 2\pi - \varphi_1] \cup [\varphi_2, \overline{\varphi_2}]$ (see Fig. 3 and section 7.1 for more details). Denote this branch by κ_*.

We define the actions

$$S_h = -i \int_{-\varphi_1}^{\varphi_1} \kappa_* d\varphi \text{ and } \quad S_v = i \int_{2\pi-\varphi_2}^{\varphi_2} (\kappa_* - \pi) d\kappa, \tag{2.10}$$

and the phase integral

$$\Phi = \frac{1}{\varepsilon} \int_{\varphi_1}^{2\pi-\varphi_1} \kappa_* d\varphi. \tag{2.11}$$

In section 9.1, we prove

Lemma 2.1. *The actions and phase integrals have the following properties*

1. *S_h, S_v and Φ take positive values on J;*
2. *they are analytic in E in a neighborhood of J;*
3. *$\Phi'(E) > 0$ for all $E \in J$;*
4. *$S_v \leq 2\pi \mathrm{Im}\,(\varphi_2)$ for $E \in J$.*

We define *tunneling coefficients*

$$t_h = \exp\left(-\frac{1}{\varepsilon} S_h\right), \; t_v = \exp\left(-\frac{1}{\varepsilon} S_v\right). \tag{2.12}$$

Due to point 1. in Lemma 2.1, the tunneling coefficients t_h and t_v are exponentially small in $1/\varepsilon$.

The actions and the phase integral can be written as contour integrals of a branch of κ along closed curves

$$S_h = -\frac{i}{2} \oint_{\gamma_v} \kappa d\varphi \quad \text{and} \quad S_v = \frac{i}{2} \oint_{\gamma_h} \kappa d\varphi, \text{ and } \Phi = \frac{1}{2\varepsilon} \oint_{\gamma_p} \kappa d\varphi. \tag{2.13}$$

The loops γ_h, γ_v and γ_p are shown on Fig. 3. We note here that the contours γ_p, γ_h and γ_v can be considered

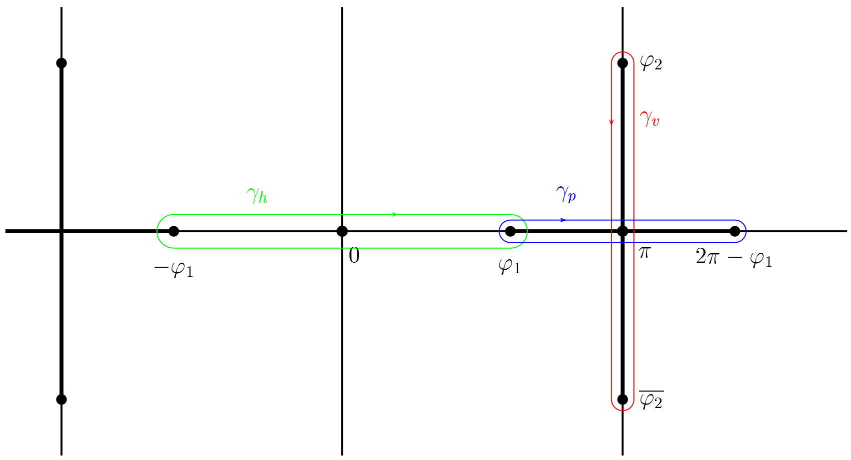

FIGURE 3. The branch points and the contours in the complex plane of φ

as projections on the complex plane of φ, of contours γ_1, γ_2, γ_3 located in Γ (see Fig. 1). This follows from the fact that, any analytic branch of κ is single valued on each of the curves γ_p, γ_h and γ_v (see section 9.1). This last property is related to the fact that all the branch points of κ are of square root type.

2.7. **Asymptotics of the monodromy matrix.** As the potential in equation (1.1) is real valued, we are able to choose a consistent basis so that the corresponding monodromy matrix has the form

$$M(\phi,E)=\begin{pmatrix} a(\phi,E) & b(\phi,E)\\ \overline{b(\overline{\phi},\overline{E})} & \overline{a(\overline{\phi},\overline{E})}\end{pmatrix}=\begin{pmatrix} a & b\\ b^* & a^*\end{pmatrix} \tag{2.14}$$

Here and from now on, for $a:\ (\phi,E)\mapsto a(\phi,E)$ a function of complex variables, a^* denotes the function $(\phi,E)\mapsto\overline{a(\overline{\phi},\overline{E})}$. The set of $GL(2,\mathbb{C})$ matrix-valued functions having the form (2.14) is denoted by $\mathcal{M}$ (regardless of the variables, these being clear from the context); note that it is an algebra over $\mathbb{R}$.
We introduce another notation that is used throughout the paper; for $a:\ \phi\mapsto a(\phi)$, a 1-periodic function, we write

$$a=a_0+\tilde{a}(\phi) \tag{2.15}$$

where a_0 is the zeroth Fourier coefficients of a.

Recall that E_1 and E_2 are the ends of the first spectral interval of the periodic Schrödinger equation (1.3). Fix $\delta>0$. We assume that

$$E-\alpha\leq E_1-\delta,\quad E_1+\delta<E+\alpha<E_2-\delta. \tag{2.16}$$

This is just a strong version of condition (1.4). In section 8, we prove

Theorem 2.2. *Pick $Y\in(\operatorname{Im}\varphi_2,\operatorname{Im}\varphi_3)$. Pick E_0 satisfying (2.16). Then, there exists $\eta>0$ and $\varepsilon_0>0$ such that, for $0<\varepsilon<\varepsilon_0$, there exists a consistent basis such that $M(E,\phi)$, the corresponding monodromy matrix is analytic in $\mathcal{V}=\{|E-E_0|<\eta\}\times\{|\operatorname{Im}\phi|\leq Y/\varepsilon\}$, belongs to $\mathcal{M}$ and, in $\mathcal{V}$, its coefficients decomposed according to (2.15) admit the asymptotics*

$$\begin{aligned} a_0=a_0(E)&=-\frac{1}{t_h}e^{i\Phi/\varepsilon}(1+o(1)),\quad \tilde{a}(\phi)=\tilde{a}(\phi,E)=\frac{t_v}{t_h}e^{2i\pi(\phi-\phi_0)}(1+o(1)),\\ b_0=b_0(E)&=\frac{i}{t_h}e^{i\Phi/\varepsilon}(1+o(1)),\quad \tilde{b}(\phi)=\tilde{b}(\phi,E)=-i\frac{t_v}{t_h}e^{2i\pi(\phi-\phi_0)}(1+o(1)).\end{aligned} \tag{2.17}$$

The functions $\Phi=\Phi(E)$, $t_h=t_h(E)$ and $t_v=t_v(E)$ are the phase and tunneling coefficient defined above. The function $\phi_0=\phi_0(E)$ is real analytic and satisfies $\phi_0'(E)=O(1)$. The asymptotics are uniform in (E,ϕ) in the set $\mathcal{V}$.

The proof of Theorem 2.2 is based on a new asymptotic method developed in [19] to study adiabatic perturbations of periodic Schrödinger operators in dimension 1. This method is based on the analysis of adiabatic asymptotics in the complex plane of φ of solutions of equations of the form

$$-\psi''(x)+V(x-\varphi,\varepsilon x)\psi(x)=E\psi(x).$$

In general, this method leads to computations similar to those typical for the standard complex WKB methods (see e.g. [15]). In our case, under condition (2.16), due to some natural symmetries, they remind of the computations made in [6] for the semi-classical analysis Harper equation.

2.8. **The spectral results.**

2.8.1. *The spectrum in the adiabatic limit.* We begin with a general observation concerning the location of the spectrum in the adiabatic limit. Recall that, as $H_{\phi,\varepsilon}$ is quasi-periodic, its spectrum does not depend on ϕ (see [1]). Let Σ_ε denote the spectrum of $H_{\phi,\varepsilon}$. One proves

Proposition 2.1. *Let $\Sigma=\sigma(H_0)+\alpha\cos(\mathbb{R})=\sigma(H_0)+[-\alpha,\alpha]$. Then, one has*

- *$\forall\varepsilon\geq 0$, $\Sigma_\varepsilon\subset\Sigma$.*
- *for any $K\subset\Sigma$ compact and any $\nu\in(0,1/2)$, for ε sufficiently small, one has*

$$\sigma_\varepsilon\cap(E-\varepsilon^\nu,E+\varepsilon^\nu)\neq\emptyset,\quad\forall E\in K.$$

2.8.2. *Location of the spectrum.* Let J be the energy interval defined by (2.16). By Lemma 2.1, the function $\Phi(E)$ is monotonically increasing on J and its derivative does not vanish there. In J, consider the points $(E^{(l)})_{l\in\mathbb{N}}$ defined by

$$\frac{1}{\varepsilon}\Phi(E^{(l)}) = \pi/2+\pi l, l\in\mathbb{N}. \tag{2.18}$$

The number of these points is of order $1/\varepsilon$; we denote the minimal and the maximal values of l for which (2.18) admits a solution in in J by L_1 and L_2. For sufficiently small ε, the distances between the points $(E^{(l)})_{l\in\mathbb{N}}$ satisfy the inequalities

$$c_1\varepsilon \le E^{(l)} - E^{(l-1)} \le c_2\varepsilon,\ l = L_1+1,\dots,L_2,$$

where c_1 and c_2 are two positive constant independent of ε. One has

Theorem 2.3. *There exists a collection of intervals $(I_l)_{L_1\le l\le L_2}$, $I_l\subset J$ such that, for $\varepsilon>0$ sufficiently small, one has*

- $\Sigma\cap J\subset\cup_{L_1\le l\le L_2} I_l$,
- *the interval I_l lies in an $o(\varepsilon)$-neighborhood of $E^{(l)}$,*
- *the measure of I_l satisfies*

$$|I_l| = 2\frac{\varepsilon(t_v(E^{(l)})+t_h(E^{(l)}))}{\Phi'(E^{(l)})}(1+o(1)). \tag{2.19}$$

Moreover, if $dN_\varepsilon(E)$ denotes the density of states measure of $H_{\phi,\varepsilon}$ at energy E, then, one has

$$\int_{I_l} dN_\varepsilon(E) = \frac{1}{2\pi}\varepsilon. \tag{2.20}$$

Note that the intervals $(I_l)_l$ are exponentially small and separated by distances of order $O(\varepsilon)$.

2.8.3. *The nature of the spectrum.* Set

$$\lambda(E) = t_v(E)/t_h(E),\quad \Delta S(E) = \varepsilon\log\lambda(E) = S_h(E) - S_v(E). \tag{2.21}$$

For $\delta>0$, define the sets

$$J_\delta^- = \{E\in J;\ \Delta S(E) < -\delta\} \text{ and } J_\delta^+ = \{E\in J;\ \Delta S(E) > \delta\}.$$

If $J_\delta^-\neq\emptyset$, then, for sufficiently small ε, the number of intervals I_l lying in J_δ^- is of order $O(1/\varepsilon)$. We prove

Theorem 2.4. *Pick $\sigma\in(0,1)$. There exists $D\subset(0,1)$, a set of Diophantine numbers such that*

- $$\frac{\text{mes}\,(D\cap(0,\varepsilon))}{\varepsilon} = 1+O\left(\varepsilon\lambda^\sigma\right)\ \textit{when}\ \varepsilon\to 0. \tag{2.22}$$
- *For $\varepsilon\in D$ sufficiently small, each of the intervals $I_l\subset J_\delta^-$ contains absolutely continuous spectrum, and for these intervals*

$$\frac{\text{mes}\,(I_l\cap\Sigma_{\text{ac}})}{\text{mes}\,(I_l)} = 1+o(1).$$

Here, Σ_{ac} is the absolutely continuous spectrum for the family of equations (1.1).

Let us now study the singular spectrum. As before, if $J_\delta^+\neq\emptyset$, then, for sufficiently small ε, the number of intervals I_l lying in J_δ^+ is of order $O(1/\varepsilon)$. We prove

Theorem 2.5. *For sufficiently small ε, each of the intervals $I_l\subset J_\delta^+$ contains only singular spectrum. Moreover, for $E\in J_\delta^+$, one has*

$$\Theta(E)\ge\frac{1}{2\pi}\left(S_h(E)-S_v(E)\right)+O(\varepsilon), \tag{2.23}$$

where Θ is the Lyapunov exponent for the family of equations (1.1)

Remark 2.1. In Theorems 2.4 and 2.5, we have fixed the value of δ independently of ε. The proofs show that one can actually take $\delta=\varepsilon^\alpha$, $0<\alpha<1$; in this case, the statements stay correct except that, in (2.22) and (2.23), the estimates of the error terms have to be corrected.

2.8.4. *Heuristics of the proof.* Let us outline the basic heuristics guiding the proof of our results. If we omit the error terms in (2.17) (i.e. the terms of lower order in ε), the monodromy matrix takes the form

$$M = M_0 + \lambda M_1, \qquad M_0 = \begin{pmatrix} -z_0 & iz_0 \\ -i\overline{z_0} & -\overline{z_0} \end{pmatrix}, \qquad M_1 = \lambda \begin{pmatrix} \zeta & -i\zeta \\ i\zeta^{-1} & \zeta^{-1} \end{pmatrix},$$
$$z_0 = \frac{1}{t_h} e^{\frac{i}{\varepsilon}\Phi}, \qquad \zeta = e^{2i\pi(\phi-\phi_0)}$$

where M_0 is a matrix with coefficients independent of ϕ, and M_1 is a first order trigonometric polynomial in ϕ. Now, consider the monodromy equation (2.5) for this matrix. The coefficient $\lambda = \lambda(E)$ plays the role of the coupling constant. If λ is small, one expects that the term λM_1 can be "omitted" using KAM theory ideas. In result, in this case, the spectrum should be absolutely continuous. If λ is large, Herman's idea ([26]) shows that the Lyapunov exponent of the solutions of the monodromy equation is positive. By Corollary 2.1, this implies the positivity of the Lyapunov exponent of the family (1.1). Hence, the spectrum is singular. Finally, since the coefficients of M_0 are much bigger than λ, and, since, by (2.3), the monodromy matrix is unimodular, the spectrum has to be located near the zeros of $\mathrm{Tr}(M_0)$, i.e. near the points where $\cos\left(\frac{\Phi(E)}{\varepsilon}\right) = 0$.

2.9. **The phase diagram.** The transitions, possible in view of Theorems 2.4 and 2.5, actually occur as we shall see now. Therefore, we study the dependence of $\Delta S(E) = S_v(E) - S_h(E)$ on E and on α. To underline the dependence on α, let us slightly change our notations and write $S_h(\alpha, E) = S_h(E)$, $S_v(\alpha, E) = S_v(E)$ and $\Delta S(\alpha, E) = \Delta S(E)$. The potential V is kept fixed. We work in the (α, E) domain

$$\mathcal{S} = \{(\alpha, E) \in]0, +\infty[\times\mathbb{R};\ E - \alpha \le E_1,\ E_1 \le E + \alpha \le E_2\}. \tag{2.24}$$

i.e. the domain where (1.4) is satisfied. We define $\mathcal{S}_\pm = \{(\alpha, E) \in \mathcal{S};\ \Delta S(\alpha, E) \gtrless 0\}$, $\mathcal{S}_0 = \{(\alpha, E) \in \mathcal{S};\ \Delta S(\alpha, E) = 0\}$, $\alpha^* := \frac{E_2 - E_1}{2}$ and $E^* := \frac{E_2 + E_1}{2}$.
The sets $\mathcal{S}_+$, $\mathcal{S}_0$ and $\mathcal{S}_-$ form a partition of $\mathcal{S}$. One has

Theorem 2.6. *Assume that V satisfy the assumptions described in the introduction. Then,*

- *$\mathcal{S}_0$ is a compact analytic curve joining the point (α^*, E^*) to some point on the half-line $E = E_1 - \alpha$, $\alpha > 0$.*
- *For any $t \in [-1, 1]$, $\mathcal{S}_0$ intersects the half-line $E = E_1 + t\alpha$ ($\alpha > 0$) at exactly one point in $\mathcal{S}$.*

The picture of the phase diagram one gets from Theorem 2.6 is essentially the one given in Fig. 4. Indeed, Theorem 2.6 implies that

- $\mathcal{S}_+$ is connected and there exists $\alpha_0 > 0$ such that $\mathcal{S} \cap \{\alpha < \alpha_0\} \subset \mathcal{S}_+$.
- $\mathcal{S}_-$ is connected and there exists $\alpha_1 > \alpha^* > 0$ such that $\mathcal{S} \cap \{\alpha > \alpha_1\} \subset \mathcal{S}_-$.

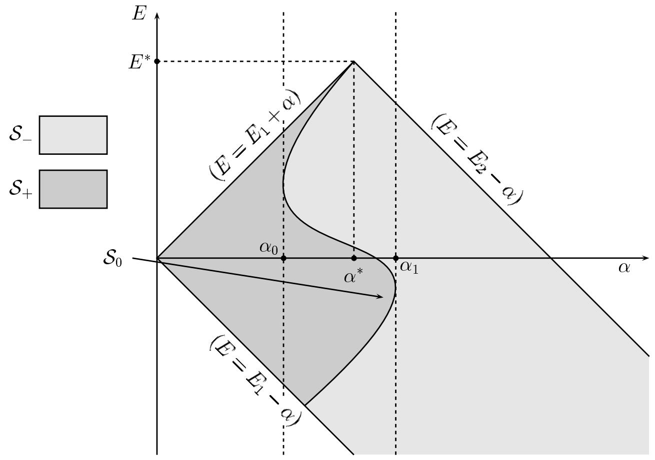

FIGURE 4. An example of phase diagram

From Theorems 2.4 and 2.6, we deduce that, for small α, for ε sufficiently small, most of the spectrum of $H_{\phi,\varepsilon}$ stays absolutely continuous (in the domain $\mathcal{S}$). This can be considered as an analogue of one of the results

of [13]; there, it is proven that, for a small quasi-periodic perturbation of $-\frac{d^2}{dx^2}$, the spectrum is purely absolutely continuous; and here, we are dealing with small perturbations of general periodic operators. For $\alpha > \alpha^*$, by Theorem 2.5, for ε sufficiently small, the spectrum is singular (in the domain $\mathcal{S}$). For $\alpha_0 < \alpha < \alpha^*$, there is at least one mobility edge.
At the point (α^*, E^*), $\mathcal{S}_0$ has the slope

$$\frac{d\alpha}{dE} = \frac{1 - c_{\text{eff}}}{1 + c_{\text{eff}}} \text{ where } c_{\text{eff}} = \sqrt{\frac{m_2}{m_1}}. \tag{2.25}$$

Here, $m_{1,2}$ are the effective masses defined by $|k'(E_i+\eta)| \sim \sqrt{m_i \cdot |\eta|}$, $i = 1, 2$, $|\eta| \ll 1$. By general estimates on the effective mass (see e.g. [32]), one always has $c_{\text{eff}} < 1$. Hence, the picture of the phase diagram that one gets near (α^*, E^*) is roughly the same as in Fig. 2. So, we see that, here, the singular spectrum grows with α for α close to α^*.
Note, that if, in (1.1), the cosine is replaced by a more general periodic potential, the derivative $\frac{d\alpha}{dE}$ can be negative. In this case, we have singular spectrum "coming from" the "center" of the spectral band of the unperturbed periodic operator.

2.10. **Outline of the paper.** In the next section, we prove Theorem 2.1 and therefore study the link between solutions of (1.1) and solutions of (2.5). Section 4 is devoted to the proof of the results stated in section 2.8 when assuming Theorem 2.2. Sections 5, 6, 7 and 8 are devoted to the proof of the asymptotics of the monodromy matrix, Theorem 2.2. In section 5, we recall some results on one dimensional periodic Schrödinger operators; in section 6, we describe the adiabatic complex WKB method. Section 7 is devoted to the description of geometrical and analytical objects of the adiabatic complex WKB method when applied to equation (1.1). In section 8, we compute the asymptotics of the monodromy matrix. In section 9, we discuss the properties of the phase and action integrals; we also prove the results on the phase diagrams. In section 10, following [40], we adapt Herman's argument to estimate the Lyapunov exponent in the case when $\Delta S(E) > 0$. And section 11 is devoted to a simple version of KAM needed to study the monodromy equation for small coupling.

3. Monodromy matrices and monodromy equation

In sections 2.1 and 2.1.2, we have introduced the monodromy matrix and the monodromy equation. We now prove

Theorem 3.1. *Let $\psi_{1,2}$ be consistent basis solutions of* (1.1), *and let M be the corresponding monodromy matrix. Fix $\phi \in \mathbb{R}$. Then, for any solution χ of equation* (2.5), *there exists a unique solution of* (1.1), *say f, such that*

$$\begin{pmatrix} f(x + 2\pi n/\varepsilon, \phi) \\ f'(x + 2\pi n/\varepsilon, \phi) \end{pmatrix} = \Psi(x, \phi - nh) \cdot \sigma \cdot \chi_{-n}, \quad \forall x \in \mathbb{R},\ n \in \mathbb{Z}, \tag{3.1}$$

where

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \frac{d\psi_1}{dx} & \frac{d\psi_2}{dx} \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad h = \frac{2\pi}{\varepsilon} \mod (1).$$

Reciprocally, for any f solution of (1.1), *there exists a unique vector χ solution of* (2.5) *satisfying* (3.1).

As the matrix $\Psi(x, \phi)$ is periodic in ϕ, bounded and unimodular, Theorem 3.1 immediately implies Theorem 2.1.

Proof. Let us recall some elementary facts on difference equations of the form (2.5) where the matrix M is unimodular. Together with vector solutions of (2.5), we also consider matrix solutions, i.e. sequences of 2×2-matrices $\{\Upsilon_n\}_{n\in\mathbb{Z}}$ such that $\Upsilon_{n+1} = M(\phi + nh)\Upsilon_n$.
A matrix solution Υ of equation (2.5) is called *fundamental* if and only if $\det \Upsilon_n \equiv 1$ for all $n \in \mathbb{Z}$. To construct a fundamental solution, we let $\Upsilon_0(\phi) = I$, $\forall \phi \in \mathbb{R}$, and then, define Υ_n for all n just by means of the equation

$$\Upsilon_{n+1}(\phi) = M\,(\phi + nh)\Upsilon_n(\phi), \quad \Upsilon_{n-1}(\phi) = M^{-1}(\phi + h(n-1))\Upsilon_n(\phi). \tag{3.2}$$

This is possible as $\det M \neq 0$. In result, one obtains a matrix solution of the difference equation (2.5). Since

$$\det \Upsilon_{n+1}(\phi) = \det M(\phi + nh) \det \Upsilon_n(\phi) = \det \Upsilon_n(\phi), \quad \forall n, \phi,$$

this matrix solution is fundamental.

Any vector solution of the monodromy equation, say χ, can be represented as

$$\chi_n = \Upsilon_n(\phi)\, p, \quad \forall n, \tag{3.3}$$

where p is a constant vector. Moreover, for any constant vector p, formula (3.3) gives a vector solution of the monodromy equation.

Now, we prove Theorem 3.1. The proof consists of several steps.
1. Let (ψ_1, ψ_2) be a consistent basis solutions of (1.1) and define $\psi = (\psi_1, \psi_2)^T$. Let $\Upsilon(\phi)$ be a fundamental solution of equation (2.5). We compute

$$\psi\,(x + 2\pi n/\varepsilon, \phi) = M\left(\phi - \frac{2\pi}{\varepsilon}\right) \cdots M\left(\phi - \frac{2\pi n}{\varepsilon}\right) \psi\,(x, \phi - 2\pi n/\varepsilon)$$
$$I = \Upsilon_0(\phi) = M\left(\phi - \frac{2\pi}{\varepsilon}\right) \cdots M\left(\phi - \frac{2\pi n}{\varepsilon}\right) \Upsilon_{-n}(\phi).$$

Hence, as $\phi \mapsto \psi(x, \phi)$ is 1-periodic, we get

$$\psi\,(x + 2\pi n/\varepsilon, \phi) = \Upsilon_{-n}^{-1}(\phi)\psi\,(x, \phi - nh). \tag{3.4}$$

2. Let Ψ be the matrix defined in Theorem 3.1. As $\det\Upsilon \equiv 1$, one has $(\Upsilon_l^{-1})^T = \sigma \Upsilon_l \sigma$; then, formula (3.4) implies the relation

$$\Psi(x + 2\pi n/\varepsilon, \phi) = \Psi(x, \phi - nh) \cdot \sigma \cdot \Upsilon_{-n}(\phi) \cdot \sigma, \tag{3.5}$$

We have obtained this formula for $n \geq 0$; one checks that it remains correct for $n < 0$.
3. Let χ be a vector solution of (2.5) for a given ϕ. Represent it in the form (3.3). Denote the components of the vector p in this representation by p_1 and p_2. The function

$$f(x, \phi) = -p_2\psi_1(x, \phi) + p_1\psi_2(x, \phi) \tag{3.6}$$

satisfies equation (1.1). Applying the left and the right hand sides of (3.5) to the vector $-\sigma p$, we get the relation (3.1). This proves the first statement of Theorem 3.1.
4 To prove the second statement, we represent the solution f as a linear combination of the linearly independent solutions $\psi_1(x, \phi)$ and $\psi_2(x, \phi)$ as in (3.6). Then, we let p be the vector $(p_1, p_2)^T$ and remark that $\chi_n = \Upsilon_n(\phi_0)p$ is a solution of (2.5). Applying (3.5) to the vector $-\sigma p$, we get (3.1).
This completes the proof of Theorem 3.1. $\square$

4. The spectral study

This section is devoted to the proofs of Proposition 2.1, and Theorems 2.3, 2.4 and 2.5 using the asymptotics of the monodromy matrix obtained in Theorem 2.2.

4.1. **Convergence to the asymptotic spectrum.**

We prove Proposition 2.1. The first statement of Proposition 2.1 follows immediately from regular perturbation theory (see e.g. [3, 38]).

We turn to the proof of the second statement. The spectrum of $H_{\phi,\varepsilon}$ is the same as the spectrum of equation $-\frac{d^2}{du^2}\psi(u) + (V(u) + \alpha\cos(\varepsilon u + \varphi))\psi(u) = E\psi(u)$, $\varphi = \varepsilon\phi$. Pick $\nu \in (0, 1)$ and $K \subset \Sigma$ a compact set. Pick $E \in K$. Then, there exists $\varphi \in [0, 2\pi)$ and $E_0 \in \sigma(H_0)$ such that $E = E_0 + \alpha\cos(\varphi)$. Hence, equation $H_0 u = Eu$ has a bounded (Bloch) solution, say u, continuous in $(x, E) \in \mathbb{R} \times \sigma(H_0)$ together with its derivative in x, see, for example, [41]. Pick $\chi \in \mathcal{C}_0^\infty(R)$ non negative and not identically vanishing. And set $\chi_\varepsilon(\cdot) = \varepsilon^{\nu/2}\chi(\varepsilon^\nu \cdot)$. One sets $u_\varepsilon = \chi_\varepsilon u$ and checks that

$$\left\| \left(-\frac{d^2}{dx^2} + V - E_0\right) u_\varepsilon \right\|_{L^2} \leq C\varepsilon^\nu.$$

The constant C is uniform in K. On the other hand, we compute

$$\|[\cos(\varepsilon \cdot + \varphi) - \cos(\varphi)]\, u_\varepsilon\|_{L^2} \leq C\varepsilon^{1-\nu}.$$

So we obtain $u \in L^2(\mathbb{R})$, $\|u\| = 1$ such that $\| -u'' + (V + \alpha\cos(\varepsilon\cdot+\varphi) - E)u\| \leq C(\varepsilon^\nu + \varepsilon^{1-\nu})$. Hence, either $E \in \sigma(H_{\phi,\varepsilon})$ or $\|(H_{\phi,\varepsilon} - E)^{-1}\| \geq C/(\varepsilon^\nu + \varepsilon^{1-\nu})$. This implies the second statement of Proposition 2.1. $\square$

4.2. **Location of the spectrum.** Here, we apply Theorem 2.2 to deduce the results of Theorem 2.3 on the location of the spectrum. We proceed as follows. First, we describe conditions guaranteeing that the monodromy equation has two linearly independent solutions rapidly decaying at infinity. Then, we use Theorem 3.1 to construct the corresponding solutions to equation (1.1). This describes the complement of the spectrum of (1.1).

4.2.1. *Bloch solutions of difference equations.* We need some results about solutions of difference equation of the form

$$\chi(\phi + h) = M(\phi)\,\chi(\phi), \quad \phi \in \mathbb{R}, \tag{4.1}$$

where h is a positive number and $M(\phi) \in \mathrm{SL}(2,\mathbb{C})$ is a given 1-periodic, matrix-valued function. The results we formulate below can be found in [7].

The set of the vector solutions of (4.1) is a two-dimensional module over the ring of h-periodic functions. For any two vector solutions χ_1 and χ_2 of (4.1), the determinant of the matrix (χ_1, χ_2) is h-periodic. The solutions χ_1 and χ_2 are linearly independent (over the ring of h-periodic functions) if and only if this determinant does not vanish. If χ_1 and χ_2 are linearly independent solutions, then, any other vector solution χ can be represented as a linear combination of χ_1 and χ_2 with h-periodic coefficients

$$\chi(\phi) = a(\phi)\chi_1(\phi) + b(\phi)\chi_2(\phi), \quad a(\phi + h) = a(\phi), \quad b(\phi + h) = b(\phi), \quad \phi \in \mathbb{R}.$$

If $\phi \mapsto \chi(\phi)$ is a solution of (4.1), so does $\phi \mapsto \chi(\phi + 1)$. Assume, moreover, that χ satisfies $\chi(\phi + 1) = u(\phi)\chi(\phi)$ for $\phi \in \mathbb{R}$ where $\phi \mapsto u(\phi)$ is an h-periodic function, then χ is called a *Bloch solution*, and the function u is its *Floquet multiplier.*
If either $|u(\phi)| \geq \text{Const} > 1$ for all $\phi \in \mathbb{R}$ or $|u(\phi)| \leq \text{Const} < 1$ for all $\phi \in \mathbb{R}$, the Bloch solution is called *monotonous.*

Remark 4.1. If $|u(\phi)| \leq C < 1$, then, the solution χ decays exponentially as $\phi \to +\infty$. Indeed, one has

$$|\chi(\phi + L)| = \prod_{l=0}^{L-1} |u(\phi + l)|\,|\chi(\phi)| \leq C^L |\chi(\phi)|.$$

If, in addition, χ is bounded on the interval $[0, 1[$, then $|\chi(\phi)| \leq C_1 e^{-C_2\cdot\phi}$ with some positive constants C_1 and C_2. Similarly, if $|u(\phi)| \geq C > 1$, then χ decays exponentially as $\phi \to -\infty$.

We finish this section by formulating a simple condition for the existence of the monotonous Bloch solutions.

Let $M = ((M_{ij}))_{1\leq i,j\leq 2}$ and set

$$\rho(\phi) = \frac{M_{12}(\phi)}{M_{12}(\phi - h)}, \quad v(\phi) = M_{11}(\phi) + \rho(\phi)\,M_{22}(\phi - h). \tag{4.2}$$

We also define

$$\rho_- = \inf_{\phi\in\mathbb{R}} |\rho(\phi)|, \quad \rho_+ = \sup_{\phi\in\mathbb{R}} |\rho(\phi)|, \quad v_- = \inf_{\phi\in\mathbb{R}} |v(\phi)|, \quad v_+ = \sup_{\mathbb{R}} |v(\phi)|. \tag{4.3}$$

For $f(\phi) \neq 0$ a periodic continuous function on $\mathbb{R}$, let $\mathrm{ind}\, f$ be the integer equal to the increment of $\dfrac{\arg f}{2\pi}$ over one period. Clearly, $\mathrm{ind} f$ is independent of the period chosen to define it. One has

Proposition 4.1 ([7]). *Assume that M is continuous in ϕ, and let $M_{12} \neq 0$. If*

$$\rho_- > 0, \quad \rho_+ < \left(\frac{v_-}{2}\right)^2, \tag{4.4}$$

$$\mathrm{ind}\, v = \mathrm{ind}\, \rho = 0, \tag{4.5}$$

then, equation (4.1) *has two monotonous Bloch solutions $\chi^\pm$ that satisfy* $\det(\chi^+, \chi^-) = 1$ *for all $\phi \in \mathbb{R}$, that are continuous in ϕ and such that the Floquet multipliers $u^\pm$ of $\chi^\pm$ satisfy the relation* $u^+(\phi)u^-(\phi) = 1$.

Remark 4.2. As $\det(\chi^+, \chi^-) = 1$, these solutions are linearly independent. As $u_+u_- = 1$, one of them decays at $+\infty$, and the other one decays at $-\infty$.

4.2.2. *Monotonous Bloch solutions and the spectrum of* (1.1). Fix an energy E. Let ψ_1 and ψ_2 form a consistent basis for the family of equations (1.1). Let $M(\phi)$ be the corresponding monodromy matrix. Here, instead of the monodromy equation (2.5), we consider its continuous analog (4.1). Clearly, if $\chi(\phi)$ is a vector solution of this equation, then the vectors

$$\chi_n = \chi(\phi + nh), \quad n \in \mathbb{Z}, \tag{4.6}$$

satisfy the monodromy equation (2.5). This and Theorem 2.1 imply

Lemma 4.1. *Let $\psi_{1,2}$ be locally bounded in $(x,\phi) \in \mathbb{R}^2$ together with their derivatives in x. If, for a given E, equation* (4.1) *has two linearly independent, locally bounded, monotonous Bloch solutions $\chi^\pm$, then E is in the resolvent set of $H_{\phi,\varepsilon}$ for any $\phi \in \mathbb{R}$.*

Proof. Fix $\phi \in \mathbb{R}$. By formula (4.6), using $\chi^\pm$, construct two vector solutions $\chi_n^\pm$ of the monodromy equation. By Theorem 3.1, construct $f^\pm$, two solutions of (1.1) satisfying the relation (3.1). Compute the Wronskian of $f^\pm$. By (3.1), one has

$$w(f^+, f^-) = \det \Psi(x, \phi - nh)\, \det \sigma\, \det \left(\chi^+_{-n}, \chi^-_{-n}\right) = \det \left(\chi^+(\phi - nh), \chi^-(x - nh)\right).$$

As $\chi^\pm(x)$ are linearly independent, the last determinant does not vanish, and $f^\pm$ are linearly independent solutions of (1.1).

As $\chi^\pm(x)$ are monotonous and locally bounded, then each of them exponentially decays either at $+\infty$ or at $-\infty$. As they are linearly independent, $\det(\chi^+, \chi^-)$ is a nonzero h-periodic function, and, therefore, one of them, say χ^+, is exponentially decaying at $+\infty$ and the second one, χ^-, is exponentially decaying at $-\infty$. So, we write $\|\chi^\pm\| \le C_1 e^{\mp C_2 x}$ $(C_1, C_2 > 0)$. This, and relations (4.6) and (3.1) imply that

$$|f^\pm(x)| \le C_3 e^{\pm C_4 x}, \quad C_3 = C_1 \cdot \left(\sup_{x\in[0,2\pi/\varepsilon],\phi\in[0,1]} \|\Psi(x,\phi)\| \right), \quad C_4 = \frac{h\varepsilon}{2\pi} C_2. \tag{4.7}$$

Here, we have used the fact that $\psi_{1,2}$ are 1-periodic in ϕ and locally bounded in (x,ϕ) together with their derivatives. As (1.1) has two linearly independent solutions $f^\pm$ satisfying estimates of the form (4.7), the energy E is in the resolvent set of $H_{\phi,\varepsilon}$ (see e.g. [9, 41]). □

4.2.3. *The functions ρ and v.* To study the functions ρ and v defined in (4.3) for the monodromy matrix described in Theorem 2.2, we discuss the coefficients a and b of the monodromy matrix M. Recall that they are 1-periodic and that we have decomposed them as $a = a_0 + \tilde a(\phi)$ and $b = b_0 + \tilde b(\phi)$ where a_0 and b_0 are the zeroth Fourier coefficients of a and b. The asymptotics of a_0, $\tilde a$, b_0 and $\tilde b$ are given in Theorem 2.2.
One proves

Lemma 4.2. *For E in a constant neighborhood of E_0, for ϕ in a constant neighborhood of $\mathbb{R}$ and for ε sufficiently small, one has*

$$\rho(\phi) = 1 - e^{-i\Phi/\varepsilon} t_v U(\phi)\,(1 - e^{-2\pi i h} + o\,(1)), \tag{4.8}$$

$$v(\phi) = F(E) + 2\lambda \cos(2\pi(\phi - \phi_0 - h)) + \lambda \cos\left(\frac{\Phi}{\varepsilon}\right) \cdot p(\phi, E) + o\,(\lambda e^{2\pi|\mathrm{Im}\,\phi_0|}), \tag{4.9}$$

$$p(\phi, E) = 2e^{-i\Phi/\varepsilon} U(\phi)(1 - e^{-2\pi i h} + o\,(1)), \tag{4.10}$$

where

$$F = a_0 + a_0^*, \quad U(\phi) = e^{2\pi i(\phi - \phi_0)}.$$

The asymptotics in (4.8) *and* (4.9) *are uniform and the error terms analytic in E.*

Proof. Using (2.17), we get

$$\tilde b / b_0 = -t_v e^{-i\Phi(E)/\varepsilon} U(\phi)(1 + o(1)), \quad \phi \in \mathbb{R}.$$

Note that this ratio is small in a constant neighborhood of E_0, and so, in this neighborhood, we can write

$$\rho = \frac{1 + \tilde b(\phi)/b_0}{1 + \tilde b(\phi - h)/b_0} = 1 + \left(\frac{\tilde b(\phi)}{b_0} - \frac{\tilde b(\phi - h)}{b_0}\right) + o\left(\frac{\tilde b(\phi)}{b_0}\right) + o\left(\frac{\tilde b(\phi - h)}{b_0}\right).$$

Substituting into this representation the asymptotics of $\tilde{b}(\phi)/b_0$, one obtains the asymptotics of ρ. The analyticity in E of the error term follows from the analyticity of the error terms in the monodromy matrix asymptotics. This completes the proof of (4.8).

Recall $\tilde{a}(\phi) = a(\phi) - a_0$. As $v(\phi) = a(\phi) + \rho(\phi)\, a^*(\phi - h)$, one has $v(\phi) = F + \tilde{a}(\phi) + (\rho(\phi) - 1)\, a^*(\phi - h) + \tilde{a}^*(\phi - h)$. Using (4.8) and the asymptotics (2.17), we obtain (4.9). This completes the proof of Lemma 4.2. □

The function $F(E) = a_0(E) + a_0^*(E)$ plays an important role in the spectral analysis of equation (1.1). We call it *the effective spectral parameter.* One has

Lemma 4.3. *The function F is real analytic in a constant neighborhood of E_0 and, there, it admits the following uniform asymptotic representation*

$$F(E) = -2\frac{1 + g_1(E)}{t_h(E)} \cos\left(\frac{1}{\varepsilon}\Phi(E) + g_2(E)\right), \quad g_{1,2} = o(1), \tag{4.11}$$

where g_1 and g_2 are real analytic, $g_{1,2}'(E), g_{1,2}''(E) = o(1)$.

Proof. The function F is real analytic. Theorem 2.2 implies that $a_0 = -\frac{e^{i\Phi/\varepsilon}}{t_h} e^{1+g(E)}$, g being analytic and satisfying the estimate $g = o(1)$ in a constant neighborhood of E_0. Let $\tilde{g}_1(E) = \frac{1}{2}(g(E) + \overline{g(\bar{E})})$, $\tilde{g}_2(E) = \frac{1}{2i}(g(E) - \overline{g(\bar{E})})$. These two functions are real analytic and admit the estimates $\tilde{g}_{1,2} = o(1)$. Clearly, $F(E) = -2\frac{e^{\tilde{g}_1(E)}}{t_h(E)} \cos(\Phi(E) - \tilde{g}_2(E))$. This implies (4.11). The estimates on the derivatives of $g_{1,2}$ follow, for example, from the Cauchy estimates for analytic functions. □

4.2.4. *Location of the spectrum up to exponentially small errors.* Now, we are ready to prove the part of the Theorem 2.3 concerning the geometry of the spectrum. Our plan is to describe the set R of values of the spectral parameter E for which the monodromy matrix described in Theorem 2.2 satisfies the hypothesis of Proposition 4.1. By Lemma 4.1, the spectrum of $H_{\phi,\varepsilon}$ lies in the complement of R.

Let V_0 be the constant neighborhood of E_0 where (2.17) and the Lemmas 4.2 and 4.3 are valid. We study the location of the spectrum of (1.1) in $J_0 = V_0 \cap J$. The asymptotics of ρ, v and F imply that the monodromy matrix described by Theorem 2.2 satisfies the assumptions of Proposition 4.1 if

$$\left|\cos\left(\frac{1}{\varepsilon}\Phi + g_2(E)\right)\right| \geq \mathrm{Const}(t_h + t_v), \tag{4.12}$$

where g_2 is the function from (4.11), and Const is independent of ε. Hence, the values of E satisfying this condition are outside of the spectrum of $H_{\phi,\varepsilon}$.

By point (4) of Lemma 2.1, one has

$$\frac{1}{\varepsilon}\Phi'(E) + g_2' \geq \frac{C}{\varepsilon}, \quad E \in J_0. \tag{4.13}$$

As the tunneling coefficients are exponentially small, inequality (4.13) implies that (4.12) is satisfied outside exponentially small intervals. These intervals are located in $o(\varepsilon)$-neighborhoods of the points $E^{(l)}$ defined by $\cos\Phi(E^{(l)}) = 0$; the distances between these points are of order ε. Near each $E^{(l)}$, there is exactly one of these intervals containing spectrum of $H_{\phi,\varepsilon}$. We call these intervals $(I_l)_{l\in L}$.

Let us discuss the intervals $(I_l)_{l\in L}$ in more details. Fix an index $l \in L$. Recalling the definitions of $t_v(E)$ and $t_h(E)$, one sees that, up to a factor $1 + o(1)$, these functions are constant as on I_l so in a $o(\varepsilon)$-neighborhood of $E^{(l)}$. By Lemmas 4.2 and 4.3, we have

$$\rho = 1 + O(t_v(E^{(l)})),$$
$$v = F(E) + 2\lambda(E^{(l)}) \cos(2\pi(\phi - \phi_0(E^{(l)}) - h)) + o(\lambda(E^{(l)})),$$
$$F(E) = -2\frac{1 + o(1)}{t_h(E^{(l)})} \cos\left(\frac{1}{\varepsilon}\Phi + g_2\right).$$

Now, we can describe the intervals $(I_l)_{l\in L}$ more precisely; instead of (4.12), they are defined by

$$\left|\cos\left(\frac{1}{\varepsilon}\Phi + g_2(E)\right)\right| \leq (t_h + t_v)(1 + o(1)),$$

where $o(1)$ only depends on ε. This implies that the length of the subinterval situated near $E^{(l)}$ is given by (2.19). Note that, any interval I_l contains precisely one zero of $F(E)$.

We have thus proved the statements of the Theorem 2.3 on the location of the spectrum in a constant interval $J_0 \subset \mathbb{R}$ containing E_0. As E_0 can be any point in J, this completes the proof of Theorem 2.3 up to the result on the integrated density of states. $\square$

4.3. **Calculation of the integrated density of states.** For $\varepsilon/\pi \notin \mathbb{Q}$, (1.1) is a metrically transitive family of equations (see e.g. [37]).

4.3.1. *Weyl solutions, Lyapunov exponents and the integrated density of states.* We begin with recalling general results on the Lyapunov exponent (see [37, 39]). Consider the family of quasi-periodic equations (1.1). For any fixed value of the spectral parameter E such that $\operatorname{Im} E > 0$, there is a unique solution to (1.1) such that $u^+(x) \in L^2(0, +\infty)$ and $u^+(0) = 1$; we call it *the Weyl solution* of (1.1). For almost all $\phi \in \mathbb{R}$, the Weyl solution satisfies the estimate

$$\sqrt{|u^+(x,E,\phi)|^2 + |du^+(x,E,\phi)/dx|^2} = e^{-x(\Theta(E)+o(1))}, \quad x \to +\infty, \tag{4.14}$$

where $\Theta(E)$ is positive, independent of ϕ and x. $\Theta(E)$ is the *Lyapunov exponent* of the family (1.1).
The Lyapunov exponent is the real part of a function $f: E \mapsto f(E)$ analytic in the upper half-plane $\operatorname{Im} E > 0$. For any $E \in \mathbb{R}$, the integrated density of states is the limit

$$N(E) = -\frac{1}{\pi} \lim_{\alpha \to +0} \operatorname{Im} f(E + i\alpha)$$

4.3.2. *More on Bloch solutions of difference equations.* Here, following [7], we discuss in more details the monotonous Bloch solutions described in Proposition 4.1. By Remarks 4.1 and 4.2, in the case of Proposition 4.1, equation (4.1) has a continuous Bloch solution $\chi(\phi)$ satisfying the estimate $\|\chi(\phi)\| \leq C_1 e^{C_2 \cdot \phi}$ $(C_{1,2} > 0)$.
This solution can be constructed in the following way. Together with (4.1), consider the finite difference *Ricatti* equation

$$G(\phi) + \frac{\rho}{G(\phi - h)} + v(\phi) = 0, \quad \phi \in \mathbb{R}, \tag{4.15}$$

where v and ρ defined by (4.2).
Under the condition (4.4), equation (4.15) has a continuous, 1-periodic solution that can be represented by the continued fraction

$$G(\phi) = v\,(\phi) - \cfrac{\rho(\phi)}{v\,(\phi - h) - \cfrac{\rho(\phi - h)}{v\,(\phi - 2h) - \cfrac{\rho(\phi - 2h)}{\cdots}}}. \tag{4.16}$$

This continued fraction converges uniformly in $\phi \in \mathbb{R}$. It satisfies

$$|G(\phi) - v(\phi)| < v_-/2 - \sqrt{\left(\frac{v_-}{2}\right)^2 - \rho_+}, \quad \phi \in \mathbb{R}, \tag{4.17}$$

where v_- and ρ_+ defined by (4.3).
For $\phi \in \mathbb{R}$, define $g(\phi) = \ln G(\phi)$. Condition (4.5) guarantees that the function g can be defined as a 1-periodic continuous function of ϕ. If v depends analytically of some parameter E in a simply connected domain, then g also is analytic in E.

Let us relate g to the Bloch solution χ. Consider a continuous solution (not necessarily periodic) of the homological equation

$$\lambda(\phi + h) - \lambda(\phi) = g(\phi).$$

The function $\chi_1(\phi) = e^{\lambda(\phi)}$ is the first component of χ, the solution of (4.1) mentioned above. Note that one has

$$G(\phi) = \frac{\chi_1(\phi + h)}{\chi_1(\phi)}. \tag{4.18}$$

For any irrational h, one has

$$\lambda(\phi) = (\theta + o(1))\frac{\phi}{h}, \quad \phi \to +\infty, \text{ where } \theta = \int_0^1 \ln G(\phi)\, d\phi, \ \operatorname{Re}\theta > 0. \tag{4.19}$$

4.3.3. *Calculation of the integrated density of states.* Let us now come back to equation (1.1).

1. We start with a description of the relation between the integrated density of states and the continued fraction G. Let Ω be a domain in the upper half plane of energies E. Assume that, on Ω, one can construct a consistent basis such that the corresponding monodromy matrix is analytic in $E \in \Omega$ and satisfies the conditions of Proposition 4.1. Then, for all $E \in \Omega$, one can construct the Bloch solution χ of (4.1) described in section 4.3.2. The vector $\chi(\phi + nh)$ satisfies the monodromy equation. Clearly, by Theorem 2.1, this solution corresponds to the Weyl solution u^+ of (1.1); the Lyapunov exponent Θ of (1.1) can be calculated by formula (2.7) with θ defined by (4.19). This leads to the formula

$$\Theta(E) = \frac{\varepsilon}{2\pi} \operatorname{Re} \int_0^1 \ln G(\phi, E)\, d\phi, \quad E \in \Omega. \tag{4.20}$$

where G is the continued fraction (4.16).
Since the Lyapunov exponent Θ is the real part of the analytic function $f(E)$, and G is analytic in $E \in \Omega$, one has

$$f(E) = \frac{\varepsilon}{2\pi} \int_0^1 \ln G(\phi, E)\, d\phi + C, \quad E \in \Omega,$$

where C is a constant independent of E. Therefore, if γ is a continuous curve in Ω connecting $a \in \mathbb{R}$ to $b \in \mathbb{R}$, then one has

$$N(b) - N(a) = -\frac{\varepsilon}{2\pi^2} \int_0^1 \arg G(\phi, E)\, d\phi \Big|_\gamma . \tag{4.21}$$

Here $f|_\gamma$ denotes the increment of f when going from a to b along γ.
We use formula (4.21) to compute the increments of the integrated density of states along the intervals I_l, $l = L_1, \dots L_2$, and, thus, prove (2.20).

2. Let us prove Theorem 2.3. Consider one of the intervals I_l, $L_1 \le l \le L_2$. Let E_* be the zero of $F(E)$ closest $E^{(l)}$. By section 4.2.4, we know that

$$E_* - E^{(l)} = o(\varepsilon), \tag{4.22}$$

that the interval I_l contains E_*, and that the length of I_l is exponentially small, $|I_l| \le e^{-C/\varepsilon}$.
We fix two constants c_1 and c_2 so that $1 < c_2 < c_1$ and choose

$$\Omega = \{E \in \mathbb{C} \,|\, \operatorname{Im} E \ge 0, \ \varepsilon^{c_1} < |E - E_*| < \varepsilon^{c_2}\}.$$

As I_l is exponentially small, one has $\Omega \cap I_l = \emptyset$. Moreover, all the other intervals containing some spectrum of $H_{\phi,\varepsilon}$ are either to the right or to the left of Ω. So, Ω is in the resolvent set of (1.1).

Pick a and b in $\Omega \cap \mathbb{R}$, $a < E_* < b$ and let γ be the semi-circle connecting a and b in Ω. To prove the statement of Theorem 2.3 on the integrated density of states, it suffices to check the formula

$$N(b) - N(a) = \frac{\varepsilon}{2\pi}. \tag{4.23}$$

To prove (4.23), we first check

Lemma 4.4. *For $E \in \Omega$, the monodromy matrix described in Theorem 2.2 satisfies the conditions of Proposition 4.1.*

So, we can calculate $N(b) - N(a)$ using (4.21). To compute the right hand side of (4.21), we use

Proposition 4.2. *For the monodromy matrix from Theorem 2.2, one has*

$$\int_0^1 \ln G(\phi, E) d\phi \Big|_\gamma = \ln F(E)|_\gamma . \tag{4.24}$$

As F is real analytic, one has $\ln F(E)|_\gamma \in \pi\mathbb{Z}$. To get the actual value of $\ln F(E)|_\gamma$, we use

Lemma 4.5. *Fix $c > 0$. Then, for ε sufficiently small, one has*

- $$F'(E_*) \sim \pm 2\frac{\Phi'(E_*)}{\varepsilon t_h(E_*)}, \tag{4.25}$$

  where the sign $\pm$ depends only on I_l;
- $$F''(E) = O\left(\frac{1}{\varepsilon^2 t_h(E_*)}\right), \text{ uniformly for } |E - E^*| \leq \varepsilon^c. \tag{4.26}$$

By Lemma 2.8, one has $\Phi' \geq \text{Const} > 0$ on J; hence, Lemma 4.5 implies that, on γ, $F(E) = F'(E_*)(E - E_*)(1+o(1))$. Therefore, $\ln F(E)|_\gamma = -\pi + o(1)$. As $\ln F(E)|_\gamma \in \pi\mathbb{Z}$, one has $\ln F(E)|_\gamma = -\pi$. This, (4.21) and Proposition 4.2 imply (4.23). This completes the proof of Theorem 2.3. □

3. We now prove Lemmas 4.4, 4.5 and Proposition 4.2.

Proof of Lemma 4.5. Formulas (4.25) and (4.26) follow from the representation (4.11). Clearly, for $E = E_*$, one has $\cos(\Phi + g_2) = 0$, and $\sin(\Phi + g_2) = \pm 1$. Therefore, one has $F'(E_*) = \pm 2\frac{1+g_1}{t_h}(\Phi'/\varepsilon + g_2')\Big|_{E=E_*}$, and as, by Lemma 2.1, $\Phi'(E_*) > 0$, we get (4.25). To estimate F'', one uses Lemma 4.3 and the estimates $\Phi'(E)$, $\Phi''(E) = O(1/\varepsilon)$, $t_h'(E) = O(t_h/\varepsilon)$ and $t_h''(E) = O(t_h/\varepsilon^2)$ which follow from the definitions of the phase integral and the tunneling coefficients. □

Proof of Lemma 4.4. Therefore, we need the following immediate corollary of Lemma 4.5

Corollary 4.1. *There exists $C > 0$ such that, for ε sufficiently small,*

$$\frac{1}{C}\frac{\varepsilon^{c_1-1}}{t_h(E_*)} \leq |F| \leq C\frac{\varepsilon^{c_2-1}}{t_h(E_*)}, \quad E \in \Omega. \tag{4.27}$$

Now, let us check that, for $E \in \Omega$, the monodromy matrix described in Theorem 2.2 satisfies the assumptions of Proposition 4.1. By Lemma 4.2, uniformly in $E \in \Omega$, one has

$$\rho = 1 + O(t_v(E_*)), \quad v = F(E) + O(\lambda(E_*)). \tag{4.28}$$

The last formula for v and Corollary 4.1 imply that

$$\frac{v - F}{F} = O\left(t_v(E_*)\varepsilon^{1-c_2}\right), \quad E \in \Omega. \tag{4.29}$$

So, as t_v is exponentially small, $(v - F)/F$ is small. We check the assumptions of Proposition 4.1:

- by (4.28), the index of the periodic function ρ is zero;
- by (4.29), one has $\operatorname{ind} v = \operatorname{ind}(F[1 + (v - F)/F]) = \operatorname{ind}(1 + (v - F)/F) = 0$; here, we have used the fact that F is independent of ϕ;
- by (4.28), one has

$$\rho_\pm = 1 + O(t_v(E_*)), \quad E \in \Omega; \tag{4.30}$$

  by (4.29) and Corollary 4.1, one has

$$v_- = |F| \min_\phi |1 + (v - F)/F| \geq C\frac{\varepsilon^{c_1-1}}{t_h(E_*)}, \quad E \in \Omega; \tag{4.31}$$

  therefore, $0 < \rho_-$ and $\rho_+ < \left(\frac{v_-}{2}\right)^2$.

So the assumptions of Proposition 4.1 are fulfilled, and, thus, the solution χ described in section 4.3.2 exists for all $E \in \Omega$. □

Proof of Proposition 4.2. The proof is done in two steps:

1. we check that $G/F = 1 + o(1)$ in Ω; hence, one can choose a continuous branch of $\ln G/F$ so that $\ln G/F = o(1)$ in Ω;
2. for this branch, we prove that $\int_0^1 \ln[G(\phi, E)/F(E)]d\phi$ is real as $E \in \Omega \cap \mathbb{R}$.

This obviously implies Proposition 4.2.

To check step 1, we write $G/F = (1+(v-F)/F)\cdot(1+(G-v)/v)$. We have already seen that $(v-F)/F = o(1)$ in Ω. The estimate $(G - v)/v = o(1)$ follows from (4.17) and (4.30)–(4.31). This implies point 1.

To check step 2, we use the self-adjointness of (1.1). It implies a relation between $G(\phi,E)$ and $G^*(\phi,E)$ that we now derive.

Note that, as the monodromy matrix is of the form (2.14), one has $\sigma M^*\sigma = M$, where $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, if χ is a solution of (4.1), so is $\sigma\chi^*$. Recall that the determinant of the matrix composed of two vector solutions of (4.1) is h-periodic (see section 4.2.1). As both χ and $\sigma\chi^*$ decay as $\phi\to-\infty$, the determinant $\det(\chi(\phi),\sigma\chi^*(\phi))$ must vanish identically. Therefore, χ and $\sigma\chi^*$ are linearly dependent over the ring of h-periodic functions (see section 4.2.1) i.e. there exists γ, an h-periodic function, such that $\chi(\phi)=\gamma(\phi)\,\sigma\chi^*(\phi)$. Denote the components of the vector χ by χ_1 and χ_2. Then, one has

$$\chi_1(\phi+h)/\chi_1(\phi) = \chi_2^*(\phi+h)/\chi_2^*(\phi). \tag{4.32}$$

Moreover, as χ satisfies equation (4.1), its first component is related to its second component by the formula

$$\chi_2(\phi) = (\chi_1(\phi+h) - a(\phi)\chi_1(\phi))/b(\phi),$$

where a and b are the coefficients of the monodromy matrix M. This and (4.18) imply that

$$\chi_2(\phi+h)/\chi_2(\phi) = \frac{b(\phi)}{b(\phi+h)}\,\frac{G(\phi+h)-a(\phi+h)}{G(\phi)-a(\phi)}\,G(\phi).$$

Substituting this formula and formula (4.18) into (4.32) and applying the operation *, we get the relation

$$G^*(\phi) = G(\phi)\,\frac{b(\phi)}{b(\phi+h)}\,\frac{G(\phi+h)-a(\phi+h)}{G(\phi)-a(\phi)}. \tag{4.33}$$

This relation implies point (2). Indeed, as F is real analytic, we can rewrite (4.33) as

$$\left(\frac{G}{F}\right)^* = \left(\frac{G}{F}\right)\frac{K(\phi+h)}{K(\phi)}, \quad K(\phi) = \frac{1-(G(\phi)-\tilde a(\phi))/a_0}{1+\tilde b(\phi)/b_0}. \tag{4.34}$$

Here, a_0, b_0 $\tilde a$, and $\tilde b$ are as in (2.15). It is easy to see that $K(\phi)=1+o(1)$; indeed, from the proof of Lemma 4.2, we get $\tilde b(\phi)/b_0 = o(1)$; from the proof of Lemma 4.2, we get $\tilde a(\phi)/a_0 = o(1)$; and, at last, from Corollary 4.1, using the asymptotics of a_0, we obtain $G/a_0 = G/F\cdot F/a_0 = (1+o(1))F/a_0 = o(1)$. Pluging this into (4.34), we get $K(\phi)=1+o(1)$. Furthermore, as, by step 1, one has $G/F = 1+o(1)$, for the branch of ln such that $\ln 1 = 0$, one has $\ln(G/F)^* = \ln(G/F)+\ln(K(\phi+h)) - \ln K(\phi)$. Integrating this relation in ϕ from zero to one and taking into account the 1-periodicity of $\ln K$, we complete step 2. This also completes the proof of Proposition 4.2. ☐

4.4. **Absolutely continuous spectrum.** We now prove Theorem 2.4. Therefore, we assume that the energy E belongs to an interval I_l where λ is small. We consider equation (4.1) where the monodromy matrix M is described in the Theorem 2.2. The aim is to put this equation into the form required to apply Proposition 11.1. We proceed as follows. We write the monodromy matrix as

$$M = M_0 + M_1(\phi), \quad M_0 = \begin{pmatrix} a_0 & b_0 \\ b_0^* & a_0^* \end{pmatrix}, \quad M_1 = \begin{pmatrix} \tilde a & \tilde b \\ \tilde b^* & \tilde a^* \end{pmatrix} \tag{4.35}$$

where a_0 and b_0 are zeroth Fourier coefficients of $a(z)$ and $b(\phi)$. The matrix M_0 is constant; for $\phi\in\mathbb{R}$ and $E\in I_l$, the matrix M_1 is of size order λ, see Theorem 2.2. Thus, it is small. We show that, on an essential part of I_l, M_0 can be diagonalized,

$$M_0 = PDP^{-1},$$

where P is a constant matrix, and D is a diagonal one. We represent a solution of (4.1) in the form $\psi = P\Psi$. Then, Ψ satisfies the equation

$$\Psi(\phi+h) = (D+\lambda_* A)\Psi(\phi), \quad A = \frac{1}{\lambda_*}PM_1(\phi)P^{-1} \tag{4.36}$$

with λ_* being a value of $\lambda(E)$ on I_l. We manage to choose P so that A be of order $O(1)$. By means of Proposition 11.1, we construct bounded solutions of (4.36), and then we use Corollary 2.1 and Ishii-Pastur-Kotani's Theorem (see e.g. [37, 8, 10]) to complete the proof of Theorem 2.4.

Let $E_0\in J_\delta^-$, so that $\lambda(E)$ be small in V_0, a constant neighborhood of E_0. We consider $I_l\subset V_0$. Let E_* be the (unique) zero of the function $F(E)$ located in I_l and let $\lambda_* = \lambda(E_*)$.

4.4.1. *Diagonalization of the monodromy matrix.* To check the assumptions of Proposition 11.1, we need that D and A belong to $\mathcal{M}$, i.e. $\sigma D^*\sigma = D$ and $\sigma A^*\sigma = A$, where $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Here, for matrices A, the mapping $A \mapsto A^*$ is defined in the same way as for functions (see section 2.7). As M_0 and M_1 belong to $\mathcal{M}$, it suffices to choose $P \in \mathcal{M}$. Assume that ν is an eigenvalue of M_0, and let p be a corresponding eigenvector. Then, σp^* is also an eigenvector of M_0 and ν^* is the corresponding eigenvalue. The matrix $P = (p, \sigma p^*)$ belongs to $\mathcal{M}$.

We take

$$p = \gamma \begin{pmatrix} b_0 \\ \nu - a_0 \end{pmatrix}, \quad \gamma = \gamma(E), \quad \gamma* = \frac{1}{\gamma}. \tag{4.37}$$

The function γ will be chosen later. With this choice of p, one checks that

$$\det P = -(a_0 - \nu)(\nu - \nu^*), \tag{4.38}$$

and, if $\det P \neq 0$, then, the matrix M_0 is diagonalizable , and

$$D = \mathrm{Diag}\,(\nu, \nu^*), \quad A = A_1 + A_2, \tag{4.39}$$

$$A_1 = \frac{1}{\lambda_* \det P} P_0 M_1 \check{P}_0, \quad A_2 = \frac{1}{\lambda_* \det P} \left(\Upsilon M_1 \check{P}_0 - P_0 M_1 \Upsilon - \Upsilon M_1 \Upsilon \right), \tag{4.40}$$

where

$$P_0 = \begin{pmatrix} \gamma b_0 & -\gamma^* a_0^* \\ -\gamma a_0 & \gamma^* b_0^* \end{pmatrix}, \quad \check{P}_0 = \begin{pmatrix} \gamma^* b_0^* & \gamma^* a_0^* \\ \gamma a_0 & \gamma b_0 \end{pmatrix}, \quad \text{and} \quad \Upsilon = \begin{pmatrix} 0 & \gamma^* \nu^* \\ \gamma \nu & 0 \end{pmatrix}.$$

4.4.2. *The eigenvalues of M_0.* Clearly, one has $\nu^2 - \mathrm{tr}\, M_0 \nu + \det M_0 = 0$. The trace of M_0 is equal to $F(E)$. Let $d(E)$ be the determinant of M_0. Then, one has

Lemma 4.6. *The function $d(E)$ is real analytic in V, a constant neighborhood of E_0; moreover, for $E \in V$, one has*

$$d(E) = 1 + o(\lambda(E)^2), \quad d'(E) = o(\lambda(E)^2/\varepsilon) \quad \text{and} \quad d''(E) = o(\lambda(E)^2/\varepsilon^2). \tag{4.41}$$

Proof. The real analyticity of d is obvious. As $\det M \equiv 1$, the representation (4.35) implies that

$$d(E) + \left[\tilde{a}\, \tilde{a}^* - \tilde{b}\, \tilde{b}^* \right]_0 = 1,$$

where $[f]_0$ denotes the zeroth Fourier coefficient of the periodic function f. By (2.17), $\tilde{a} \sim \lambda U$, and $\tilde{b} \sim -i\lambda U$. This implies the asymptotics of d. To estimate the first and the second derivatives of d, we note that the asymptotics of d can be rewritten in the form $d(E) = 1 + o(1)\lambda(E)^2$ with where $o(1)$ is analytic in E. The Cauchy estimates imply that, in a (smaller) constant neighborhood of E_0, one has $\frac{do(1)}{dE}, \frac{d^2 o(1)}{dE^2} = o(1)$. This implies (4.41), and completes the proof of Lemma 4.6. ☐

To study the eigenvalues of M_0, we use

Lemma 4.7. *Let $f(E) = \frac{F(E)}{\sqrt{d(E)}}$ where the branch of the square root is fixed by the condition $\sqrt{d(E_*)} \sim 1$. Fix $c > 1$ and $C > 0$. One has*

1. *for $|E - E_*| \leq \varepsilon^c$, one has $f'(E) = F'(E_*)(1 + o(1))$ when $\varepsilon \to 0$;*
2. *for $|w| \leq C$, the equation $f(E) = w$ has only one solution in the domain $|E - E_*| < \varepsilon^c$;*
3. *this solution satisfies $E - E_* = \frac{w}{F'(E_*)}(1 + o(1))$ when $\varepsilon \to 0$.*

Proof. Point 4 of Lemma 2.1, Lemmas 4.5 and 4.6 imply that

$$f'(E) \sim F'(E_*) \text{ and } f''(E) = O\left(\frac{1}{\varepsilon^2 t_h(E_*)} \right) \text{ for } |E - E_*| \leq \varepsilon^c. \tag{4.42}$$

The first of the above two estimates is just point 1 of Lemma 4.7.

Assume that $|E - E_*| = \varepsilon^c$ and $|w| \leq C$. Using (4.42) and (4.25), we obtain

$$\left| \frac{f(E) - f'(E_*)(E - E_*)}{f'(E_*)(E - E_*) - w} \right| = O\left(\varepsilon^{c-1} \right).$$

Applying Rouché's Theorem to the functions $f(E) - w$ and $f'(E_*)(E - E_*)$, we see that, in the disc $|E - E_*| \leq \varepsilon^c$, the function $f(E) - w$ has a single root and that this root is simple. This proves point 2 of Lemma 4.7.

To get the asymptotics of the root of $f(E) - w$ in the disc $|E - E_*| \leq \varepsilon^c$, we let $E_w = E_* + \frac{w}{f'(E_*)}$ and pick $\alpha \in (0, 1)$. On the circle $|E - E_w| = \alpha|E_w - E_*|$, one has

$$\begin{aligned}\left|\frac{f(E) - f'(E_*)(E - E_*)}{f'(E_*)(E - E_*) - w}\right| &\leq \frac{\max_{|E-E_*|\leq\varepsilon^c} |f''(E)|}{2|f'(E_*)|} \frac{|E - E_*|^2}{|E - E_w|} \leq \frac{C}{\varepsilon} \frac{|E - E_*|^2}{|E - E_w|} \\ &\leq \frac{C(1 + \alpha)^2}{\varepsilon\alpha}|E_w - E_*| = \frac{C(1 + \alpha)^2}{\varepsilon\alpha}\left|\frac{w}{f'(E_*)}\right| \leq C\frac{t_h(E_*)}{\alpha}.\end{aligned}$$

Here, C denotes constants that are independent of ε and α. We have used (4.42), (4.25), and Lemma 2.1. If $t_h(E_*) = o(\alpha)$, then, the right hand side is small; so, by Rouché's Theorem, the root of $f(E) - w$ in the disc $|E - E_*| \leq \varepsilon^c$ is inside the disc $|E - E_w| \leq \alpha|E_w - E_*|$. This gives the following estimate for this root

$$\left|E - E_* - \frac{w}{f'(E_*)}\right| = |E - E_w| < \alpha|E_w - E_*| = \alpha\left|\frac{w}{f'(E_*)}\right|.$$

Now, as α can be taken, say, of order of ε, this implies point 3 of Lemma 4.7, and completes the proof of Lemma 4.7. $\square$

The eigenvalues of M_0 are equal to $\frac{1}{2}F(E) \pm \sqrt{G(E)}$, where $G(E) = \left(\frac{F(E)}{2}\right)^2 - d(E)$. As we can rewrite $G(E) = \frac{d(E)}{4}(f(E) - 2)(f(E) + 2)$, Lemma 4.7 implies that

- on the interval $[E_* - \varepsilon^c, E_* + \varepsilon^c]$, the function $G(E)$ has two simple roots $E_{*,\pm}$ described by the equations $f(E_{*,\pm}) = \pm 2$; for the sake of definiteness, we set $E_{*,-} < E_{*,+}$.
- $E_{*,\pm}$ admit the asymptotics

$$E_{*,\pm} = E_* \pm \frac{2}{|F'(E_*)|}(1 + o(1)); \tag{4.43}$$

- in the interior of $I_* := [E_{*,-}, E_{*,+}]$, the function G is negative;
- outside of I_*, the function G is positive.

Let us discuss the eigenvalues of M_0 on the interval I_*. The third property of G implies, that, in I_*, they are complex and complex conjugate of each other. So, if ν is one of them, the second is equal to ν^*. As $\nu\nu^* = 1$, and d is positive on $I_* \subset J$, one gets

$$\nu(E) = \sqrt{d(E)}e^{i\eta}, \quad \nu^* = \sqrt{d(E)}e^{-i\eta}, \tag{4.44}$$

where $\sqrt{d(E)} > 0$, and $\eta = \eta(E)$ is real.

As $\dfrac{\sqrt{-G}}{F/2} = \dfrac{\sqrt{4 - f^2}}{f}$, for $E \in I_*$, we can choose

$$\eta(E) = -\arctan\left(\frac{\sqrt{4 - f^2(E)}}{f(E)}\right) \operatorname{sign} f'(E_*).$$

Here, the square root is positive, and the branch of arctan is chosen so that, for $E \in I_*$, one has $0 \leq \eta(E) \leq \pi$. With this choice, $\eta(E)$ is continuous in $E \in I_*$.
One computes

$$\eta'(E) = \frac{f'(E)}{\sqrt{4 - f^2(E)}} \operatorname{sign} f'(E_*).$$

Hence, $\eta(E)$ is monotonously increasing on I_* from 0 to π.

Finally, we note that, the relation $F(E) = \nu + \nu^*$ implies that

$$\cos\eta(E) = f(E). \tag{4.45}$$

4.4.3. *A new parameterization of the monodromy matrix.* As $d = 1 + O(\lambda^2)$, on the interval I_*, the matrix D admits the representation

$$D = D_0 + D_1, \quad D_0 = \begin{pmatrix} e^{i\eta(E)} & 0 \\ 0 & e^{-i\eta(E)} \end{pmatrix}, \quad D_1 = o(\lambda_*^2). \tag{4.46}$$

Proposition 11.1 is applied to equation (11.3) with matrices D and A depending on a parameter η (see (11.2)). So, we shall consider the monodromy matrix as a function of η. As $\eta(E)$ is monotonous, we can introduce the inverse function $E(\eta)$ and consider all the objects as functions of η. Let us study $E(\eta)$. Clearly, it is a monotonous continuous function mapping the interval $[0, \pi]$ onto the interval I_*. Furthermore, (4.45) implies

Lemma 4.8. *The function $E(\eta)$ can be analytically continued to $V_{(0,\pi)}$, a constant neighborhood of the interval $(0, \pi)$; and there exists $C > 0$ such that, in $V_{(0,\pi)}$, one has*

$$|E(\eta) - E_*| \leq C\varepsilon\, t_h(E_*), \ \ \text{and}\ E'(\eta) \sim -\frac{\sin\eta}{F'(E_*)}. \tag{4.47}$$

Proof. Fix $C > 0$. Let $D_C = \{|E - E_*| \leq C\,\varepsilon\, t_h(E_*)\}$. Lemma 4.7 and formula (4.25) imply that

- there exists $C_1 > 0$ such that, in the domain D_C, one has $|f(E)| \leq C_1$ for ε sufficiently small;
- f bijectively maps D_C onto its image;
- if C is large enough then, $f(D_C)$ contains the interval $(0, \pi)$;
- for $E \in D_C$, one has $f'(E) \sim F'(E_*)$.

The function $E(\eta)$ can be constructed as $f^{-1}(\cos\eta)$, where f^{-1} is the inverse to f. The information collected on f implies Lemma 4.8. □

We finish this section with three immediate corollaries of Lemma 4.8.

Corollary 4.2. *The eigenvalues ν and ν^* of the matrix M_0 are analytic in η in $V_{(0,\pi)}$; for $\eta \in V_{(0,\pi)}$, one has*

$$\nu = e^{i\eta} + o(\lambda_*^2), \quad \nu^* = e^{-i\eta} + o(\lambda_*^2).$$

Proof. The statements is a consequence of (4.44) and Lemmas 4.8 and 4.6. □

Corollary 4.3. $\det P$ *and* $\sin\eta/\det P$ *are analytic in* $\eta \in V_{(0,\pi)}$; *for* $E \in V_{(0,\pi)}$, *one has*

$$\det P \sim \ 2i\frac{e^{i\Phi(E_*)/\varepsilon}}{t_h(E_*)}\sin\eta.$$

Proof. Corollary 4.3 is a consequence of formulas (4.38) and (4.44), Lemma 4.8 and the asymptotics of a_0. □

Corollary 4.4. *The matrix* $D_1 = D - \mathrm{Diag}\left(e^{i\eta}, e^{-i\eta}\right)$ *is analytic in* $V_{(0,\pi)}$; *in* $V_{(0,\pi)}$, *one has* $D_1 = O(\lambda_*^2)$.

Proof. Corollary 4.4 is a consequence of Lemma 4.8 and Corollary 4.2. □

4.4.4. *Estimates of the matrices A_1 and A_2.* Recall that the matrices A_1 and A_2 are defined in (4.40). We now estimate them assuming that $(\phi, \eta) \in R$, where $R = \{|\mathrm{Im}\,\phi| \leq r\} \times V_{(0,\pi)}$. Here, r is a fixed constant, independent of ε.

Trying to estimate A_1, there is one difficulty. Indeed, note that, in R, one has

$$|a_0|, |b_0| \sim 1/t_h(E_*), \quad |\tilde{a}|, |\tilde{b}| \sim \lambda_*.$$

These estimates are a consequence of Theorem 2.2 and Lemma 4.8. Furthermore, by Corollary 4.3, one has $\left|\dfrac{1}{\det P}\right| \sim \left|\dfrac{t_h(E_*)}{2\sin\eta}\right|$. So, even if $|\gamma| \leq$ Const, straightforward norm estimates yield only that $\|A_1\| = O\left(\dfrac{1}{t_h(E_*)\cdot\sin\eta}\right)$. Thus, the norm of $\lambda_* A_1$ is not necessarily small.

To overcome this difficulty, we recall that we still have one free parameter in this construction, namely γ (see (4.37)). Choose

$$\gamma = -\frac{b_0^*}{b_0}. \tag{4.48}$$

Let us show that, for this choice, one has

$$\|A_1\| = O\left(\frac{1}{\sin\eta}\right), \quad (\phi,\eta) \in R. \tag{4.49}$$

As $A_1 \in \mathcal{M}$, it suffices to estimate its coefficients $(A_1)_{11}$ and $(A_1)_{12}$. One computes

$$\begin{aligned}
(A_1)_{11} &= \frac{1}{\lambda_* \det P}\left\{a_0(a_0^*\tilde{a} + a_0\tilde{a}^* - b_0^*\tilde{b} - b_0\tilde{b}^*) + F\,(b_0\tilde{b}^* - a_0\tilde{a}^*) - d\,\tilde{a},\right\},\\
(A_1)_{12} &= \frac{1}{\lambda_* \det P}\left\{b_0(a_0^*\tilde{a} + a_0\tilde{a}^* - b_0^*\tilde{b} - b_0\tilde{b}^*) + F\,(\frac{b_0}{b_0^*}\,a_0^*\tilde{b}^* - b_0\tilde{a}^*) - d\,\frac{b_0}{b_0^*}\tilde{b}^*,\right\},
\end{aligned}$$

where $F = a_0 + a_0^*$, and $d = a_0 a_0^* - b_0 b_0^*$. Note that, by Lemma 4.6 and Lemma 4.8, for $\eta \in V_{(0,\pi)}$, one has $d \sim 1$, and $F \sim \cos\eta$ (as $F = \nu + \nu^* = \sqrt{d(E)}\cos\eta$). So, to prove (4.49), it suffices to show that

$$a_0^*\tilde{a} + a_0\tilde{a}^* - b_0^*\tilde{b} - b_0\tilde{b}^* = O(\lambda_*). \tag{4.50}$$

As M is 1-periodic, the equality $\det M \equiv 1$ implies that all the non constant terms of the Fourier series of $\det M$ vanish. Therefore, one has

$$a_0\tilde{a}^* + a_0^*\tilde{a} - b_0\tilde{b}^* - b_0^*\tilde{b} = \left\{\tilde{a}\tilde{a}^* - \tilde{b}\tilde{b}^*\right\},$$

where $\{f\}$ denotes the sum of all non constant terms of the Fourier series of a periodic function f. By (2.17), in R, $\tilde{a}$ and $\tilde{b}$ are $O(\lambda_*)$. This implies (4.50), hence, (4.49).

Estimating A_2 is straightforward; one just uses norm estimates on the matrices composing A_2 to get

$$\|(A_2)\| \le O\left(\frac{1}{\sin\eta}\right). \tag{4.51}$$

So, we have proved

Lemma 4.9. *If γ is given by formula* (4.48), *then* $\sin\eta \cdot A$ *is analytic and bounded in* $(\phi,\eta) \in R$.

Proof. As $A = A_1 + A_2$, the boundedness follows from (4.49) and (4.51), and the analyticity from Corollary 4.3. ☐

4.4.5. *Bounded solutions of the monodromy equation.* Let us summarize the above results to check that one can apply Proposition 11.1 to the "continuous" monodromy equation (4.1) with the monodromy matrix described in Theorem 2.2. We have transformed the "continuous" monodromy equation into

$$\Psi(\phi + h) = \left(D_0 + \frac{\lambda_*}{\sin\eta}\tilde{A}(\phi)\right)\Psi(\phi) \tag{4.52}$$

with

$$D_0 = \mathrm{Diag}\,(e^{i\eta}, e^{-i\eta}), \quad \tilde{A} = \sin\eta\,(A + D_1/\lambda_*), \tag{4.53}$$

where $D_1 = D - D_0$. The matrices D_0 and $\tilde{A}$ belong to $\mathcal{M}$. As $\det M \equiv 1$, one has $\det(D_0 + \frac{\lambda_*}{\sin\eta}\tilde{A}) = 1$. By Corollary 4.4 and Lemma 4.9, the matrix $\tilde{A}$ is analytic and uniformly bounded in R; so, the norm $\|\tilde{A}\|_{r,(0,\pi)}$ is bounded by a constant uniformly in ε (for the definition of this norm, see (11.1)). So,we apply Proposition 11.1 to equation (4.52) except near energies where $\sin(\eta(E))$ vanishes. Therefore, we "cut" the ends of the interval $\eta \in (0,\pi)$.
Fix $0 < \alpha < 1/2$ and consider the interval $I_\alpha = (\lambda_*^\alpha, \pi - \lambda_*^\alpha)$. On this interval, the Lipschitz norm of the function $\frac{1}{\sin\eta}$ is bounded by Const$\lambda_*^{-2\alpha}$, and thus, we can apply Proposition 11.1 with the "effective" coupling constant equal to $\lambda_*^{1-2\alpha}$. In result, we proved

Proposition 4.3. *Fix positive $\alpha < 1/2$ and $\sigma' < 1$. Let $\sigma = (1-2\alpha)\sigma'$. If ε is small enough, η is outside a subset Φ_∞ of I_α of the measure $O(\lambda_*^{\sigma/2})$, and h satisfies the Diophantine condition* (11.4), *then the monodromy equation* (2.5) *has bounded solutions.*

4.4.6. *Back to equation* (1.1). Let us reformulate the conditions of Proposition 4.3 in terms of the initial equation (1.1).

1. Let us first check that h satisfies the Diophantine condition (11.4) if and only if ε belongs to a set D satisfying the property described in Theorem 2.4. Therefore, define D as the set $\varepsilon \in (0,1)$ such that $h = \frac{2\pi}{\varepsilon}$ mod 1 satisfies (11.4). When estimating mes$\,(D)$, the Lebesgue measure of the set D, C denotes different positive constants independent of ε. One has

$$\text{mes}\,((0,\varepsilon)\setminus D) \le \sum_{n=L(\varepsilon)+1}^{\infty} \int_{(0,1)\setminus H_\sigma} \frac{2\pi}{(h+n)^2} dh.$$

Here, the set H_σ is the set of h satisfying (11.4), and $L(\varepsilon)$ is equal to the integer part of $2\pi/\varepsilon$. Recall that $\lambda_* = \exp(-\frac{S}{\varepsilon})$ where S is positive and independent of ε. Therefore, one computes

$$\begin{aligned}\text{mes}\,(\{(0,\varepsilon)\setminus D\}) &\le 2\pi \sum_{n=L(\varepsilon)}^{\infty} \frac{1}{n^2} \sum_{k=1}^{\infty}\sum_{l=0}^{k} \frac{1}{k^3} \exp\left(-\frac{\sigma n S}{2\pi}\right) \\ &\le C \sum_{n=L(\varepsilon)}^{\infty} \frac{1}{n^2} \exp\left(-\frac{\sigma n S}{2\pi}\right) \le C \frac{1}{L(\varepsilon)^2} \exp\left(-\frac{\sigma L(\varepsilon) S}{2\pi}\right) \le C\varepsilon^2 \lambda^\sigma.\end{aligned}$$

This proves that the measure of D satisfies estimate (2.22).

2. By (4.43), the length $|I_*|$ of the interval I_* has the asymptotics $|I_*| \sim \frac{4}{|F'(E_*)|}$. As λ_* is small, one see that $|I_l|$ has the same asymptotics, i.e. $|I_l|/|I_*| \sim 1$ when $\varepsilon \to 0$.

3. Let $E_\infty = E(B)$, where the set $B = (0, \lambda_*^\alpha) \cup (\pi - \lambda_*^\alpha, \pi) \cup \Phi_\infty$, and the function $E(\eta)$ is the inverse of $\eta(E)$. Clearly, mes$\,(E_\infty) = \int_B E'(\eta) d\eta$. This, estimate (11.6) and (4.47) imply that mes$\,(E_\infty) = \frac{1}{|F'(E_*)|} \cdot (O(\lambda_*^{\sigma/2}) + O(\lambda_*^{2\alpha})) = o(|I_l|)$.

4. Let us combine 1., 2. and 3. As the interval I_l contains the spectrum situated on I_* and $|I_l|/|I_*| \sim 1$, we see that, if ε belongs to the set D and is small enough, equation (4.1) has bounded solutions for $E \in I_l$ outside some subset of I_l of measure $o(|I_l|)$. This means that the monodromy equation itself has bounded solutions for these values of E. Then, Corollary 2.1 implies that, for these energies, the Lyapunov exponent vanishes. Now, applying Ishii-Pastur-Kotani's Theorem, we see that the measure of the absolutely continuous spectrum of (1.1) situated on I_l has the same asymptotics as the length of I_l. As σ can be made as close to 1 as desired, this completes the proof of Theorem 2.4. $\square$

4.5. **Singular spectrum.** Here, we prove Theorem 2.5. Therefore, we estimate the Lyapunov exponent of (1.1) on J_δ^+ using the asymptotics (2.17) of the monodromy matrix described in Theorem 2.2.

Let $E_0 \in J_\delta^+$, let V_0 be a neighborhood of E_0 as in Theorem 2.2. Recall that the asymptotics (2.17) are uniform in E in V_0 and in ϕ in the strip $|\text{Im}\,\phi| \le Y/\varepsilon$, and that $Y > \text{Im}\,\varphi_2(E_0)$.

Let $Y_1 = \frac{1}{2\pi} S_v(E)$. By point 4 of Lemma 2.1, one has $Y_1 < \text{Im}\,\varphi_2(E)$ for $E \in J$. Clearly, there is a (constant) real neighborhood V_1 of E_0 where $Y - Y_1 > \delta$ for some positive constant δ. For E in this neighborhood and for ϕ in the strip $-Y/\varepsilon \le \text{Im}\,\phi \le -(Y-\delta)/\varepsilon$, the product $t_v e^{2\pi i \phi}$ is exponentially large.

As Φ takes real values on J, see Lemma 2.1, the last observation and formula (2.17) imply that, for $E \in V_0 \cap V_1$ and $-Y/\varepsilon \le \text{Im}\,\phi \le -(Y-\delta)/\varepsilon$, one has

$$a = \lambda e^{2i\pi(\phi-\phi_0)}(1+o(1)), \quad b/a = -i + o(1), \quad a^*/a = o(1), \quad b^*/a = o(1).$$

Therefore,

$$M = \lambda e^{2i\pi(\phi-\phi_0)} \left[\begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} + o(1) \right].$$

Recall that λ is exponentially large on J_δ^+. Let E belongs to the (constant) interval $J_0 = J_\delta^+ \cap V_0 \cap V_1$. Then, the monodromy matrix satisfies the assumptions of Proposition 10.1, and we get the following estimate for the Lyapunov exponent of the matrix cocycle associated to the pair (M, h)

$$\theta(M,h) \ge \log \lambda + o(1).$$

The Lyapunov exponent $\Theta(E)$ for equation (1.1) is related to $\theta(M,h)$ by Corollary 2.1. Therefore,

$$\Theta(E) \geq \frac{\varepsilon}{2\pi} \log \lambda + o(\varepsilon).$$

Expressing the coupling constant in terms of the tunneling actions, $\log \lambda = \frac{1}{\varepsilon}(S_h(E) - S_v(E)) + o(1)$, we obtain to (2.23). On the interval $J_0 \subset J_\delta^+$, the leading term in (2.23) is positive, hence, so is the Lyapunov exponent for sufficiently small ε. Therefore, by Ishii-Pastur-Kotani's Theorem, the spectrum of (1.1) situated on J_0 is singular. Since E_0 can be taken arbitrary in J_δ^+, this completes the proof of Theorem 2.5

5. Periodic Schrödinger operators

In this subsection, we discuss the periodic Schrödinger operator (5.1) where V is a 1-periodic, real valued, L^2_{loc}-function. We collect all the results needed for this paper. The proofs of these results as well as more details can be found, for example, in [12, 34, 35, 41].

5.1. **Bloch solutions.**

Let ψ be a solution of the equation

$$H_0\psi = -\frac{d^2}{dx^2}\psi(x) + V(x) = E\psi(x), \quad x \in \mathbb{R}. \tag{5.1}$$

satisfying the relation

$$\psi(x+1) = \lambda\, \psi(x), \quad \forall x \in \mathbb{R}, \tag{5.2}$$

with λ independent of x. Such a solution is called a *Bloch* solution, and the number λ is called the *Floquet multiplier*. Let us discuss the analytic properties of Bloch solutions.

As in section 2.3, we denote the spectral bands of the periodic Schrödinger equation by $[E_1, E_2]$, $[E_3, E_4]$, $\ldots$, $[E_{2n+1}, E_{2n+2}]$, $\ldots$. Consider $\mathcal{G}_\pm$, two copies of the complex plane $E \in \mathbb{C}$ cut along the spectral zones. Paste them together to get a Riemann surface with square root branch points. We call this Riemann surface $\mathcal{G}$.

One can construct a Bloch solution $\psi(x,E)$ of equation (5.1) meromorphic on this Riemann surface. It can be normalized by the condition $\psi(1,E) \equiv 1$. The poles of this solution are located in the spectral gaps. More precisely, each spectral gap contains precisely one simple pole. It is located either on $\mathcal{G}_+$ or on $\mathcal{G}_-$. The position of the pole is independent of x.

Outside the edges of the spectrum, the two branches of ψ are linearly independent solutions of (5.1)

Finally, we note that, in the spectral gaps, both branches of ψ are real valued functions of x, and, on the spectral bands, they differ only by complex conjugation.

5.2. **The Bloch quasi-momentum.**

Consider the Bloch solution $\psi(x,E)$. The corresponding Floquet multiplier $\lambda(E)$ is analytic on $\mathcal{G}$. Represent it in the form

$$\lambda(E) = \exp(ik(E)). \tag{5.3}$$

The function $k(E)$ is the *Bloch quasi-momentum.*

5.2.1. *The Bloch quasi-momentum as an analytic multi-valued function.* The Bloch quasi-momentum is an analytic multi-valued function of E. It has the same branch points as $\psi(x,E)$. Let D be a simply connected domain containing no branch points of the Bloch quasi-momentum. In D, one can fix an analytic single-valued branch of k, say k_0. All the other single-valued branches of k that are analytic in $E \in D$ are related to k_0 by the formulae

$$k_{\pm,l}(E) = \pm k_0(E) + 2\pi l, \quad l \in \mathbb{Z}. \tag{5.4}$$

The Riemann surface of k is more complicated than the one for ψ. However, on the complex plane cut along the spectral gaps of the periodic operator, one can fix a single-valued analytic branch of k.

5.2.2. *The main branch of the Bloch quasi-momentum.* Consider the complex plane cut along the real line from E_1 to $+\infty$. Denote the cut plane by $\mathbb{C}_0$. On $\mathbb{C}_0$, one can fix a single valued analytic branch of the quasi-momentum by the condition

$$-ik_0(E) > 0, \quad E < E_1. \tag{5.5}$$

We call k_0 the main branch of the Bloch quasi-momentum. The function k_0 conformally maps $\mathbb{C}_0$ onto the upper half of the complex plane with some vertical slits beginning at the points πl, $l \in \mathbb{Z}$, and having finite lengths. It is a bijection.

In Fig. 5, we drew two curves in $\mathbb{C}_0$ and their images under the transformation $E \mapsto k_0(E)$. Consider k_0

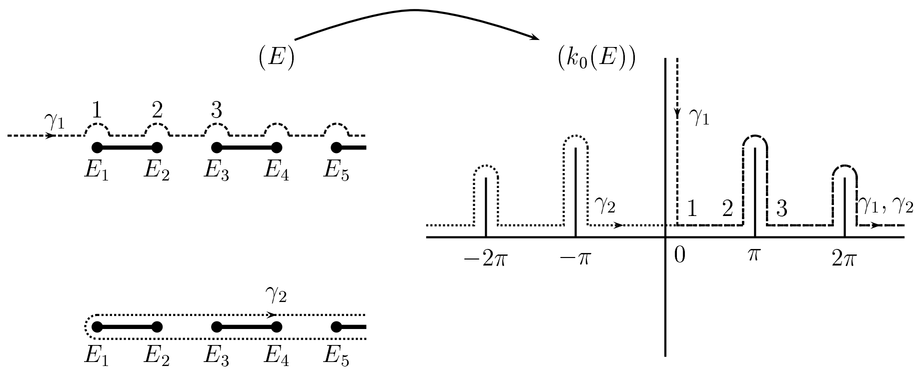

FIGURE 5. The action of mapping k on some curves

along the curve γ_1. The quasi-momentum $k_0(E)$ is real and monotonically increasing along the spectral zones; along the spectral gaps, it takes complex values and its real part is constant; in particular, we have

$$k_0(E_1) = 0, \quad k_0(E_{2l} + i0) = k_0(E_{2l+1} + i0) = \pi l, \quad l = 1, 2, 3 \ldots \tag{5.6}$$

If $E_{2n} < E_{2n+1}$, one says that the n-th gap is open. Along any open gap, $\operatorname{Im} k_0(E + i0)$ is a non-constant function with only one non-degenerate maximum.

The values of the quasi-momentum k_0 on the different sides of the cut $E_1 < E < +\infty$ are related to each other by the formula

$$k_0(E + i0) = -\overline{k_0(E - i0)}, \quad E_1 \le E. \tag{5.7}$$

All the branch point of k_0 are of square root type: let E_m be one of the branch points, then, in a sufficiently small neighborhood of E_m, k_0 is analytic in $\sqrt{E - E_l}$, and

$$k_0(E) - k_0(E_l) = c_l\sqrt{E - E_l} + O(E - E_l), \quad c_l \ne 0. \tag{5.8}$$

The constants $m_l = |c_l|^2/2$ are called the *effective masses* associated to E_l.

5.3. **Periodic components of the Bloch solution.** Let $D \subset \mathbb{C}$ be a simply connected domain that does not contain any branch point of k. On D, we fix an analytic branch k of the Bloch quasi-momentum. There are two disjoint domains in $\mathcal{G}$ denoted by $D_\pm$ that project onto D. Define $\psi_\pm$ to be the restrictions of the Bloch solution ψ to $D_\pm$. The functions $\psi_\pm$ are indexed so that k be the Bloch quasi-momentum of ψ_+. The Bloch solutions $\psi_\pm$ can be represented in the form

$$\psi_\pm(x, E) = e^{\pm ik(E)x} p_\pm(x, E), \quad E \in D, \tag{5.9}$$

where $p^\pm(x, E)$ are functions periodic in x,

$$p_\pm(x + 1, E) = p_\pm(x, E), \quad \forall x \in \mathbb{R}. \tag{5.10}$$

We call $p_\pm$ the *periodic components* of $\psi_\pm$ with respect to the branch k of the Bloch quasi-momentum. One has

$$(5.11)\qquad \int_0^1 p_+(t,E)p_-(t,E)dt = \int_0^1 \psi_+(t,E)\psi_-(t,E)dt = -ik'(E)w(E), \quad E \in D,$$

where $w(f(x),g(x)) = f'(x)g(x) - g'(x)f(x)$ and $w(E) = w(\psi_+(x,E),\psi_-(x,E))$. For a proof, we refer to [19], section 2.

6. Main theorem of the complex WKB method

In this section, we describe the complex WKB method for adiabatically perturbed periodic Schrödinger equations. The reader can find more details and proofs of the results of this section in [19]. This method was developed for the asymptotic study of equations of the form

$$(6.1)\qquad -\frac{d^2}{dx^2}\psi(x) + (V(x) + W(\varepsilon x))\psi(x) = E\psi(x), \quad x \in \mathbb{R},$$

where V is a real valued 1-periodic function of x, and ε is a small parameter (fixed positive number). This method is designed to study exponentially small effects due to the complex tunneling. Usually, these effects are measured by exponentially small coefficients of certain transition matrices (scattering matrices, monodromy matrices, etc) relating two distinguished bases of solutions. To calculate such exponentially small terms, one assumes that W is analytic and introduces an additional parameter φ into equation (6.1) so that it becomes

$$(6.2)\qquad -\frac{d^2}{dx^2}\psi(x) + (V(x) + W(\varepsilon x + \varphi))\psi(x) = E\psi(x), \quad x \in \mathbb{R}.$$

The idea is that the terms that are exponentially small when φ is real, may become dominant for complex values of φ, and that having computed these terms for complex values of φ, one can try to recover their values for φ on the real axis.

Of course, to realize these ideas, one has to have a good enough control of the dependence on φ of solutions of equation (6.2). However, there is no equation controlling this dependence. But, there is a natural condition which replaces such an equation. We say that $(\psi_\pm)$ two solutions of (6.2) form *a consistent basis* if their Wronskian is independent of φ and if

$$(6.3)\qquad \psi_\pm(x+1,\varphi) = \psi_\pm(x,\varphi+\varepsilon) \quad \forall\varphi.$$

Equation (6.3) is called the *consistency condition.* To clarify this condition, we pass to the variables $\phi = \varphi/\varepsilon$ and $t = x + \phi$. In terms of these variables, (6.2) takes the form

$$(6.4)\qquad -\frac{d^2}{dt^2}\tilde\psi + (V(t-\phi) + W(\varepsilon t))\tilde\psi = E\tilde\psi,$$

and (6.3) just becomes the 1-periodicity $\tilde\psi$ in ϕ i.e.

$$(6.5)\qquad \tilde\psi_\pm(t,\phi+1) = \tilde\psi_\pm(t,\phi).$$

This condition plays a crucial role for the asymptotic analysis of (6.2).

Remark 6.1. Note that, if $W = \alpha\cos(.)$, then equation (6.4) is nothing but equation (1.1) and the condition (6.5) is precisely the condition (2.1) that we required to define the monodromy matrices. We use the complex WKB method to calculate the Fourier coefficients of the monodromy matrices.

Now, following [19], we shall describe the main constructions of the complex WKB method. Below, we assume that $V \in L^2_{\mathrm{loc}}$, and that W is analytic in a neighborhood $\mathcal{D}(W)$ of the real line.

6.1. **Complex momentum.** The central analytic object of the complex WKB method is the *complex momentum* $\kappa(\varphi)$. It is defined in terms of the Bloch quasi-momentum of (5.1) by the formula

$$(6.6)\qquad \kappa(\varphi) = k(E - W(\varphi))$$

in $\mathcal{D}(W)$ the domain of analyticity of the function W.

The complex momentum κ is a multi-valued analytic function. Its branch points are related to the branch points of the quasi-momentum by the relations

$$(6.7)\qquad E_l = E - W(\varphi), \quad l \in \mathbb{N},$$

where E_l are the ends of the spectral zones of the operator H_0.

Let D be a simply connected domain containing no branch points of κ. Then, in D, one can fix an analytic branch κ_0 of this function. By (6.10), all the other analytic branches are described by the formulas

$$\kappa_m^{\pm} = \pm\kappa_0 + 2\pi m, \tag{6.8}$$

where $\pm$ and m are indexing the branches.

6.2. **Canonical domains.** The canonical domain notion is the main geometric notion of the complex WKB method.

Throughout the paper, we say that a set is *regular* if it is in the domain of analyticity of W and contains no branch points of the complex momentum. We use the terms: regular domain, regular curve, regular point. When speaking about regular curves, we suppose in addition that they are smooth and connected.

A piecewise smooth, connected curve $\gamma \subset \mathbb{C}$ is called *vertical* if it intersects the lines $\{\operatorname{Im}\varphi = \text{Const}\}$ at non-zero angles θ,

$$0 < \theta < \pi. \tag{6.9}$$

Thus, vertical line are naturally parameterized by $\operatorname{Im}\varphi$.

Let γ be a regular, vertical curve in the domain of analyticity of W. On γ, fix a continuous branch of the momentum κ. The curve γ is said to be *canonical* if, along γ,

- $\operatorname{Im}\int^{\varphi}\kappa d\varphi$ is strictly increasing with $\operatorname{Im}\varphi$;
- $\operatorname{Im}\int^{\varphi}(\kappa-\pi)d\varphi$ is strictly decreasing with $\operatorname{Im}\varphi$.

From now on, in this section, φ_1 and φ_2 are two regular points such that $\operatorname{Im}\varphi_1 < \operatorname{Im}\varphi_2$.

Definition 6.1. Let $K \subset \mathcal{D}(W)$ be a regular, simply connected domain. On K, fix a continuous branch of the quasi-momentum, say κ. The domain K is called *canonical* if it is the union of curves canonical with respect to κ and connecting φ_1 and φ_2 located on ∂K.

Note that the boundary of K may contain branch points of the complex momentum.

6.3. **Canonical Bloch solutions.** Consider the periodic Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = \mathcal{E}\psi(x), \quad \mathcal{E} = E - W(\varphi), \quad x \in \mathbb{R}. \tag{6.10}$$

Here, φ plays the role of a parameter. We need two linearly independent Bloch solutions of (6.10) that are analytic in φ in a given canonical domain. This property defines these solutions uniquely up to an analytic, non-vanishing factor depending on φ.

We construct these solutions in terms of $\psi(x,\mathcal{E})$, the Bloch solution of (5.1) meromorphic on the Riemann surface $\mathcal{G}$ (see section 5.1) and its Bloch quasi-momentum $k(\mathcal{E})$. Recall that k is defined modulo 2π.

Pick φ^0, a regular point such that $k'(E - W(\varphi_0)) \neq 0$. Begin with a local construction in V^0, a small enough neighborhood of φ^0. One has,

- in V^0, we fix an analytic branch of $\kappa(\varphi)$, the complex momentum;
- there are two different branches of the function $\psi(x, E - W(\varphi))$ meromorphic in $\varphi \in V^0$. They are linearly independent Bloch solutions of equation (6.10). We denote them by $\psi_{\pm}(x,\varphi)$ so that their respective Bloch quasi-momentum be equal to $\pm\kappa(\varphi)$;
- in V^0, fix an analytic branch of the function

$$q(\varphi) = \sqrt{k'(E - W(\varphi))}. \tag{6.11}$$

Consider $p_{\pm}(x,\varphi)$ the periodic components of the Bloch solutions $\psi_{\pm}(x,\varphi)$. One has

$$\psi_{\pm}(x,\varphi) = e^{\pm i\kappa(\varphi)x}p_{\pm}(x,\varphi), \quad p_{\pm}(x+1,\varphi) = p_{\pm}(x,\varphi), \quad x \in \mathbb{R}. \tag{6.12}$$

Let

$$\omega_{\pm}(\varphi) = -\frac{\int_0^1 p_{\mp}(x,\varphi)\frac{\partial p_{\pm}}{\partial\varphi}(x,\varphi)dx}{\int_0^1 p_+(x,\varphi)p_-(x,\varphi)dx}. \tag{6.13}$$

The functions

$$\Psi_\pm(x,\varphi)=q(\varphi)e^{\int^\varphi \omega_\pm(\varphi)d\varphi}\psi_\pm(x,\varphi) \tag{6.14}$$

are called the *canonical* Bloch solutions. In section 3 of [19], we have proved the

Lemma 6.1. *The canonical Bloch solutions are analytic in any regular simply connected domain D containing φ^0.*

Normalization point. Let D be a regular simply connected domain. Construct the canonical Bloch solutions analytic in D. We normalize them by integrating in (6.14) from a given point φ_0, located in $\overline{D}$. In this case, we speak of the canonical Bloch solution *normalized at the point* φ_0 and the point φ_0 is called *the normalization point.*

Indexing of the canonical Bloch solutions. Let D be a simply connected regular domain. In D, fix κ a continuous branch of the complex momentum. Consider the canonical Bloch solutions $\Psi_\pm$ corresponding to D. They can be indexed by $\pm$ so that $\kappa(\varphi)$ be the Bloch quasi-momentum corresponding to the solution $\Psi_+(x,\varphi)$.

The Wronskian of the canonical Bloch solutions. Here, we compute the Wronskian of $\Psi_\pm$ normalized at φ_0. One has

Lemma 6.2. *One has*

$$w(\Psi_+,\Psi_-)=q^2(\varphi_0)w(\psi_+(x,\varphi_0),\psi_-(x,\varphi_0))=i\int_0^1\psi_+(x,\varphi_0)\psi_-(x,\varphi_0)dx.$$

The representation of the Wronskian given in Lemma 6.2 shows that, for any regular φ_0 such that $q^2(\varphi_0)=k'(E-W(\varphi_0))\neq 0$, the canonical Bloch solutions are linearly independent.

Note that, in this case, neither Ψ_+ nor Ψ_- can be identically zero for any given $\varphi\in D$.

The second representation given by Lemma 6.2 allows us to chose as φ_0 a branch point of the complex momentum. In this case, one has

$$w(\Psi_+,\Psi_-)=i\int_0^1|\psi_+(x,\varphi_0)|^2dx\neq 0. \tag{6.15}$$

Indeed, at the ends of any spectral gap, the branches of the Bloch solution $\psi(x,\mathcal{E})$ coincide and are real valued in x.
Proof of Lemma 6.2. The definition of the canonical Bloch solutions and of the functions $\omega_\pm$ imply

$$w(\Psi_+,\Psi_-)=q^2(\varphi)\exp\left(-\int_{\varphi_0}^{\varphi}\frac{\partial}{\partial\varphi}\log\left(\int_0^1 p_+(x,\varphi)p_-(x,\varphi)dx\right)\right)w(\psi_+(x,\varphi),\psi_-(x,\varphi)).$$

Equation (5.11) then gives both representations for the Wronskian given in Lemma 6.2. □

6.4. **The main theorem of the WKB method.** Now, we can formulate the main theorem of the complex WKB method for (6.2):

Theorem 6.1 ([19]). *Fix $X>0$. Fix $E=E_0\in\mathbb{C}$. Let K be a bounded canonical domain for the family of equations (6.2), and let κ be the branch of the complex momentum with respect to which K is canonical. For sufficiently small positive ε, there exists a consistent basis $(f_\pm)$ defined for $x\in\mathbb{R}$ and $\varphi\in K$ and having the following properties:*

- *for any fixed $x\in\mathbb{R}$, the functions $f_\pm(x,\varphi)$ are analytic in $\varphi\in K$.*
- *For $-X\le x\le X$, and $\varphi\in K$, the functions $f_\pm(x,\varphi)$ have the asymptotic representations*

$$f_\pm(x,\varphi)=e^{\pm\frac{i}{\varepsilon}\int^\varphi\kappa d\varphi}(\Psi_\pm(x,\varphi)+o(1)),\quad \varepsilon\to 0. \tag{6.16}$$

  Here, $\Psi_\pm$ are the canonical Bloch solutions corresponding to the domain K, indexed so that $\kappa(\varphi)$ be the Bloch quasi-momentum corresponding to the solution $\Psi_+(x,\varphi)$.
- *The error estimates in (6.16) may be differentiated once in x. Moreover, they are uniform in $x\in[-X,X]$ and locally uniform in φ in the interior of K.*

This theorem was proved in [19].

In the sequel, we normalize the solutions $f_\pm$ by integrating in (6.14) and in (6.16) from a point φ_0 being either in K or on its boundary. When we need to indicate explicitly the point φ_0, we write $f_\pm(x,\varphi,\varphi_0)$.

One easily calculates the Wronskian of the solutions $f_\pm(x,\varphi,\varphi_0)$:

$$w(f_+,f_-)=w(\Psi_+,\Psi_-)+o(1). \tag{6.17}$$

By Lemma 6.2, the solutions $f_\pm$ are linearly independent if $k'(E-W(\varphi_0))\neq 0$, or if φ_0 is one of the branch points of κ.

6.5. **Dependence on the spectral parameter and admissible subdomains.** To simplify the statement of Theorem 6.1, we have not considered the dependence of the solutions on the spectral parameter E. Therefore, we introduce the notion of admissible subdomains of a canonical domain. Let K be a compact canonical domain and pick $\delta>0$. Let A be the complementary in K of the δ-neighborhood of ∂K, the boundary of K. The set A is the *δ-admissible subdomain* of the canonical domain K. One proves

Proposition 6.1. *In the setting of Theorem 6.1, the solutions $f_\pm$ are analytic in E in V_0, a neighborhood of E_0. For A, an admissible subdomain of K, there exists $V_A\subset V_0$, a neighborhood of E_0 such that the asymptotics* (6.16) *is uniform in $(\varphi,E,x)\in A\times V_A\times[-X,X]$. It can be once differentiated in x without loosing its uniformity properties.*

In [19], we have not discussed explicitly the dependence of the solutions $f_\pm$ on the spectral parameter E. In the proof of Theorem 6.1, all the estimates are locally uniform in E. To prove Proposition 6.1, one just has to follow the proof of Theorem 6.1 given in [19], keeping in mind the following additional observation

Lemma 6.3. *Let γ be a compact canonical curve for a given value E_0 of the spectral parameter E. Then, there exists $V_1\subset\mathbb{C}$, a neighborhood of E_0 and a constant $\delta>0$ such that, for any $E\in V_1$, the curve γ remains canonical, and along γ, one has*

$$\frac{d}{dy}\left(Im\int^{\varphi}\kappa d\varphi\right)>\delta \text{ and } \frac{d}{dy}\left(Im\int^{\varphi}(\kappa-\pi)d\varphi\right)<-\delta,\quad y=\mathrm{Im}\varphi, \tag{6.18}$$

uniformly in $E\in V_1$.

Proof. Since γ is canonical and compact, for $E=E_0$, along γ, one has

$$\frac{d}{dy}\left(\mathrm{Im}\int^{\varphi}\kappa d\varphi\right)>\delta_0 \text{ and } \frac{d}{dy}\left(\mathrm{Im}\int^{\varphi}(\kappa-\pi)d\varphi\right)<-\delta_0,\quad y=\mathrm{Im}\varphi,$$

for some positive constant δ_0. The branch points of the complex momentum depend continuously on the spectral parameter. So, γ being regular and compact, there exists a closed neighborhood of E_0, say $V_0\subset\mathbb{C}$, such that for all $E\in V_0$, the branch points of the complex momentum stay at a non-vanishing distance d of γ. The values of the two derivatives estimates in (6.18) depend continuously on $(\varphi,E)\in\gamma\times V_0$. So, for any constant $\delta<\delta_0$, there is $V_1\subset V_0$, a neighborhood of E_0 such that, for $E\in V_1$, one obtains the estimates (6.18). This completes the proof of Lemma 6.3. ☐

6.6. **How to use the complex WKB method.** Let us outline the ideas we use when applying the complex WKB method. Having described canonical domains and the corresponding bases of solutions with the standard asymptotic behavior, one tries to relate the bases corresponding to different canonical domains. Pick two canonical domains $K^{1,2}$ and the associated basis of solutions constructed in Theorem 6.1; denote it by $(f_\pm^{1,2})$. Equation (6.2) being linear of second order, there exists a transfer matrix T_{12}, depending only on φ, such that

$$\begin{pmatrix} f_+^2\\ f_-^2\end{pmatrix}=T_{12}\begin{pmatrix} f_+^1\\ f_-^1\end{pmatrix},\quad \varphi\in K^1\cap K^2.$$

The consistency condition (2.1) implies that T_{12} is ε-periodic in φ. Moreover, $(f_\pm^{1,2})$ being analytic in φ, T_{12} is also analytic.

In $K^1\cap K^2$, we can use the asymptotics (6.16) for both bases to obtain the asymptotics of T_{12}. Using the ε-periodicity of T_{12}, we see that these asymptotics are valid in a horizontal strip containing $K^1\cap K^2$. Now, assume that we know the asymptotics of T_{12} near the boundary of a complex strip containing $\mathbb{R}$. Then, we can easily get the asymptotics of the Fourier coefficients of T_{12} corresponding to the terms of the Fourier

series which are large at this boundary. This enables us to control the Fourier series terms of T_{12} which are of order $\exp(-\mathrm{Const}/\varepsilon)$ on the real line.
The consistency condition directly relates the x-behavior of the solution to its φ-behavior. This means that, one can express the transition matrices appearing in the spectral study of the equation family (6.2) (for example, monodromy matrices) as products of transfer matrices.

6.7. **Canonical domains.** We finish the section on the complex WKB method by describing a simple general approach to "constructing" canonical domains. We use this approach in this paper. Below, we assume that D is a regular simply connected domain and that κ is a branch of the complex momentum analytic in D.

The main elements of the construction are the pre-canonical lines. Pre-canonical lines are made of elementary lines that we discuss now.

6.7.1. *Lines of Stokes type.* Let $\gamma \subset D$ be a smooth curve. We say that γ is a line *of Stokes type* with respect to κ if, along γ, either

$$\mathrm{Im}\left(\int^{\varphi} \kappa d\varphi\right) = \mathrm{Const} \quad \text{or} \quad \mathrm{Im}\left(\int^{\varphi} (\kappa - \pi) d\varphi\right) = \mathrm{Const}.$$

6.7.2. *Canonical lines.* We identify $\mathbb{C}$ and $\mathbb{R}^2$ in the usual way. For $\varphi \in D$, denote by $S(\varphi) \subset \mathbb{C}$ the cone situated between the vectors $\overline{\kappa(\varphi)}$ and $\overline{\kappa(\varphi)} - \pi$ and such that, for any vector $z \in S(\varphi)$, one has $\mathrm{Im}\,(\overline{\kappa(\varphi)}\, z) > 0$, and $\mathrm{Im}\,((\overline{\kappa(\varphi)} - \pi)\, z) < 0$. The definition of a canonical line implies that a smooth vertical curve $\gamma \in D$ is canonical with respect to κ if and only if for all $\varphi \in \gamma$, the vector $t(\varphi)$ tangent to γ and oriented upwards belongs to $S(\varphi)$.

6.7.3. *Pre-canonical lines.* Let $\gamma \subset D$ be a vertical curve. The curve γ is a *pre-canonical line* if it consists of a finite union of bounded segments of either canonical lines or lines of Stokes type.

One of the main properties of pre-canonical lines is that they can be used to construct canonical lines.

Proposition 6.2. *Let γ be a pre-canonical curve. Denote the ends of γ by φ_a and φ_b. For $V \subset D$, a neighborhood of γ and $V_{\varphi_a} \subset D$, a neighborhood of φ_a, there exists a canonical line connecting in V the point φ_b to a point in V_{φ_a}.*

Proof. The proof of Proposition 6.2 is done in two steps. The pre-canonical curve γ can contain segments of lines of Stokes type. In the first step, we "replace" these segments by nearby segments of canonical curves to prove that, near a given pre-canonical curve, there is a pre-canonical curve consisting only of segments of canonical curves. This curve is not yet a canonical one, since canonical curves are smooth. So, in the second step, we smooth out the constructed pre-canonical curve to get a canonical one. If the given canonical curve contains no segments of lines of Stokes type, we do not need to make the first step. Otherwise, we use the following lemma.

Lemma 6.4. *Let β be either a bounded vertical line of Stokes type or a bounded canonical line. Denote the ends of β by φ_a and φ_b. Let $V \subset D$ be a neighborhood of β. For $V_{\varphi_a} \subset V$, a neighborhood of φ_a, there exists $V_{\varphi_b} \subset V$, a neighborhood of φ_b such that, for each point $\varphi_2 \in V_{\varphi_b}$, there is a point $\varphi_1 \in V_{\varphi_a}$, connected in V with φ_2 by a curve canonical with respect to κ.*

Proof. If β is a canonical line, the result follows from the fact that any line C^1-close to β is also canonical. Let us turn to the case of a line of Stokes type. Assume that $\mathrm{Im}\,\left(\int^{\varphi} \kappa d\varphi\right) = \mathrm{Const}$ along β. The case when $\mathrm{Im}\,\left(\int^{\varphi}(\kappa - \pi)d\varphi\right) = \mathrm{Const}$ along β is treated similarly. Define $I(\varphi) = \int_{\varphi_a}^{\varphi} \kappa d\varphi$, $X(\varphi) = \mathrm{Re}\,[I(\varphi)]$, and $Y(\varphi) = \mathrm{Im}\,[I(\varphi)]$.

As V does not contain any branch point of κ, we know that $dI/d\varphi = \kappa$ does not vanish in V. Moreover, along β, $X(\varphi)$ is a strictly monotonous function of $\mathrm{Im}\,\varphi$. This implies that there exists $\mathcal{V} \subset \mathbb{C}$ a neighborhood of β and $\mathcal{U} \subset \mathbb{C}$ a neighborhood of the interval $[a,b] \equiv X(\beta) \subset \mathbb{R}$ such that $\Theta :\ \mathcal{V} \to \mathcal{U}$, $\Theta(\varphi) = X(\varphi) + iY(\varphi)$, is a diffeomorphism.

Assume that X is increasing along β (the case where X is decreasing is treated similarly). Consider the smooth curves $\alpha \subset \mathcal{U}$ along which Y is a monotonously increasing function of X, $Y = F_\alpha(X)$,

$$F'_\alpha(X) > 0, \quad X \in [a,b] \quad \text{and} \quad \|F_\alpha\|_{C^1([a,b])} \leq \delta.$$

Along the curves $\Theta^{-1}(\alpha)$, the function $\mathrm{Im}\int^{\varphi}\kappa d\varphi$ is strictly increasing. As $\delta\to 0$, the curves $\Theta^{-1}(\alpha)$ are C^1-close to β. As β is vertical and as the function $\mathrm{Im}\int_{\varphi_a}^{\varphi}\kappa(\varphi)d\varphi$ is constant along β, $\mathrm{Im}\int_{\varphi_a}^{\varphi}(\kappa(\varphi)-\pi)d\varphi$ is decreasing along β. So, if δ is small enough, the curves $\Theta^{-1}(\alpha)$ are vertical, and $\mathrm{Im}\int^{\varphi}(\kappa-\pi)d\varphi$ is still decreasing along $\Theta^{-1}(\alpha)$. Thus, the curves $\Theta^{-1}(\alpha)$ are canonical. This completes the proof of Lemma 6.4. $\square$

Lemma 6.4 allows us to replace any given pre-canonical line with a neighboring pre-canonical line consisting only of segments of vertical canonical lines. This new pre-canonical curve begins at φ_a one of the ends of β and it connects it in V with a point in any given neighborhood V_b of φ_b the other end of φ.

To make the second step of the proof of Proposition 6.2, we have to smooth out the pre-canonical line $\hat{\beta}$ obtained in the first step. In fact, we do it only in small neighborhoods of the common ends of the segments of canonical lines making up $\hat{\beta}$. Let φ_0 be one of such points, and let $\gamma_j\subset\hat{\beta}$, $j=1,2$ be the segments of canonical lines with the common end φ_0. As in section 6.7.2 define the cones $S(\varphi)$ for $\varphi\in V$. There exists V_0, a sufficiently small neighborhood of φ_0 such that $S_0\equiv\cap_{\varphi\in V_0}S(\varphi)\neq\emptyset$ and such that, in V_0, both the tangent vectors to γ_1 and the tangent vectors to γ_2 belong to S_0. We replace the part of $\hat{\beta}$ in V_0 by a smooth curve so that the new curve $\hat{\beta}$ be smooth and the vectors tangent to this curve in V_0 be in S_0. We carry out this smoothening for all the common ends of the segments of canonical lines making up $\hat{\beta}$. The smoothened curve $\hat{\beta}$ is canonical. This completes the proof of Proposition 6.2. $\square$

By Proposition 6.2, we can construct a canonical curve arbitrarily close to a given pre-canonical curve. But, in general, the ends of these curves can not coincide. Let us single out one case where these curves do can have common ends:

Lemma 6.5. *Let γ be a pre-canonical curve. Assume that it begins and ends by γ_a and γ_b segments of canonical curves. Denote by φ_a and φ_b those ends of these segments which are internal points of γ ordered so that* $\mathrm{Im}\,\varphi_a<\mathrm{Im}\,\varphi_b$. *Then, for any positive δ, there is a pre-canonical curve situated in the δ-neighborhood of γ, and containing γ_a and γ_b without the δ-neighborhood of φ_a and the δ-neighborhood of φ_b.*

Proof. If the curve γ consists only of segments of canonical curves, then one proves the lemma just by smoothing out the curve γ as in the proof of Proposition 6.2. Otherwise, we apply Proposition 6.2 to the pre-canonical curve γ' the part of γ between φ_a and φ_b. This allows us to construct a canonical curve γ'' arbitrarily close to γ' and connecting φ_a to a point φ_b' arbitrarily close to φ_b. But, if this point is sufficiently close to φ_b, then we can use the C^1-stability of canonical curves and slightly deform γ_b in the δ-neighborhood of φ_b to obtain a new canonical curve γ_b' that ends at φ_b'. Now, it suffices to smooth out the pre-canonical curve $\gamma_a\cup\gamma''\cup\gamma_b'$. This completes the proof of Lemma 6.5. $\square$

6.8. **Enclosing canonical domains.** Let $\gamma\subset D$ be a line canonical with respect to κ. Denote by φ_a and φ_b the ends of γ so that $\mathrm{Im}\,\varphi_a<\mathrm{Im}\,\varphi_b$. Let a domain $K\subset D$ be a canonical domain corresponding to the triple $\varphi_a,\varphi_b,\kappa$. If $\gamma\in K$, then K is called a canonical domain *enclosing* γ. As any line C^1-close to γ is canonical, one can always construct a canonical domain enclosing any given canonical curve. We shall call such canonical domains *local.* In practice, one constructs canonical domains by means of

Proposition 6.3. *Let γ be a canonical line with respect to κ. Assume that $K\subset D$ is a simply connected domain containing γ (without its ends). The domain K is a canonical domain enclosing γ if it is the union of pre-canonical lines each of which is obtained from γ by replacing some its internal segments by a line pre-canonical with respect to κ.*

Proof. It suffices to prove that, for any $\varphi\in K$, there is a pre-canonical curve β lying in K, connecting φ_d and φ_u two internal points of γ such that $\mathrm{Im}\,\varphi_u>\mathrm{Im}\,\varphi_d$ and such that φ is an internal point of a segment of β being a canonical line. Indeed, consider the pre-canonical curve going (1) along γ from φ_a to φ_d, then (2) along β from φ_d to φ_u, and then (3) along γ from φ_u to φ_b. Consider the segments of this curve from φ_a to φ and from φ to φ_b. The proof of Proposition 6.3 is then completed by smoothing them out using Lemma 6.5 (with δ small enough).

We know that there is a pre-canonical curve β'' containing φ and connecting two internal points of γ. Let us modify this curve to construct the curve β. The point φ divides β'' into two pre-canonical curves β_u'' and β_d'' connecting φ respectively to the points φ_u'' and φ_d'' of γ. We assume that β_d'' is below of β_u''. Begin with describing the necessary transformation of β_u''. We can continue β_u'' somewhat beyond the point φ_u'' so that the new line remain pre-canonical. For this new line, we keep the old notation. By Proposition 6.2, in any neighborhood of β''_u, there is a canonical line β'_u beginning at φ and ending in any given neighborhood of the other end of β''_u. We can choose these two neighborhoods sufficiently small so that $\beta'_u \subset K$ and so that β'_u intersects γ at, say, φ_u. By construction, one has $\operatorname{Im}\varphi < \operatorname{Im}\varphi_u$. Similarly, starting from β''_d, we construct a canonical line $\beta'_d \subset K$ connecting φ with a point $\varphi_d \in \gamma$ situated below φ. The pre-canonical line $\beta'_d \cup \beta'_u$ would be the one we need, but φ is the common end of the two canonical lines β'_u and β'_d. To finish the proof we deform these curves so that φ be an internal point of one of them. To this end, we continue β'_u somewhat beyond φ so that the new curve β_u remain canonical. Then, using the C^1-stability of the canonical lines, we slightly deform β'_d to a new canonical curve β_d connecting φ_d to a point of β_u situated somewhat below of φ. The pre-canonical line going along β_d from φ_d to β_u, and then along β_u to φ_u is the needed one. This completes the proof of Proposition 6.3. □

7. Constructions of the complex WKB method for equation (1.1)

To use the complex WKB method, we rewrite (1.1) in terms of the variables $u = x - \phi$ and $\varphi = \varepsilon\phi$,

$$-\frac{d^2}{du^2}\psi(u) + (V(u) + \alpha\cos(\varepsilon u + \varphi))\psi(u) = E\psi(u), \quad u \in \mathbb{R}. \tag{7.1}$$

We describe the complex momentum and canonical domains for (7.1); in section 8, these are used to compute the monodromy matrix.

7.1. Complex momentum.

Properties of κ depend on the value of the spectral parameter E and on the lengths of the spectral zones and the spectral gaps of the periodic operator H_0. Recall that we assume condition (2.16) to hold.

Some general properties of the complex momentum were discussed in section 6.1. In section 2.5, we have stated some properties of the set of branch points of the complex momentum for equation (1.1). These properties are obvious consequences of the description of the Bloch quasi-momentum given in section 5.2.

Consider the strip $\{|\operatorname{Re}\varphi| < \pi\}$. Cut it from $-\pi$ to $-\varphi_1$ and from φ_1 to π. Denote the domain thus obtained by D_0. It is regular and simply connected. In D_0, one can fix a single-valued branch of the complex momentum by the conditions

$$\operatorname{Im}\kappa_0(0) > 0, \quad \operatorname{Re}\kappa_0(0) = 0. \tag{7.2}$$

Indeed, the domain D_0 is mapped by $\mathcal{E} : \varphi \to E - \alpha\cos\varphi$ onto the upper half plane of the complex plane. The branch κ_0 is related to the main branch of Bloch quasi-momentum by the formula

$$\kappa_0(\varphi) = k_0(E - \alpha\cos\varphi). \tag{7.3}$$

The branch κ_0 (defined on D_0) is called *the main branch* of the complex momentum. We now describe four

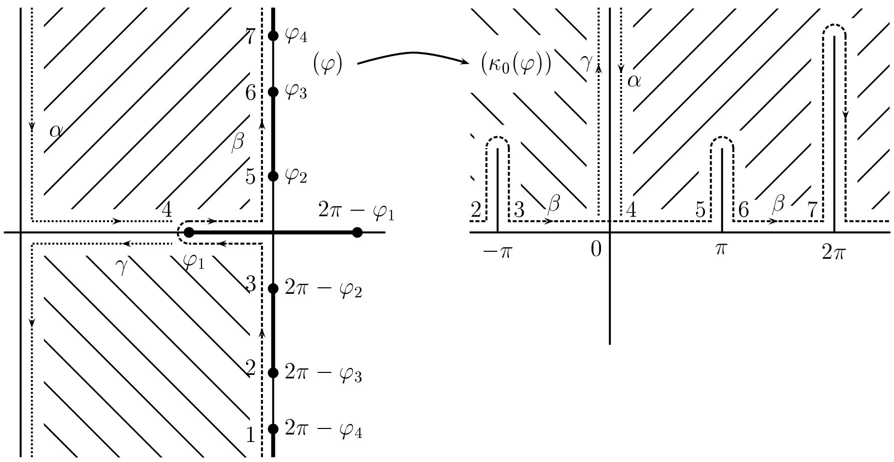

Figure 6. The action of the mapping κ

properties of the main branch of the complex momentum; they follow from (7.3) and from the properties of

k_0 described in section 5.2.2. The first two properties are

$$\kappa_0(-\varphi) = \kappa_0(\varphi), \tag{7.4}$$

$$\kappa_0(\overline{\varphi}) = -\overline{\kappa_0(\varphi)}. \tag{7.5}$$

Note that (7.4) is also valid for complex values of E.

The domain $\{\varphi \in D_0; \mathrm{Re}\,\varphi \geq 0\}$ is mapped by κ_0 onto the upper half plane with vertical cuts as in Fig. 6 (n this figure, we have drawn two curves on the complex plane $\varphi \in \mathbb{C}$, and their images by $\kappa_0(\varphi)$). Finally,

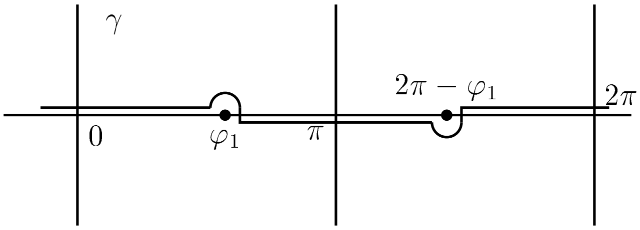

FIGURE 7. The period γ

we consider the 2π-periodic curve γ going along the real line around the branch points as shown in Fig. 7. Continuing κ_0 analytically along this curve, we get

$$\kappa_0(\varphi + 2\pi) = \kappa_0(\varphi), \quad \varphi \in \gamma. \tag{7.6}$$

7.2. **The Stokes lines.** The definition of the Stokes lines is fairly standard (see e.g. [15]). The integral $\int \kappa\, d\varphi$ has the same branch points as the complex momentum. Let φ_0 be one of them. Consider the curves beginning at φ_0 described by

$$\mathrm{Im} \int_{\varphi_0}^{\varphi} (\kappa(\xi) - \kappa(\varphi_0))\, d\xi = 0. \tag{7.7}$$

These curves are the *Stokes lines* beginning at φ_0. It follows from (6.8) that the Stokes line definition is independent of the choice of the branch of κ in (7.7).

As the branch points of the complex momentum are of square root type, exactly three Stokes lines begin at any branch point. At this point, the angle between any two nearest neighbor Stokes lines are equal to $\frac{2\pi}{3}$. These Stokes lines may be finite: they connect pairs of finite branch points; they may also be infinite: they go from finite branch points to infinity.

If φ_0 is a branch point for κ, then so is the point $\varphi_0 + 2\pi$. The Stokes lines starting at $\varphi_0 + 2\pi$ are just the 2π-translates of the Stokes lines starting at φ_0. Furthermore, the whole picture of the Stokes lines in the domain $0 \leq \mathrm{Re}\,\varphi \leq 2\pi$ is symmetric with respect to the real line as well as with respect to the line $\mathrm{Re}\,\varphi = \pi$. In Fig. 8, we have shown some of the Stokes lines; they are represented by dotted lines. Let us

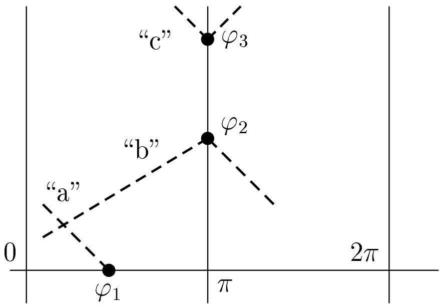

FIGURE 8. The Stokes lines

discuss them briefly.

Consider the Stokes lines beginning at the branch point φ_1. As κ_0 is real on the interval $[\varphi_1, \pi]$, this interval is a part of the Stokes line beginning at the point φ_1. There are two other Stokes lines beginning at this point. One of them is going upwards. We denote it by "a".

Consider the Stokes lines beginning at the branch point φ_2. As $\kappa_0 - \pi$ is purely imaginary on the segment $[\varphi_2, \varphi_3]$ of the line $\pi + i\mathbb{R}$, this segment coincides with the Stokes line connecting the points φ_2 and φ_3. There are two other Stokes lines starting at φ_2. We denote by "b" the Stokes line going to the left.

As already noted, one of the Stokes lines beginning at φ_3 coincides with the segment $[\varphi_2, \varphi_3]$ of the line $\pi + i\mathbb{R}$. Let "c" be the Stokes line starting at φ_3 going up to the left.

The global behavior of the Stokes lines "a", "b" and "c" is described by

Lemma 7.1.
1. *The Stokes line "a" stays vertical; it does not intersect the lines $i\mathbb{R}$ and $\pi + i\mathbb{R}$ and stays between them.*
2. *The Stokes line "b" intersects "a" above φ_1; the segment between its beginning and the intersection is vertical and stays between $i\mathbb{R}$ and $\pi + i\mathbb{R}$.*
3. *after intersecting "a", the Stokes line "b" goes downward staying vertical, then, not intersecting "a" anymore, it intersects either $\mathbb{R}$ or the Stokes line "d" symmetric of "a" with respect to $i\mathbb{R}$;*
4. *The Stokes line "c" stays vertical; it does not intersect the lines "a" and $\pi + i\mathbb{R}$ and stays between these two curves.*

Proof. First, consider the Stokes line "a". A Stokes line can become horizontal only at a point where $\operatorname{Im}\kappa = 0$, i.e. at a point of the pre-image of a spectral band. Therefore, "a" stays vertical as long as it stays (strictly) between $i\mathbb{R}$ and $\pi + i\mathbb{R}$. Assume that "a" goes upward between $i\mathbb{R}$ and $\pi + i\mathbb{R}$ and, then, intersects $\pi + i\mathbb{R}$ at φ_a. Denote the segment of "a" between φ_1 and φ_a by $\tilde{a}$. By the definition of Stokes lines and as the interval $[\varphi_1, \pi]$ is a Stokes line, one has

$$0 = \operatorname{Im} \int_{\tilde{a}} \kappa_0(\zeta) d\zeta = \operatorname{Im} \int_{\pi}^{\varphi_a} \kappa_0(\zeta) d\zeta = \int_{\pi, \text{ along } \pi + i\mathbb{R}}^{\varphi_a} \operatorname{Re} \kappa_0(\zeta) d\zeta. \tag{7.8}$$

But, on the line $\pi + i\mathbb{R}$, one has $\operatorname{Re}\kappa_0(\zeta) > 0$. So, the right-hand side of (7.8) is non-zero. Thus, the Stokes line "a" does not intersect $\pi + i\mathbb{R}$ and stays vertical if it does not intersect $i\mathbb{R}$.

So, now we assume that "a" intersects $i\mathbb{R}$ at φ_a. Denote the segment of "a" between φ_1 and φ_a by $\tilde{a}$. Then, as $\kappa_0 \in i\mathbb{R}$ on $i\mathbb{R}$,

$$0 = \operatorname{Im} \int_{\tilde{a}} \kappa_0(\zeta) d\zeta = \operatorname{Im} \int_{\varphi_1}^{0} \kappa_0(\zeta) d\zeta = \int_{\varphi_1, \text{ along } \mathbb{R}}^{0} \operatorname{Im} \kappa_0(\zeta) d\zeta.$$

The last integral is non-zero as $\operatorname{Im}\kappa_0 > 0$ on the real line between 0 and φ_1. The above two observations prove point 1 of Lemma 7.1.

Now, consider the Stokes line "b". If the lines "b" and "a" do not intersect one another, then "b" intersects either the interval $[\varphi_1^+, \pi]$ or the segment $[\pi, \varphi_2]$ of the line $\pi + i\mathbb{R}$. Assume that there are such intersections. Denote by φ_b the point where it happens for the first time. If $\varphi_b \in [\pi, \varphi_2]$, then, as $\kappa_0(\varphi_2) = \pi$,

$$0 = \operatorname{Im} \int_{\varphi_b, \text{ along "}b\text{"}}^{\varphi_2} (\kappa_0(\zeta) - \pi) d\zeta = \operatorname{Im} \int_{\varphi_b, \text{ along } \pi + i\mathbb{R}}^{\varphi_2} (\kappa_0(\zeta) - \pi) d\zeta.$$

As, along $[\pi, \varphi_2)$, one has $0 < \kappa_0 < \pi$, the right hand side in the above formula is non-zero. If $\varphi_b \in [\varphi_1, \pi]$, one has

$$0 = \operatorname{Im} \int_{\varphi_b, \text{ along "b"}}^{\varphi_2} (\kappa_0(\zeta) - \pi) d\zeta = \operatorname{Im} \int_{\pi, \text{ along } \pi + i\mathbb{R}}^{\varphi_2} (\kappa_0(\zeta) - \pi) d\zeta.$$

As before, we see that the right hand side in the last formula is non-zero. In result, we see that "b" must intersect "a" at a point φ_{ab} above $\mathbb{R}$; it stays between "a" and $\pi + i\mathbb{R}$ before the first intersection. This proves point 2 of Lemma 7.1.

To prove point 3, we need only to check that "b" can not intersect "a" once more before intersecting either "d" or $\mathbb{R}$. Indeed, assume that it intersects "a". Denote the intersection point by ζ_{ab}, then

$$0 = \operatorname{Im} \int_{\zeta_{ab}, \text{ along "b"}}^{\varphi_{ab}} (\kappa_0(\zeta) - \pi) d\zeta = \operatorname{Im} \int_{\zeta_{ab}, \text{ along "a"}}^{\varphi_{ab}} (\kappa_0(\zeta) - \pi) d\zeta = -\pi(\varphi_{ab} - \zeta_{ab})$$

which is impossible.

Check the last statement. Using almost the same argument as in the case of "a", one proves that "c" can not intersect $\pi + i\mathbb{R}$ before intersecting $i\mathbb{R}$. So, it suffices to prove that "c" stays to the right of "a". Assume that the Stokes lines "a" and "c" intersect. Let φ_{ac} be their first common point. Then,

$$0 = \mathrm{Im}\int_{\varphi_1,\text{ along "a"}}^{\varphi_{ac}} \kappa_0(\zeta)d\zeta = \mathrm{Im}\int_{\pi,\text{ along }\pi+i\mathbb{R}}^{\varphi_2} \kappa_0 d\zeta + \mathrm{Im}\int_{\varphi_2,\text{ along "c"}}^{\varphi_{ac}} \kappa_0 d\zeta,$$

Now, using the definition of the Stokes line "c", we transform the right hand side into

$$\int_{\pi,\text{ along }\pi+i\mathbb{R}}^{\varphi_2} \mathrm{Re}\,\kappa_0 d\mathrm{Im}\,\zeta + \pi\mathrm{Im}\,(\varphi_{ac}-\varphi_2).$$

This expression is positive. So, "a" and "c" can not intersect. This implies point 4 and completes the proof of Lemma 7.1. □

7.3. **Canonical domains.** Now, we describe the canonical domains used in the next section to compute the monodromy matrix.

7.3.1. *The domain K_0.* Consider the simply connected regular domain K_0^∞ corresponding to Fig. 9, part A. In this figure, the lines $\mathrm{Im}\int_{\zeta_0}^{\varphi}\kappa_0 d\zeta = \mathrm{Const}$ are represented by continuous curves, and the lines $\mathrm{Im}\int_{\zeta_0}^{\varphi}(\kappa_0 - \pi)d\zeta = \mathrm{Const}$ are shown as dotted curves. The boundary of K_0^∞ consists of Stokes lines. This domain exists for all E on the interval J defined by (1.4). On K_0^∞, we fix an analytic branch κ_0 of the complex momentum as the analytic continuation of the main branch of the complex momentum. Consider its subdomain K_0

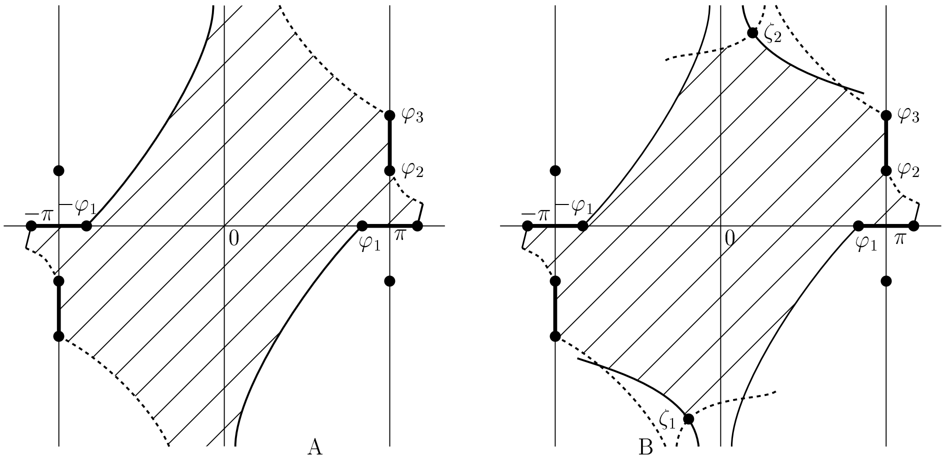

FIGURE 9. Canonical domains: K_0^∞ and K_0

corresponding to Fig. 9, part B. Its boundary contains the lines of Stokes type (with respect to κ_0) passing by ζ_1 and ζ_2 (as shown in the same figure). Assuming that such a domain exists, we define the horizontal strip $|\mathrm{Im}\,\varphi| \le Y(K_0)$ where K_0 and K_0^∞ coincide. One has

Proposition 7.1. *One can choose the points ζ_1 and ζ_2 so that the domain K_0 exist and so that $Y(K_0) > \mathrm{Im}\,\varphi_3$ be as large as desired. The domain K_0 is canonical with respect to the branch κ_0 and the points ζ_1 and ζ_2.*

Proof. As usual, we identify $\mathbb{R}^2$ and $\mathbb{C}$. Recall that the lines of Stokes type $\mathrm{Im}\int^{\varphi}\kappa_0 d\zeta = \mathrm{Const}$ are integral curves of the vector field $\overline{\kappa_0(\varphi)}$, and that the lines of Stokes type $\mathrm{Im}\int^{\varphi}(\kappa_0-\pi)d\zeta = \mathrm{Const}$ are integral curves of the vector field $\overline{\kappa_0(\varphi)} - \pi$.

First, choosing the points ζ_1 and ζ_2 in K_0^∞ properly, we show that there exists a domain K_0 bounded by the lines of Stokes type as shown in Fig. 9, and that $Y(K_0)$ can be made as large as desired. The proof is split into a few steps.

1. Pick a point A on the Stokes line "c". Recall that along "c", $\mathrm{Im}\int_{\varphi_3}^{\varphi}(\kappa_0-\pi)d\zeta=0$, see Fig. 10. Consider γ_A, the line of Stokes type $\mathrm{Im}\int_A^{\varphi}\kappa_0 d\zeta=0$, passing by A. It is transversal to the Stokes line "c" at A. Show that, above A, it is vertical and stays between "a" and "c".

At A, γ_A is tangent to the vector $\overline{\kappa_0(A)}$; in a sufficiently small neighborhood of A, above A, it goes up and stays to the left of "c" and to the right of "a". Above A, it can not intersect "a" (at least, without intersecting "c") as "a" is also a line of Stokes type along which $\mathrm{Im}\int^{\varphi}\kappa_0 d\zeta=\mathrm{Const}$. Assumes that γ_A intersects "c" above A. Denote the intersection point by ζ_i. Then,

$$0=\mathrm{Im}\int_{A,\ \text{along}\ \gamma_A}^{\zeta_i}\kappa_0 d\zeta$$

$$=\mathrm{Im}\int_{A,\ \text{along "c"}}^{\zeta_i}\kappa_0 d\zeta=\pi\mathrm{Im}\,(\zeta_i-A)$$

which is impossible. So, γ_A stays between "a" and "c" which also implies that it is vertical above A.

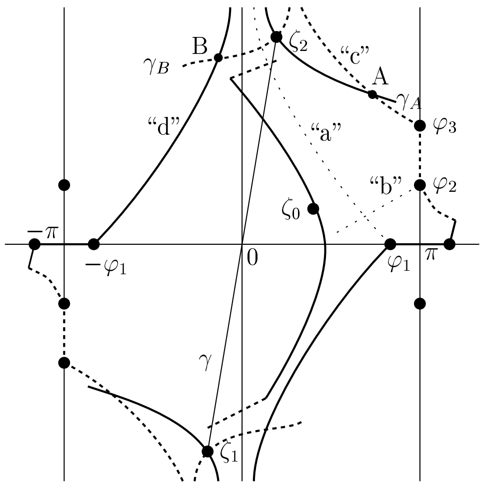

FIGURE 10. The construction of K_0

2. We pick ζ_2 on γ_A above A. Consider γ_B, the line of Stokes type $\mathrm{Im}\int_{\zeta_2}^{\varphi}(\kappa_0-\pi)d\zeta=0$ passing by ζ_2. At ζ_2, it is transversal to γ_A and vertical. One shows that, below ζ_2,

- γ_B goes down staying to the left of the lines γ_A, "c", the segment $[\varphi_2,\varphi_3]$ and the line "b",
- then, it intersects either $\mathbb{R}$ or "d", the Stokes line symmetric to "a" with respect $i\mathbb{R}$,
- it stays vertical between ζ_2 and B, the intersection point with either $\mathbb{R}$ or "d".

Indeed, being tangent to the vector $\overline{\kappa_0(\zeta_2)-\pi}$ at ζ_2, γ_B goes down from ζ_2 and, in a sufficiently small neighborhood of ζ_2, stays to the left of γ_A and to the right of "d". Then, it stays vertical and goes down at least while between "d" and γ_A.
Let R be the "rectangle" bounded by "d" , the line $\mathrm{Im}\,\varphi=\mathrm{Im}\,\zeta_2$, γ_A and the line $\mathrm{Im}\,\varphi=\mathrm{Im}\,A$. If γ_B leaves R via "d", the second step is completed. If it leaves R via ζ_i, a point of γ_A, then

$$\pi\mathrm{Im}\,(\zeta_i-\zeta_2)=\mathrm{Im}\int_{\zeta_2,\ \text{along}\ \gamma_B}^{\zeta_i}\kappa_0 d\zeta=\mathrm{Im}\int_{\zeta_2,\ \text{along}\ \gamma_A}^{\zeta_i}\kappa_0 d\zeta=0$$

which is impossible. So, we are left with the case where it leaves R through a point of the line $\mathrm{Im}\,\varphi=\mathrm{Im}\,A$ staying between "d" and γ_A.
Now, consider "P", the "polygon" bounded by the lines "d", $\mathrm{Im}\,\varphi=\mathrm{Im}\,A$, "c", the segment $[\varphi_2,\varphi_3]$ of the line $\pi+i\mathbb{R}$, the line "b" and, possibly, $\mathbb{R}$. Inside P, γ_B stays vertical and goes down. If γ_B leaves P via "d" or $\mathbb{R}$, the second step is completed. This is the case as it can not intersect the other lines forming the boundary of P since all of them are lines of Stokes type along which $\mathrm{Im}\int^{\varphi}(\kappa_0-\pi)d\zeta=\mathrm{Const}$.
Note that, if B is real, then it is located between $-\varphi_1$, the starting point of "d", and φ_1, the starting point of "a".

3. Show that, by choosing $\mathrm{Im}\,\zeta_2$ large enough, one can make B belong to "d" and $\mathrm{Im}\,B$ as large as desired. First assume that $B\in\mathbb{R}$. Using the definitions of the lines of Stokes type, one computes

$$\pi\mathrm{Im}\,(\zeta_2)=\mathrm{Im}\int_{B,\ \text{along}\ \gamma_B}^{\zeta_2}\kappa_0 d\zeta=\mathrm{Im}\int_{B,\ \text{along}\ \mathbb{R}}^{\varphi_1}\kappa_0 d\zeta+\mathrm{Im}\int_{\varphi_1,\ \text{in}\ K_0^{\infty}}^{A}\kappa_0 d\zeta.$$

The second term in the right-hand side is constant, and the first one increases with $\mathrm{Im}\,\zeta_2$. So, if $\mathrm{Im}\,\zeta_2$ is large enough, then B becomes a point of "d". Assume $B\in$"d". Along both "d" and γ_A, $\mathrm{Im}\int^{\varphi}\kappa_0 d\zeta=\mathrm{Const}$; so, one gets

$$\pi\mathrm{Im}\,(\zeta_2-B)=\mathrm{Im}\int_{-\varphi_1,\ \text{in}\ K_0^{\infty}}^{A}\kappa_0 d\zeta.$$

So, $\mathrm{Im}\,B$ linearly increases with $\mathrm{Im}\,\zeta_2$. This completes the third step.

4. One chooses the point ζ_1 symmetric to ζ_2 with respect to 0. As κ_0 and the cosine are even ((7.4)), the families of lines $\mathrm{Im}\int^{\varphi}(\kappa_0-\pi)d\zeta=\mathrm{Const}$ and $\mathrm{Im}\int^{\varphi}\kappa_0 d\zeta=\mathrm{Const}$ are symmetric with respect to $\varphi=0$. In result, we see that the domain K_0 exists and that $Y(K_0)$, the size of the horizontal strip, where K_0 coincides with K_0^{∞}, can be made as large as wanted.

By means of Proposition 6.3, we show that K_0 is canonical. The proof is again split into a few steps.

5. We say that the lines of a family $(l_v)_{v\in U}$ *fibrate* a domain D if D is the disjoint union of the lines $(l_v)_{v\in U}$. We use the fact that the families lines of Stokes type $\mathrm{Im}\int^{\varphi}\kappa_0 d\zeta=\mathrm{Const}$ and $\mathrm{Im}\int^{\varphi}(\kappa_0-\pi)d\zeta=\mathrm{Const}$ both fibrate K_0. Indeed, consider, for example, the first family. Pick a point ζ_0 in K_0. If there are two lines of Stokes type passing through ζ_0, then ζ_0 is a critical point for the harmonic function $\mathrm{Im}\int_0^{\varphi}\kappa_0 d\zeta=\mathrm{Const}$. Then $\kappa_0(\zeta_0)=0$. This is possible only at a branch point of κ_0 and there are no branch points inside of K_0.

6. We construct a line $\gamma\in K_0$ connecting ζ_1 and ζ_2 and canonical with respect to κ_0. Therefore, we use Lemma 6.5 and make some preparations that we present now.
First, in a neighborhood of ζ_2, pick a segment γ_2 of a vertical, say, strait line starting at ζ_2 and going down between γ_A and γ_B. At ζ_2, the last two lines are tangent respectively to the vectors $\overline{\kappa_0(\zeta_2)}$ and $\overline{\kappa_0(\zeta_2)}-\pi$. So, at least, the part of γ_2 situated in a small enough neighborhood of ζ_2 is a canonical line. To keep the notations simple, we denote it also by γ_2. The line γ_1 starting at ζ_1 and symmetric to γ_2 with respect to 0 is also canonical.

Describe a pre-canonical line $\beta\subset K_0$ connecting a_1, an internal point of γ_1 to a_2, an internal point of γ_2. Consider σ_2, the line of Stokes type containing a_2 and satisfying the relation $\mathrm{Im}\int_{\varphi}^{a_2}(\kappa_0-\pi)d\zeta=0$. As the lines of the Stokes type $\mathrm{Im}\int^{\varphi}(\kappa_0-\pi)d\zeta=\mathrm{Const}$ fibrate K_0, choosing a_2 close enough to γ_B, we get that **(1)** σ_2 stays in the part of K_0 where $\mathrm{Im}\,\kappa_0>0$ and, so is vertical in K_0; **(2)** the segment of σ_2 going down from a_2 intersects "d". In particular, together with γ_B, σ_2 intersects the line $i\mathbb{R}$ at a point b_2 having a positive imaginary part.

The pre-canonical line β is symmetric with respect to 0. It begins at a_2, then, along σ_2, it goes down to its intersection with $i\mathbb{R}$, and then goes down along $i\mathbb{R}$ to the origin.

The line $\tilde\beta=\gamma_1\cup\beta\cup\gamma_2$ is pre-canonical. The pre-canonical line $\tilde\beta$ found, Lemma 6.5 implies that, as close to β as desired, there exists γ, a canonical line connecting ζ_1 to ζ_2 so that, in a sufficiently small neighborhood of ζ_1, γ coincides with γ_1, and, in a sufficiently small neighborhood of ζ_2, it coincides with γ_2. We choose γ so that it belong to K_0.

7. By means of Proposition 6.3, we show that K_0 is a canonical domain enclosing γ (actually, it is the maximal canonical domain, but, we do not need this fact). Pick a point ζ_0 in K_0. We need only to check that, in K_0, there is a pre-canonical curve β connecting two internal points of γ and containing ζ_0. We assume that $\mathrm{Im}\,\zeta_0\geq 0$. Due to the symmetry of K_0, β and $\gamma_{1,2}$ with respect to the origin, for $\mathrm{Im}\,\zeta_0<0$, the analysis is similar. There are four cases to be considered.

7a. First, we assume that ζ_0 is to the left of "a". Consider σ_0, a line of Stokes type $\mathrm{Im}\int_{\zeta_0}^{\varphi}\kappa_0 d\zeta=0$, containing ζ_0. As ζ_0 is between "d" and "a", and as the lines of Stokes type fibrate K_0, the line σ_0 stays between the lines "a" and "d", and also between "$\overline{\text{a}}$" and "$\overline{\text{d}}$", the lines symmetric to "a" and "d" with respect to the real line. So, it stays vertical and has to intersect one of the lines γ_A and γ_B and one of the lines $-\gamma_A$ and $-\gamma_B$, the symmetrics of γ_A and γ_B with respect to the origin. In fact, as both γ_A and $-\gamma_A$ belong to the same family of lines of Stokes type as σ_0, it intersects γ_B and $-\gamma_B$.

To construct the needed pre-canonical curve, we pick c_1 and c_2, two internal points of σ_0, so that $\mathrm{Im}\,c_1<\mathrm{Im}\,\zeta_0<\mathrm{Im}\,c_2$.
Consider α_2, the line of Stokes type from the same family as γ_B and beginning at c_2. As the lines of Stokes type fibrate K_0, choosing c_2 close enough to γ_B, we get that **(1)** α_2 stays in the part of K_0 where $\mathrm{Im}\,\kappa_0>0$, and, thus, is vertical; **(2)** its segment beginning at c_2 and going up intersects γ_A. As γ_A is a part of the right boundary of K_0, this segment intersects also γ.
Similarly, by properly choosing c_1, one constructs α_1, a vertical line of Stokes type in K_0, connecting the point c_1 to an internal point of γ situated below c_1. The needed pre-canonical line β goes from this point of γ along α_1 to σ_0, then, along σ_0, to α_2, and, then along α_2 to an internal point of γ. The construction of β is illustrated it by Fig. 10.

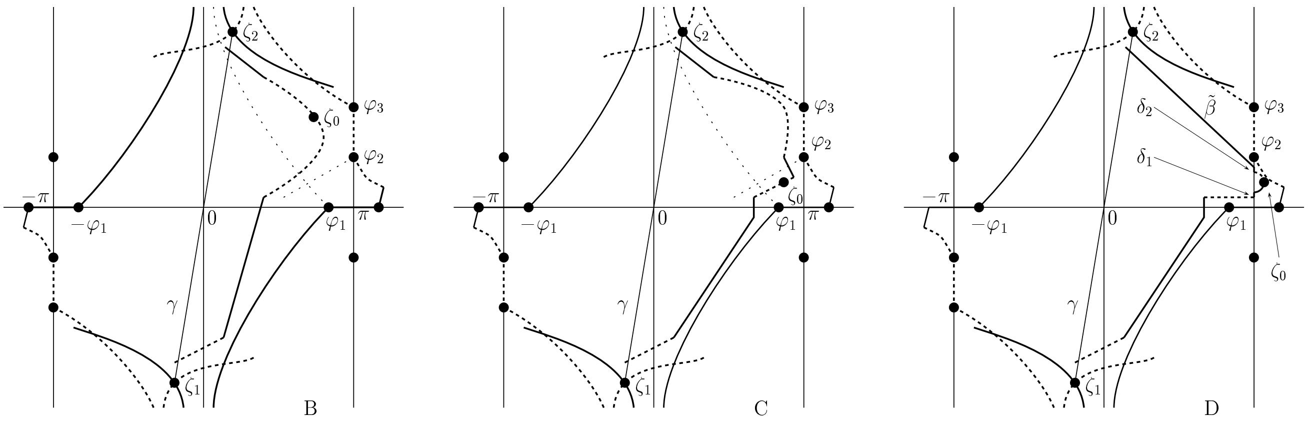

FIGURE 11. The pre-canonical line β

7b. Assume that ζ_0 is situated to the right of or on "a" and above "b". The construction of the pre-canonical curve corresponds to Fig. 11, part B.

7c. If ζ_0 is to the left of the line $\pi + i\mathbb{R}$ and either below "b" or on "b", we construct β as in Fig. 11, part C.

7d. If ζ_0 is either to the right of the line $\pi + i\mathbb{R}$ or on this line, then, we first construct a line $\tilde{\beta}$ as shown in Fig. 11, part D. This line contains δ_1 and δ_2, two segments of the line $\pi + i\mathbb{R}$. As, on the line $\pi + i\mathbb{R}$, $0 < \kappa_0 < \pi$, this line and its segments δ_1 and δ_2 are canonical. But, the line $\tilde{\beta}$ is not yet pre-canonical : as, on $\pi + i\mathbb{R}$, κ_0 is real, the lines of Stokes type are horizontal (i.e. are not vertical) at the points of $\pi + i\mathbb{R}$. To correct this, we use the C^1-stability of the canonical lines. We slightly deform the canonical line $\pi + i\mathbb{R}$ so as to get a pre-canonical line from its deformed segments δ_1 and δ_2 and $\tilde{\beta}$.

We have shown that, for any point $\zeta_0 \in K_0$, there exists a pre-canonical line β containing ζ_0 as an internal point and connecting to internal points of the canonical line γ. As explained, this implies that K_0 is canonical. This completes the proof. □

We finish this section by noting that the relations (7.6) and (7.5) imply that any domain obtained from K_0 by means of 2π-translations and the reflection with respect to the real line is canonical.

7.3.2. *The domain K_1.* Consider the simply connected regular domain K_1^∞ corresponding to Fig. 12, part A. The boundary of K_1^∞ consists of Stokes lines. This domain exists for all E in the interval J defined by (1.4). Let κ_1 be the analytic continuation of κ_0 from K_0^∞ to K_1^∞ through their common part. Consider the subdomain $K_1 \subset K_1^\infty$ corresponding to Fig. 12, part B. Its boundary contains the lines of Stokes type (with respect to κ_1) passing by ζ_1 and ζ_2 (as shown in the same figure). Consider the horizontal strip $|\mathrm{Im}\,\varphi| \le Y(K_1)$ where K_1 and K_1^∞ coincide. One has

Proposition 7.2. *One can choose the points ζ_1 and ζ_2 so that the domain K_1 exist and so that $Y(K_1) > \mathrm{Im}\,\varphi_3$ be as large as desired. The domain K_1 is canonical with respect to the branch κ_1 and the points ζ_1 and ζ_2.*

The proof of this statement being analogous to one of Proposition 7.1, we omit it.

We finish this section by noting that κ_1 satisfies the following symmetry relations

$$\kappa_1(2\pi - \varphi) = \kappa_1(\varphi), \tag{7.9}$$

$$\kappa_1(\overline{\varphi}) = \overline{\kappa_1(\varphi)}. \tag{7.10}$$

These relations follow from the definition of κ_1. Let $\psi_\pm^1(x,\varphi)$ be the Bloch solutions, meromorphic in K_1 such that κ_1 be the quasi-momentum of ψ_+^1. One has

$$\psi_-^1(x,\varphi) = \overline{\psi_+^1(x,\overline{\varphi})}, \quad \psi_\pm^1(x,2\pi-\varphi) = \psi_\pm^1(x,\varphi). \tag{7.11}$$

Indeed, the first symmetry holds as the interval $[\varphi_1, 2\pi - \varphi_1]$ is a connected component of the pre-image of a spectral band, for $\varphi \in [\varphi_1, 2\pi - \varphi_1]$, κ_1 is real. The second symmetry holds as $\cos(2\pi - x) = \cos(x)$.

Using (6.13), define the corresponding functions $\omega^1_\pm$. Equation (7.11) then implies

$$\overline{\omega^1_+(\varphi)} = \omega^1_-(\overline{\varphi}), \quad \omega^1_\pm(2\pi - \varphi) = -\omega^1_\pm(\varphi). \tag{7.12}$$

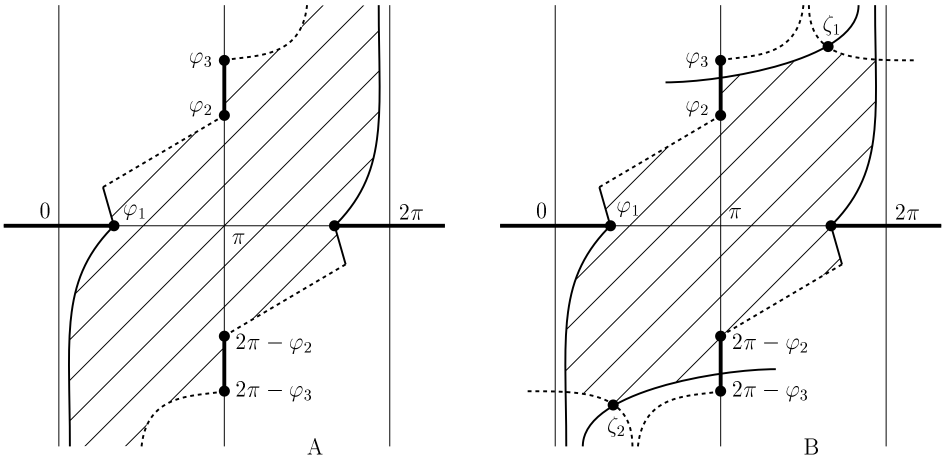

FIGURE 12. Canonical domains: K_1^∞ and K_1

7.4. **Canonical domain** $K_2 = K_0 + 2\pi$**.** We check that the domain $K_2 = K_0 + 2\pi$ is canonical, and discuss the solutions constructed by Theorem 6.1 on this domain.

7.4.1. *Bloch solutions along periodic curves.* We need the following simple observations. Consider the two branches of $\psi(x, E - \alpha\cos\varphi)$, the Bloch solution of (6.10) with $W(\varphi) = \alpha\cos\varphi$. Consider them on the domain D_0 where the main branch κ_0 was defined. We index these branches by $\psi^0_\pm(x,\varphi)$ so that their Bloch quasi-momenta be equal to $\pm\kappa_0(\varphi)$.

Continue $\psi_\pm$ analytically along the periodic curve γ corresponding to Fig. 7. Then, along γ,

$$\psi^0_\pm(x, \varphi + 2\pi) = \psi^0_\pm(x,\varphi), \tag{7.13}$$

$$\omega^0_\pm(\varphi + 2\pi) = \omega^0_\pm(\varphi), \tag{7.14}$$

where $\omega^0_\pm$ are constructed by (6.13) in terms of $\psi^0_\pm$. Equation (7.14) follows from the definition of the functions $\omega_\pm$ and from (7.13). And (7.13) holds as (1) there are only two branches of the function $\psi(x, E - \alpha\cos\varphi)$; (2) κ_0 is the Bloch quasi-momentum of ψ_+; (3) κ_0 is 2π-periodic along γ.

On γ, fix an analytic branch of $q_0 = \sqrt{k_0'(E - \alpha\cos\varphi)}$. Recall that $k'(\mathcal{E}) \in -i\mathbb{R}_+$ for $\mathcal{E} < E_1$. We fix the branch of q_0 so that $q_0 \in e^{-i\pi/4}\mathbb{R}_+$. One has

$$q_0(\varphi + 2\pi) = q_0(\varphi), \quad \varphi \in \gamma. \tag{7.15}$$

This relation easily follows from analytic properties of the main branch of the Bloch quasi-momentum k_0 ($k'(\mathcal{E})$ can vanish only in spectral gaps).

7.4.2. *The domain* K_2. Define the domain K_2 by $K_2 = K_0 + 2\pi$ and, in K_2, define κ_2, $\psi^2_\pm(x,\varphi)$, $\omega^2_\pm$ and q_2 as the analytic continuations of the functions κ_0, $\psi^0_\pm(x, E - \alpha\cos\varphi)$, $\omega^0_\pm$ and q_0 to K_2 along the periodic curve γ corresponding to Fig. 13.
Recall that the analytic continuations of κ_0, $\psi^0_\pm(x, E - \alpha\cos\varphi)$, $\omega^0_\pm$ and q_0 along γ are 2π-periodic. This implies

Lemma 7.2. *The domain K_2 is canonical with respect to κ_2. Moreover, if $f^0_\pm$ is the consistent basis solutions constructed for K_0 by Theorem 6.1 and normalized at the point $\varphi = 0$, then the functions $f^2_\pm(\varphi) = f^0_\pm(\varphi - 2\pi)$ have on K_2 the same asymptotics as the consistent basis solutions constructed for K_2 by Theorem 6.1 and normalized at the point $\varphi = 2\pi$.*

8. The proof of Theorem 2.2

This section is devoted to the proof of Theorem 2.2.

8.1. **The monodromy matrix.** Let $(f^0_\pm)$ be the solutions of (7.1) constructed by Theorem 6.1 for the canonical domain K_0. To this basis, we associate the matrix $\tilde{M}$ defined by

$$F^0(u,\varphi+2\pi)=\tilde{M}(\varphi)F^0(u,\varphi),\quad F^0=\begin{pmatrix} f^0_+\\ f^0_-\end{pmatrix}. \tag{8.1}$$

In (8.1), changing the variables (u,φ) to the variables (x,ϕ) of the input equation (1.1), we see that $\tilde{M}$ is related to the monodromy matrix M by the formula $\tilde{M}(\varphi)=M(\phi)$, $\varphi=\varepsilon\phi$. The matrix $\tilde{M}$ is unimodular and ε-periodic (compare with (2.3)). We define the coefficients of the matrix $\tilde{M}(\varphi)$ by

$$\tilde{M}(\varphi)=\begin{pmatrix} m_{11}(\varphi) & m_{12}(\varphi)\\ m_{21}(\varphi) & m_{22}(\varphi)\end{pmatrix}.$$

We prove

Proposition 8.1. *Under the assumptions of Theorem 2.2, $\tilde{M}$ is analytic in $\mathcal{V}=\{|E-E_0|<\eta\}\times\{|\mathrm{Im}\,\phi|\le Y/\varepsilon\}$ and, in $\mathcal{V}$, its coefficients admit the uniform asymptotics*

$$m_{11}(\varphi)=t_h\cdot e^{i\Phi/\varepsilon}(1+o(1)),$$
$$m_{12}(\varphi)=iG\cdot\left(e^{i\Phi/\varepsilon}(1+o(1))-t_v\cdot e^{\frac{2i\pi}{\varepsilon}(\varphi-\varphi_{(0)})}(1+o(1))\right),$$
$$m_{21}(\varphi)=-iG^{-1}\cdot\left(e^{i\Phi/\varepsilon}(1+o(1))-t_v\cdot e^{-\frac{2i\pi}{\varepsilon}(\varphi-\varphi_{(0)})}(1+o(1))\right),$$
$$m_{22}(\varphi)=\frac{1}{t_h}e^{-i\Phi/\varepsilon}(1+o(1))+\frac{1}{t_h}e^{i\Phi/\varepsilon}(1+o(1))-\frac{t_v}{t_h}\left(e^{i\frac{2\pi}{\varepsilon}(\varphi-\varphi_{(0)})}(1+o(1))+e^{-i\frac{2\pi}{\varepsilon}(\varphi-\varphi_{(0)})}(1+o(1))\right).$$

Here, the phase Φ, the coefficients t_h, t_v are defined in (2.10) *and* (2.11), *and*

$$G=\exp\left(\int_0^{\varphi_1}(\omega_+-\omega_-)d\varphi\right),\quad \varphi_{(0)}=-\frac{i\varepsilon}{2\pi}\int_{\varphi_1}^{\varphi_2}(\omega_--\omega_+)d\varphi-\pi, \tag{8.2}$$

where the integrals are taken along curves situated in K_0, and the functions $\omega_\pm$ are the ones defined for K_0 in 7.3.1. The function $\varphi_{(0)}$ is real analytic.

We deduce Theorem 2.2 from Proposition 8.1 by changing the consistent basis; this is done in section 8.6. The next sections are devoted to the proof of Proposition 8.1.

8.2. **Canonical domains K_0, K_1, K_2 and transition matrices.** It is easy to compute the asymptotics of $\tilde{M}$ in a subset of K_0 where we know both the asymptotic behavior of $F^0(u,\varphi+2\pi)$ and $F^0(u,\varphi)$. Unfortunately, a quick look at section 7.3 and figure Fig. 13 shows us that $K_0\cap(K_0+2\pi)=\emptyset$. Therefore, we need to introduce additional canonical domains.

The first domain we need is K_1 (see Fig. 12 B and subsection 7.3). Let $(f^1_\pm)$ be the consistent basis constructed by Theorem 6.1 in K_1. As $(f^0_\pm)$ and $(f^1_\pm)$ are both basis of solutions of (7.1), we can write

$$F^0(u,\varphi)=T_1(\varphi)F^1(u,\varphi),\quad F^1=\begin{pmatrix} f^1_+\\ f^1_-\end{pmatrix}. \tag{8.3}$$

where the matrix T_1 is independent of u. As $(f^0_\pm)$ and $(f^1_\pm)$ satisfy the consistency condition 6.3, the matrix T_1 is ε-periodic.

The second canonical domain we need is K_2, see section 7.4.2. Define $f^2_\pm(\varphi)=f^0_\pm(\varphi-2\pi)$. It is a consistent basis of solutions with standard asymptotic behavior on K_2. As $(f^2_\pm)$ and $(f^1_\pm)$ are two bases of solutions of (7.1), we have

$$F^1(u,\varphi)=T_2(\varphi)F^2(u,\varphi),\quad F^2=\begin{pmatrix} f^2_+\\ f^2_-\end{pmatrix}. \tag{8.4}$$

where the matrix T_2 is independent of u and ε-periodic in φ.

Putting (8.3) and (8.4) together, we get

$$F^0(u,\varphi)=T_1(\varphi)T_2(\varphi)F^2(u,\varphi)=T_1(\varphi)T_2(\varphi)F^0(u,\varphi-2\pi).$$

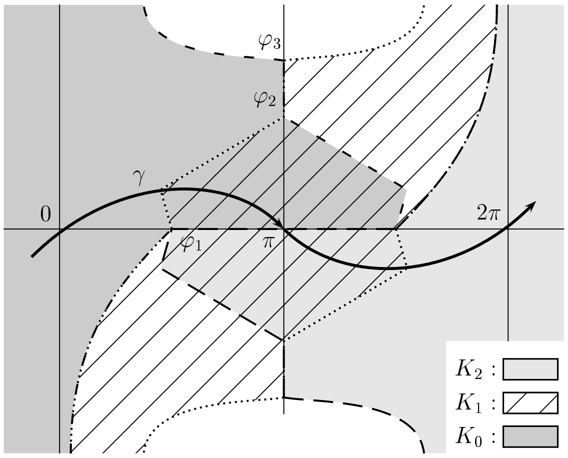

FIGURE 13. The three canonical domains

Hence, the monodromy matrix reads

$$\tilde{M}(\varphi) = T_1(\varphi + 2\pi)T_2(\varphi + 2\pi). \tag{8.5}$$

To prove Proposition 8.1, we study the matrices T_1 and T_2. We denote their coefficients as follows

$$T_1(\varphi) = \begin{pmatrix} a_1(\varphi) & b_1(\varphi) \\ c_1(\varphi) & d_1(\varphi) \end{pmatrix}, \quad T_2(\varphi) = \begin{pmatrix} a_2(\varphi) & b_2(\varphi) \\ c_2(\varphi) & d_2(\varphi) \end{pmatrix}. \tag{8.6}$$

Below, when describing the asymptotics of the matrices T_1 and T_2 and when computing these asymptotics, the contour integrals of κ_j and $\omega_\pm^j$ $j = 0, 1, 2$, are taken along curves in K_j in all the cases when no other choice is described explicitly.

One proves

Proposition 8.2. *Under the assumptions of Theorem 2.2, T_1 is analytic in $\mathcal{V} = \{|E - E_0| < \eta\} \times \{|\mathrm{Im}\,\phi| \leq Y/\varepsilon\}$ and, in $\mathcal{V}$, its coefficients admit the uniform asymptotics*

$$a_1(\varphi) = e^{\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi + \int_0^\pi \omega_+^0 d\varphi}(1 + o(1)), \tag{8.7}$$

$$b_1(\varphi) = e^{\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi} \cdot o(1), \tag{8.8}$$

$$\begin{aligned} c_1(\varphi) = -ie^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi} \Big(& e^{2\frac{i}{\varepsilon}\int_{\varphi_1}^\pi \kappa_1 d\varphi} e^{\int_0^{\varphi_1} \omega_-^0 d\varphi - \int_\pi^{\varphi_1} \omega_+^1 d\varphi}(1 + o(1)) \\ & - e^{-2\frac{i}{\varepsilon}\int_\pi^{\varphi_2}(\kappa_0 - \pi) d\varphi} e^{\int_0^{\varphi_2} \omega_-^0 d\varphi - \int_\pi^{\varphi_2} \omega_+^0 d\varphi} e^{-2\pi\frac{i}{\varepsilon}(\varphi - \pi)}(1 + o(1)) \Big), \end{aligned} \tag{8.9}$$

$$d_1(\varphi) = e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi + \int_0^\pi \omega_-^0 d\varphi}(1 + o(1)). \tag{8.10}$$

and

Proposition 8.3. *Under the assumptions of Theorem 2.2, T_2 is analytic in $\mathcal{V} = \{|E - E_0| < \eta\} \times \{|\mathrm{Im}\,\phi| \leq Y/\varepsilon\}$ and, in $\mathcal{V}$, its coefficients admit the uniform asymptotics*

$$a_2(\varphi) = e^{\frac{i}{\varepsilon}\int_\pi^{2\pi} \kappa_2 d\varphi + \int_\pi^{2\pi} \omega_+^2 d\varphi}(1 + o(1)), \tag{8.11}$$

$$\begin{aligned} b_2(\varphi) = ie^{-\frac{i}{\varepsilon}\int_\pi^{2\pi} \kappa_2 d\varphi} \Big(& e^{2\frac{i}{\varepsilon}\int_\pi^{2\pi - \varphi_1} \kappa_1 d\varphi} e^{\int_\pi^{2\pi - \varphi_1} \omega_+^1 d\varphi - \int_{2\pi}^{2\pi - \varphi_1} \omega_-^2 d\varphi}(1 + o(1)) \\ & - e^{2\frac{i}{\varepsilon}\int_\pi^{\bar{\varphi}_2}(\kappa_2 - \pi) d\varphi} e^{\int_{\bar{\varphi}_2}^{2\pi} \omega_-^2 d\varphi - \int_{\bar{\varphi}_2}^{\pi} \omega_+^2 d\varphi} e^{2\pi\frac{i}{\varepsilon}(\varphi - \pi)}(1 + o(1)) \Big), \end{aligned} \tag{8.12}$$

$$c_2(\varphi) = e^{-\frac{i}{\varepsilon}\int_{2\pi}^{2\pi - \varphi_1} \kappa_2 d\varphi} \cdot o(1), \tag{8.13}$$

$$d_2(\varphi) = e^{-\frac{i}{\varepsilon}\int_\pi^{2\pi} \kappa_2 d\varphi + \int_\pi^{2\pi} \omega_-^2 d\varphi}(1 + o(1)). \tag{8.14}$$

The next sections are devoted to the proof of Propositions 8.2 and 8.3. As it is similar to the one of Proposition 8.2, we do not give a detailed proof of Proposition 8.3; we just describe the starting points and partial results.

As a last remark, let us say that, from now on, C denotes a positive constant independent of φ and ε.

8.3. **The asymptotics of T_1.** By definition, we have

$$a_1(\varphi) = \frac{1}{w_1} w(f_+^0, f_-^1), \quad b_1(\varphi) = -\frac{1}{w_1} w(f_+^0, f_+^1),$$
$$c_1(\varphi) = \frac{1}{w_1} w(f_-^0, f_-^1), \quad d_1(\varphi) = -\frac{1}{w_1} w(f_-^0, f_+^1), \tag{8.15}$$
$$w_1 = w(f_+^1, f_-^1) = f_-^1 \frac{d}{du} f_+^1 - f_+^1 \frac{d}{du} f_-^1.$$

The solutions $(f_\pm^j)$ are normalized at the points ν_j, $j = 0, 1$ where

$$\nu_0 = 0, \quad \nu_1 = \pi.$$

For convenience, we recall the asymptotics of $f_\pm^j$

$$f_\pm^j(u,\varphi) = e^{\pm \frac{i}{\varepsilon} \int_{\nu_j}^{\varphi} \kappa_j \, d\varphi} (\Psi_\pm^j(u,\varphi) + o(1)), \quad \varphi \in K_j, \tag{8.16}$$

$$\Psi_\pm^j(x,\varphi) = q_j(\varphi) e^{\int_{\nu_j}^{\varphi} \omega_\pm^j \, d\varphi} \psi_\pm^j(x,\varphi). \tag{8.17}$$

Recall that κ_1, $\psi_\pm^1$, $\omega_\pm^1$ and q_1 are the analytic continuations of respectively κ_0, $\psi_\pm^0$, $\omega_\pm^0$ and q_0 along the periodic curve γ from K_0 to K_1 (see section 7.4.1).

Note that, in view of Lemma 6.2, one has

$$w_j = w(f_+^j, f_-^j) = q_j^2 w(\psi_+^j, \psi_-^j)|_{\varphi = \nu_j} + o(1). \tag{8.18}$$

These Wronskians are non-zero. Indeed,

- as ν_j are different from branch points of the complex momentum, ψ_+^j and ψ_-^j are linearly independent;
- $q^2 = k'(E - \alpha \cos\varphi)$ only vanishes at points in the pre-image of the spectral gaps $\cup_{l=1}^{\infty}(E_{2l}, E_{2l+1})$, of the periodic Schrödinger operator with respect to the mapping $\mathcal{E} : \varphi \mapsto E - \alpha\cos\varphi$ (see section 5.2.2), and ν_j do not belong to this pre-image.

8.3.1. *Periodicity.* Recall that the coefficients of T_1 are ε-periodic. Hence, to get their asymptotics, say, on a line $\mathbb{R} + iC$, one needs only to get them on a segment of this line of length ε.

8.3.2. *Three substrips.* Fix $\operatorname{Im}\varphi_2 < Y < \operatorname{Im}\varphi_3$. Pick K_0 and K_1 so that

$$Y < Y(K_0) < \operatorname{Im}\varphi_3, \quad \text{and} \quad Y < Y(K_1) < \operatorname{Im}\varphi_3,$$

the definitions and properties of $Y(K_0)$ and $Y(K_1)$ are described in sections 7.3.1 and 7.3.2.

Let $Y_0 = \min\{Y(K_0), Y(K_1)\}$. We study the transition matrix T_1 in the strip $-Y_0 \le \operatorname{Im}\varphi \le Y_0$. When computing the asymptotics of T_1, we divide this strip into three different smaller substrips called (I), (II) and (III) (see Fig. 14). Each of these strips requires a different type of computation.

8.3.3. *Properties of the analytic objects of the complex WKB method in the substrips.* In the section, we compare κ_j, $\psi_\pm^j$, $\omega_\pm^j$ and q_j for $j = 0$ and $j = 1$ in each of the different strips (I), (II) and (III).

In the strip (II), we have a non-empty intersection for K_0 and K_1. In the intersection, just by the definitions, we have

$$\kappa_1 = \kappa_0, \quad \psi_\pm^1 = \psi_\pm^0, \quad \omega_\pm^1 = \omega_\pm^0, \quad \text{and} \quad q_1 = q_0. \tag{8.19}$$

In the strip (I), consider the common boundary of K_0 and K_1 (see Fig. 14). It is the Stokes line beginning at $\varphi = \varphi_1$ and going downwards. Along this line, we get

$$\kappa_1(\varphi + 0) = -\kappa_0(\varphi - 0), \tag{8.20}$$
$$\psi_\pm^1(u, \varphi + 0) = \psi_\mp^0(u, \varphi - 0), \tag{8.21}$$
$$\omega_\pm^1(\varphi + 0) = \omega_\mp^0(\varphi - 0), \tag{8.22}$$
$$q_1(\varphi + 0) = i q_0(\varphi - 0), \tag{8.23}$$

where $\varphi + 0$ (resp. $\varphi - 0$) denotes the limit taken from the right (resp. left).

These formulae hold as φ_1 is a pre-image of E_1 (the infimum of the first band of the periodic Schrödinger operator) by the mapping $\mathcal{E} : \varphi \mapsto E - \alpha \cos\varphi$. Indeed, formula (8.20) holds as φ_1 is a branch point of the complex momentum, as φ_1 is of square root type and as $\kappa_0(\varphi_1) = 0$. Formula (8.21) holds as $\psi^j_\pm$ are just branches of the Bloch solution $\psi(x, \mathcal{E})$, $\mathcal{E} = E - \alpha\cos\varphi$, which has only two different branches, and as E_1 is a branch point of this function. The relation for $\omega^j_\pm$ follow from (8.21) and the definition of $\omega_\pm$. The relation for q follows from the definition of q as E_1 is a square root branch point of the Bloch quasi-momentum.
We also notice that, as the imaginary part of the main branch κ_0 is positive in K_0 along the common boundary of K_0 and K_1, the relation (8.20) implies that, along this boundary, one has

$$\operatorname{Im}\kappa_1(\varphi + 0) = -\operatorname{Im}\kappa_0(\varphi - 0) < 0. \tag{8.24}$$

In the strip (III), the common boundary of K_0 and K_1 consists in the interval $[\varphi_2, \varphi_3]$. This interval is the

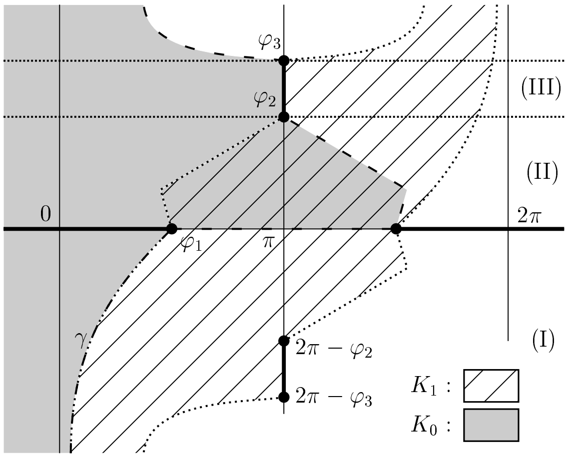

FIGURE 14. Going from K_0 to K_1

Stokes line joining φ_2 to φ_3. Along $[\varphi_2, \varphi_3]$, we have

$$\kappa_1(\varphi + 0) = 2\pi - \kappa_0(\varphi - 0), \tag{8.25}$$

$$\psi^1_\pm(u, \varphi + 0) = \psi^0_\mp(u, \varphi - 0), \tag{8.26}$$

$$\omega^1_\pm(\varphi + 0) = \omega^0_\mp(\varphi - 0), \tag{8.27}$$

$$q_1(\varphi + 0) = -iq_0(\varphi - 0), \tag{8.28}$$

and

$$\operatorname{Im}\kappa_1(\varphi + 0) = -\operatorname{Im}\kappa_0(\varphi - 0) < 0. \tag{8.29}$$

8.3.4. *A continuation lemma.* In the strips (I) and (III), the δ-admissible sub-domains of canonical domains K_0 and K_1 are separated by a distance 2δ. To get uniform asymptotics for $f^0_\pm$ in K_1, we use

Lemma 8.1. *Let $\varphi_-, \varphi_+, \varphi_0$ be fixed points such that (see Fig. 16)*

- $\operatorname{Im}\varphi_- = \operatorname{Im}\varphi_+$;
- *there is no branch point of $\varphi \mapsto \kappa(\varphi)$ on the interval $[\varphi_-, \varphi_+]$;*
- $\varphi_0 \in (\varphi_-, \varphi_+)$, $q(\varphi_0) \neq 0$.

Fix a continuous branch of κ on $[\varphi_-, \varphi_+]$. Let $f(u, \varphi)$, $f_\pm(u, \varphi)$ be solutions of (7.1) *for $\varphi \in [\varphi_-, \varphi_+]$ and $u \in [-U, U]$ satisfying condition (6.3), and such that:*

1. $f(u, \varphi) = e^{\frac{i}{\varepsilon}\int_{\varphi_0}^{\varphi} \kappa d\varphi}(\Psi_+(u, \varphi) + o(1))$ *for $\varphi \in [\varphi_-, \varphi_0]$ when $\varepsilon \to 0$, and the asymptotic is differentiable in u;*
2. $f_\pm(u, \varphi) = e^{\pm\frac{i}{\varepsilon}\int_{\varphi_0}^{\varphi} \kappa d\varphi}(\Psi_\pm(u, \varphi) + o(1))$ *for $\varphi \in [\varphi_-, \varphi_+]$ when $\varepsilon \to 0$, and the asymptotic is differentiable in u.*

Here, $(\Psi_\pm)$ are canonical Bloch solutions with the Bloch quasi-momenta $\pm\kappa$.
Then,

- *if $Im(\kappa(\varphi)) > 0$ for all $\varphi \in [\varphi_-, \varphi_+]$, there exists $C > 0$ such that, for $\varepsilon > 0$ small enough,*

$$\left|\frac{df}{du}(u,\varphi)\right| + |f(u,\varphi)| \leq Ce^{\frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi}|Im\kappa|d\varphi}, \quad \varphi \in [\varphi_0, \varphi_+]; \tag{8.30}$$

- *if $Im(\kappa(\varphi)) < 0$ for all $\varphi \in [\varphi_-, \varphi_+]$, then*

$$f(u,\varphi) = e^{\frac{i}{\varepsilon}\int_{\varphi_0}^{\varphi}\kappa d\varphi}(\Psi_+(u,\varphi) + o(1)), \quad \varphi \in [\varphi_0, \varphi_+], \tag{8.31}$$

and the asymptotic is differentiable in u.

Remark 8.1. This lemma reflects a heuristic that says that the WKB asymptotics of a solution stays valid as long as its leading term is exponentially increasing. A similar statement for a class of difference of equations can be found in[6].

Proof. Note that

$$w(f_+, f_-) = w(\Psi_+, \Psi_-)|_{\varphi=\varphi_0} + o(1).$$

As $q(\varphi_0) \neq 0$, by Lemma 6.2, the leading term in this formula is a non-zero constant, and $f_\pm$ are linearly independent. So, we can write

$$f = a(\varphi)f_+ + b(\varphi)f_-, \tag{8.32}$$

where

$$a(\varphi) = \frac{w(f, f_-)}{w(f_+, f_-)} \quad \text{and} \quad b(\varphi) = \frac{w(f_+, f)}{w(f_+, f_-)}.$$

As, on $[\varphi_-, \varphi_0]$, the solutions f and f_+ have the same asymptotics, one computes

$$a(\varphi) = 1 + o(1). \tag{8.33}$$

The coefficient a is ε-periodic in φ so that (8.33) holds on $[\varphi_-, \varphi_+]$.
Let us now estimate the coefficient b. We start with the case when $\mathrm{Im}(\kappa(\varphi)) > 0$ for all $\varphi \in [\varphi_-, \varphi_+]$. Then, on $[\varphi_-, \varphi_0]$, we have

$$|w(f_+, f)| \leq C\left|e^{\frac{2i}{\varepsilon}\int_{\varphi_0}^{\varphi}\kappa d\varphi}\right| \leq Ce^{\frac{2}{\varepsilon}\int_{\varphi}^{\varphi_0}\mathrm{Im}(\kappa)d\varphi}.$$

So, for $\varphi \in [\varphi_0 - \varepsilon, \varphi_0]$, we get that $|b(\varphi)| \leq C$; hence, by ε-periodicity, we have $|b(\varphi)| \leq C$ for $\varphi \in [\varphi_-, \varphi_+]$. On the other hand, on $[\varphi_0, \varphi_+]$, we have

$$|f_+| \leq Ce^{-\frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi}\mathrm{Im}(\kappa)d\varphi}, |f_-| \leq Ce^{\frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi}\mathrm{Im}(\kappa)d\varphi}.$$

So that, by (8.32), we get (8.30) on $[\varphi_0, \varphi_+]$.

If $\mathrm{Im}(\kappa(\varphi)) < 0$ for all $\varphi \in [\varphi_-, \varphi_+]$ then, for $\varphi \in [\varphi_-, \varphi_0]$, we get $|b(\varphi)| \leq C\left|e^{\frac{2i}{\varepsilon}\int_{\varphi_0}^{\varphi}\kappa d\varphi}\right|$. Using this estimate for $\varphi \in [\varphi_-, \varphi_- + \varepsilon]$ and the ε-periodicity, for $\varphi \in [\varphi_-, \varphi_+]$, we get

$$|b(\varphi)| \leq Ce^{-\frac{2}{\varepsilon}\int_{\varphi_-}^{\varphi_0}|\mathrm{Im}(\kappa)|d\varphi}.$$

Hence, by (8.32), on $[\varphi_0, \varphi_+]$, we have

$$\begin{aligned}
f &= a(\varphi)f_+ + b(\varphi)f_- \\
&= e^{\frac{i}{\varepsilon}\int_{\varphi_0}^{\varphi}\kappa d\varphi}\left(\Psi_+(u,\varphi) + O\left(e^{-\frac{2i}{\varepsilon}\int_{\varphi_0}^{\varphi}\kappa d\varphi}e^{-\frac{2}{\varepsilon}\int_{\varphi_-}^{\varphi_0}|\mathrm{Im}(\kappa)|d\varphi}\right) + o(1)\right) \\
&= e^{\frac{i}{\varepsilon}\int_{\varphi_0}^{\varphi}\kappa d\varphi}\left(\Psi_+(u,\varphi) + o(1)\right).
\end{aligned}$$

This proves (8.31) and completes the proof of Lemma 8.1. □

Remark 8.2. Assume that, for $E = E_0$, **(1)** the functions $f_\pm$ are two solutions constructed by Theorem 6.1 on some canonical domain, **(2)** the interval $[\varphi_-, \varphi_+]$ is in the δ-admissible subdomain, **(3)** the function f is a solution constructed by Theorem 6.1 on a canonical domain, **(4)** the interval $[\varphi_-, \varphi_0]$ is located in its δ-admissible subdomain. Then, as seen from the proof of Lemma 8.1, the asymptotics and estimates of f obtained by Lemma 8.1 are uniform in a constant neighborhood of E_0.

8.3.5. *Complex values of E and admissible subdomains.* Let $E_0 \in J$ be the point fixed in Theorem 2.2. And let K_0 and K_1 be the canonical domains corresponding to this value of E. Fix $\delta > 0$. We describe the precise choice of δ later.

As already underlined, we do not work with K_0 and K_1, but with their δ-admissible subdomains, see section 6.5. More precisely, we use the asymptotics of $f_\pm^j$, $j = 1, 2$, given by Theorem 6.1 only for φ in these admissible subdomains. Then, in view of Proposition 6.1, these asymptotics are valid and uniform in E in a constant complex neighborhood of the point E_0 fixed in Theorem 2.2. This guarantees that the asymptotics of the coefficients of T_1 are valid and uniform in this complex neighborhood.

Let Y_0 be chosen as in section 8.3.2. In the sequel, we fix δ sufficiently small so that $\delta < Y_0 - Y$.

In the sequel, if no other choice is indicated explicitly, K_0 and K_1 denote the δ-admissible subdomains of K_0 and K_1.

8.3.6. *Auxiliary canonical domains.* To use the Continuation Lemma, Lemma 8.1, "to bridge" the gap between the admissible subdomains of K_1 and K_0 in the strip (III), we need two linearly independent solutions of equation (1.1) with the standard asymptotic behavior in the strip (III) between these admissible subdomains. So, we introduce an auxiliary canonical domain $\tilde K_0$ corresponding to Fig. 15, part A. Let $\tilde\kappa_0$ be the analytic continuation of κ_0 through the interval (φ_2, φ_3). The domain $\tilde K_0$ is canonical with respect to $\tilde\kappa_0$. We construct $\tilde K_0$ by means of Proposition 6.3. This construction being similar to the one of K_0 done

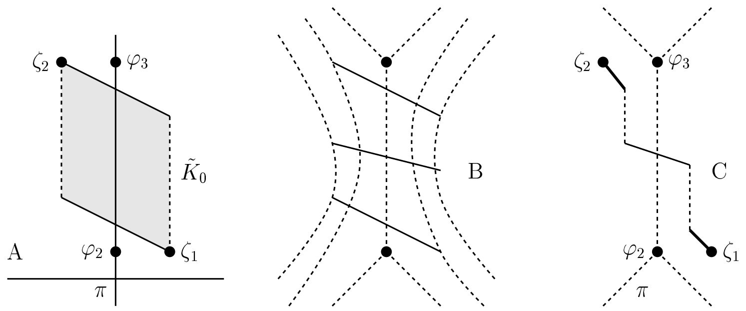

FIGURE 15. The local canonical domain

in section 7.3.1, we only illustrate it in parts B and C of Fig. 15. Here, the dotted lines are lines of Stokes type $\mathrm{Im}\int(\tilde\kappa_0 - \pi)d\varphi = \mathrm{Const}$, the thin continuous lines of Stokes type $\mathrm{Im}\int\tilde\kappa_0 d\varphi = \mathrm{Const}$, and the bold lines are canonical. Along the continuous lines $\mathrm{Im}\,\varphi$ decays as $\mathrm{Re}\,\varphi$ increases. Note that the domain $\tilde K_0$ can be constructed so that it would comprise any given subinterval of (φ_2, φ_3).

Also, we need to apply Lemma 8.1 to bridge the gap between the admissible subdomains of K_0 and K_1 in the strip (I). In this case, the auxiliary canonical domain is $\overline{K_0}$, the symmetric of K_0 with respect to the real line.

We are now ready to start with the computations of the coefficients of T_1.

8.3.7. *The asymptotics of $d_1(\varphi)$.* By (8.15), we need to compute the asymptotics of $w(f_-^0, f_+^1)$ when $\varepsilon \to 0$.

Strip (I). In view of (8.24), Lemma 8.1 allows us to "continue" the asymptotics of f_-^0 from K_0 to K_1 across the common boundary of the canonical domains in the strip (I). The consistent basis needed to apply Lemma 8.1 is the basis corresponding to the auxiliary canonical domain $\overline{K_0}$.
Let the interval $[\varphi_-, \varphi_+]$ be as in Fig. 16 i.e. the points φ_- and φ_0 are in the admissible subdomain of K_0, the point φ_+ is situated in the admissible subdomain of K_1, and the point φ_b is on the common boundary of the canonical domains. Lemma 8.1 implies that the asymptotics (8.16) of f_-^0 stays valid on $[\varphi_b, \varphi_+]$. We rewrite it in the form

$$f_-^0 = e^{-\frac{i}{\varepsilon}\int_0^{\varphi_1}\kappa_0 d\varphi - \frac{i}{\varepsilon}\int_{\varphi_1}^{\varphi_b}\kappa_0 d\varphi} e^{-\frac{i}{\varepsilon}\int_{\varphi_b}^{\varphi}\kappa_0 d\varphi}(\Psi_-^0 + o(1)),$$

where the last integral is taken along $[\varphi_-, \varphi_+]$. By (8.16) and as $\kappa_1 = \kappa_0$ in the strip (II), and $\kappa_1 = -\kappa_0$ in the strip (I), for $\varphi \in [\varphi_-, \varphi_+] \cap K_1$, we can write

$$f_+^1 = e^{\frac{i}{\varepsilon}\int_\pi^{\varphi_1} \kappa_0 d\varphi - \frac{i}{\varepsilon}\int_{\varphi_1}^{\varphi_b} \kappa_0 d\varphi} e^{-\frac{i}{\varepsilon}\int_{\varphi_b}^{\varphi} \kappa_0 d\varphi} (\Psi_+^1 + o(1)),$$

where the last integral is taken along $[\varphi_-, \varphi_+]$. So,

$$|w\,(f_-^0, f_+^1)| \leq C |e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}| \cdot |e^{-\frac{2i}{\varepsilon}\int_{\varphi_1}^{\varphi_b} \kappa_0 d\varphi}| \cdot |e^{-\frac{2i}{\varepsilon}\int_{\varphi_b}^{\varphi} \kappa_0 d\varphi}|.$$

As the points φ_1 and φ_b are on the Stokes line $\mathrm{Im}\int_{\varphi_1}^{\varphi} \kappa_0 d\varphi = 0$, one has $|e^{-\frac{i}{\varepsilon}\int_{\varphi_1}^{\varphi_b} \kappa_0 d\varphi}| = 1$.

Now, as φ has to be in an admissible sub-domain of K_1, we know that $\mathrm{Re}\,(\varphi - \varphi_b) \geq \delta$. Assuming that $\delta < \mathrm{Re}\,(\varphi - \varphi_b) \leq \delta + \varepsilon$, we get

$$|w(f_-^0, f_+^1)| \leq C |e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}| e^{C\delta/\varepsilon}. \tag{8.34}$$

By the ε-periodicity of the Wronskian, it holds everywhere on the line $\{\mathrm{Im}\,(\varphi) = \mathrm{Im}\,(\varphi_b)\}$. The estimate (8.34) is valid as long as the interval $[\varphi_-, \varphi_+]$ stays off the δ-neighborhood of the branch points. Hence, for $-Y_0 + \delta \leq \mathrm{Im}\,\varphi \leq -\delta$, we have

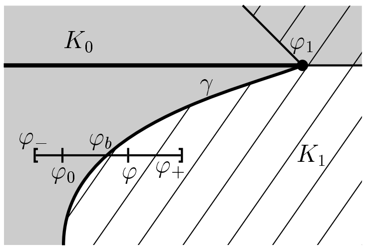

FIGURE 16. Continuation below φ_1

$$|d_1(\varphi)| \leq C |e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}| e^{C\delta/\varepsilon}. \tag{8.35}$$

This estimates is uniform along horizontal lines $\mathrm{Im}\,\varphi = C$.

Strip (II). In this case, by (8.19), for $\varphi \in K_0 \cap K_1$, we have

$$f_-^0 = e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi + \int_0^\pi \omega_-^0 d\varphi} e^{-\frac{i}{\varepsilon}\int_\pi^\varphi \kappa_0 d\varphi} (\Psi_-^1(u, \varphi) + o(1)).$$

Comparing this with the formula

$$f_+^1 = e^{\frac{i}{\varepsilon}\int_\pi^\varphi \kappa_0 d\varphi} (\Psi_+^1(u, \varphi) + o(1)),$$

one easily obtains

$$w(f_-^0, f_+^1) = -e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi} e^{\int_0^\pi \omega_-^0 d\varphi} (w(f_+^1, f_-^1) + o(1)).$$

Hence,

$$d_1(\varphi) = e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi} e^{\int_0^\pi \omega_-^0 d\varphi} (1 + o(1)). \tag{8.36}$$

By the ε-periodicity of the Wronskian, (8.36) is uniform along horizontal in the strip $\delta \leq \mathrm{Im}\,\varphi \leq \mathrm{Im}\,\varphi_2 - \delta$.

Strip (III). In this region, we satisfy ourselves with an estimate on $w(f_-^0, f_+^1)$. In view of (8.29), by Lemma 8.1, we can continue the asymptotics of f_-^0 from $[\varphi_-, \varphi_0]$ to the whole interval $[\varphi_-, \varphi_+]$, see Fig. 17. The consistent basis needed to apply Lemma 8.1 is obtained by applying Theorem 6.1 for the auxiliary canonical domain $\tilde{K}_0$. This domain comprises a part of the common boundary of the canonical domains K_0 and K_1, i.e. of the segment (φ_2, φ_3) (see Fig. 14, 15 and 17). On the segment $[\varphi_b, \varphi_+]$, we have

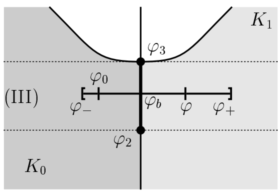

FIGURE 17. Continuation through $[\varphi_2, \varphi_3]$

$$f_-^0 = e^{-\frac{i}{\varepsilon}\int_0^{\varphi_2} \kappa_0 d\varphi - \frac{i}{\varepsilon}\int_{\varphi_2}^{\varphi_b} \kappa_0 d\varphi} e^{-\frac{i}{\varepsilon}\int_{\varphi_b}^{\varphi} \kappa_0 d\varphi} (\Psi_-^0 + o(1)), \tag{8.37}$$

where the last integral is taken along $[\varphi_-, \varphi_+]$. In view of (8.19) and (8.25), we also have

$$f_+^1 = e^{\frac{i}{\varepsilon}\int_\pi^{\varphi_2} \kappa_0 d\varphi - \frac{i}{\varepsilon}\int_{\varphi_2}^{\varphi} \kappa_0 d\varphi} e^{\frac{i}{\varepsilon}(\varphi - \varphi_2)2\pi} (\Psi_+^1 + o(1)), \quad \varphi \in [\varphi_-, \varphi_+] \cap K_1. \tag{8.38}$$

where, in the last integral, we integrate the analytic continuation of κ_0 from K_0 to K_1 through (φ_2, φ_3), and this integral is taken along a curve in K_1. Representations (8.37) and (8.38) imply

$$|d_1(\varphi)| \leq C \left| e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi} \right| \cdot \left| e^{-2\frac{i}{\varepsilon}\int_{\varphi_2}^{\varphi} (\kappa_0 - \pi) d\varphi} \right|.$$

Recall that for $\varphi \in [\varphi_2, \varphi_3]$, one has $\kappa_0 = \pi + it$ where $t \geq 0$. Therefore, if $|\varphi - \varphi_b|$ is of order δ, we obtain

$$|d_1(\varphi)| \leq C|e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}|e^{C\delta/\varepsilon}. \tag{8.39}$$

Since d_1 is ε-periodic, this estimate is valid uniformly along horizontal lines in the strip $\{\mathrm{Im}\,\varphi_2 + \delta \leq \mathrm{Im}\,\varphi \leq Y_0 - \delta\}$.

Uniform asymptotics. As $d_1(\varphi)$ is ε-periodic in φ and analytic in a strip $\{-Y_0 \leq \mathrm{Im}\varphi \leq Y_0\}$, we can expand d_1 in a Fourier series with exponentially decreasing coefficients

$$d_1(\varphi) = \sum_{n\in\mathbb{Z}} \delta_n e^{2i\pi n\frac{\varphi}{\varepsilon}} \text{ where } \delta_n = \frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi_0+\varepsilon} d_1(\varphi)e^{-2i\pi n\frac{\varphi}{\varepsilon}}d\varphi \tag{8.40}$$

for any $\varphi_0 \in \{-Y_0 \leq \mathrm{Im}\varphi \leq Y_0\}$. For $n \neq 0$, we get

$$|\delta_n| \leq C|e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}|e^{-2\pi|n|(Y-\delta)/\varepsilon}e^{C\delta/\varepsilon}. \tag{8.41}$$

To prove this estimate for $n > 0$, one uses estimate (8.35) and (8.40) with $\mathrm{Im}\,\varphi_0 = -Y_0 + \delta$. In the case of $n < 0$, one uses (8.39) and (8.40) with $\mathrm{Im}\,\varphi_0 = Y_0 - \delta$. Furthermore, by means of the estimate (8.36) and (8.40) with $\mathrm{Im}\,\varphi_0 = 0$, we get

$$\delta_0 = e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}e^{\int_0^\pi \omega_-^0 d\varphi}(1 + o(1)). \tag{8.42}$$

Choose $\delta < \min\{2\pi/C, Y_0 - Y\}$. Then, (8.41) and (8.42) imply

$$d_1(\varphi) = e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi + \int_0^\pi \omega_-^0 d\varphi}(1 + o(1)),$$

uniformly along horizontal lines in the strip $\{|\mathrm{Im}\,\varphi| \leq Y\}$.

8.3.8. *An estimate for $b_1(\varphi)$.* We only estimate b_1 when $\varepsilon \to 0$.

Strip (I). Let γ be the Stokes line beginning at $\varphi = \varphi_1$ and bordering K_0 and K_1. In view of (8.24), Lemma 8.1 only gives us an estimation on f_+^0 when we cross γ along $[\varphi_-, \varphi_+]$ (see Fig. 16). This estimation is

$$|f_+^0| \leq C|e^{\frac{i}{\varepsilon}\int_0^{\varphi_1}\kappa_0 d\varphi + \frac{i}{\varepsilon}\int_{\varphi_1}^{\varphi_b}\kappa_0 d\varphi + \frac{i}{\varepsilon}\int_{\varphi_b}^{\varphi_0}\kappa_0 d\varphi}|e^{\frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi}|\mathrm{Im}\kappa_0|d\varphi}, \ \varphi \in [\varphi_-, \varphi_+] \cap K_1.$$

Recall that $\kappa_0 = \kappa_1$ in the strip (II) and that, in the strip (I), one has $\kappa_0 = -\kappa_1$. Therefore

$$|f_+^1| \leq C|e^{\frac{i}{\varepsilon}\int_\pi^{\varphi_1}\kappa_0 d\varphi - \frac{i}{\varepsilon}\int_{\varphi_1}^{\varphi_b}\kappa_0 d\varphi - \frac{i}{\varepsilon}\int_{\varphi_b}^{\varphi_0}\kappa_0 d\varphi}|e^{\frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi}|\mathrm{Im}\,\kappa_0|d\varphi}, \ \varphi \in [\varphi_-, \varphi_+] \cap K_1.$$

Note that the first derivatives with respect to x of the solutions satisfy the same estimates. Therefore, for φ on $[\varphi_b, \varphi_+]$ and inside the admissible subdomain of K_1, we obtain

$$|w(f_+^0, f_+^1)| \leq C|e^{\frac{i}{\varepsilon}\int_0^{\varphi_1}\kappa_0 d\varphi}|e^{\frac{2}{\varepsilon}\int_{\varphi_0}^{\varphi}|\mathrm{Im}\kappa_0|d\varphi}.$$

We have used the fact that κ is real on $[\varphi_1, \pi]$. Now, the only restriction we have on φ_0 is that it has to be in the admissible subdomain of K_0; hence, we choose $|\varphi_0 - \varphi| \sim 2\delta$ where 2δ is the distance between the admissible subdomains of K_0 and K_1. So, for $-Y_0 + \delta \leq \mathrm{Im}\,\varphi \leq -\delta$, uniformly along horizontal lines, we have

$$|b_1(\varphi)| \leq Ce^{C\frac{\delta}{\varepsilon}}|e^{\frac{i}{\varepsilon}\int_0^{\varphi_1}\kappa_0 d\varphi}| = e^{C\frac{\delta}{\varepsilon}}|e^{\frac{i}{\varepsilon}\int_0^{\pi}\kappa_0 d\varphi}|. \tag{8.43}$$

Strip (II). We don't need to to study b_1 here.

Strip (III). In view of (8.29), Lemma 8.1 gives us only an estimate on f_+^0 in K_1. Using the notations of Fig. 17), we get

$$|f_+^0| \leq C|e^{\frac{i}{\varepsilon}\int_0^{\pi}\kappa_0 d\varphi + \frac{i}{\varepsilon}\int_\pi^{\varphi_2}\kappa_0 d\varphi + \frac{i}{\varepsilon}\int_{\varphi_2}^{\varphi_b}\kappa_0 d\varphi}e^{\frac{i}{\varepsilon}\int_{\varphi_b}^{\varphi_0}\kappa_0 d\varphi}|e^{\frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi}|\mathrm{Im}\,\kappa_0|d\varphi}, \ \varphi \in [\varphi_-, \varphi_+] \cap K_1. \tag{8.44}$$

where the last integral is taken along $[\varphi_-, \varphi_+]$. We also have

$$|f_+^1| \leq C|e^{\frac{i}{\varepsilon}\int_\pi^{\varphi_2}\kappa_1 d\varphi + \frac{i}{\varepsilon}\int_{\varphi_2}^{\varphi_b}\kappa_1 d\varphi + \frac{i}{\varepsilon}\int_{\varphi_b}^{\varphi_0}\kappa_1 d\varphi}|e^{\frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi}|\mathrm{Im}\,\kappa_1|d\varphi}, \ \varphi \in [\varphi_-, \varphi_+] \cap K_1.. \tag{8.45}$$

In the last two integrals, κ_1 is the analytic continuation of the branch κ_1 through $[\varphi_2,\varphi_3]$. The derivatives of f_+^0 and f_+^1 also satisfy (8.44) and (8.45). This and the relations (8.19) and (8.25) imply that

$$|w(f_+^0,f_+^1)| \le C|e^{\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}||e^{2\frac{i}{\varepsilon}\int_\pi^{\varphi_2}\kappa_0 d\varphi + \frac{2\pi i}{\varepsilon}(\varphi_0-\varphi_2)}|e^{\frac{2}{\varepsilon}\int_{\varphi_0}^{\varphi}|\mathrm{Im}\kappa_0|d\varphi}.$$

The main branch κ_0 is positive along $[\pi,\varphi_2]$ and $\mathrm{Im}(\varphi_b-\varphi_2)>0$. Hence, $\mathrm{Im}\,(\int_\pi^{\varphi_2}\kappa_0 d\varphi + \pi\int_{\varphi_2}^{\varphi_b}d\varphi)>0$. So, there exists $\eta>0$ (independent of ε and of δ) such that

$$|b_1(\varphi)| \le Ce^{C\frac{\delta}{\varepsilon}}|e^{\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}|e^{-\eta/\varepsilon},$$

uniformly along horizontal lines in (III).

Uniform asymptotics. Using the same method as for d_1, in the strip $\{|\mathrm{Im}\varphi|\le Y\}$, we get

$$b_1(\varphi) = e^{\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}\cdot o(1).$$

8.3.9. *The asymptotics of* $a_1(\varphi)$. By (8.15), we need to compute the asymptotics of $w(f_+^0,f_-^1)$ when $\varepsilon\to 0$. The computations being similar to those made for b_1 and d_1, we give only the results.

Strip (I). For $-Y_0+\delta\le \mathrm{Im}\,\varphi\le -\delta$,

$$|a_1(\varphi)| \le Ce^{C\frac{\delta}{\varepsilon}}|e^{\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}|.$$

Strip (II). For $0<\delta\le \mathrm{Im}\,\varphi\le \mathrm{Im}\,\varphi_2-\delta$, we get

$$a_1(\varphi) = e^{\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}e^{\int_0^\pi \omega_+^0 d\varphi}(1+o(1)).$$

Strip (III). For $\mathrm{Im}\,\varphi_2+\delta\le \mathrm{Im}\,\varphi\le Y_0-\delta$, we get

$$|a_1(\varphi)| \le Ce^{C\frac{\delta}{\varepsilon}}|e^{\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}|.$$

Uniform asymptotics. Estimating the Fourier coefficients of a_1, we get that, uniformly in the strip $\{|\mathrm{Im}\varphi|\le Y\}$,

$$a_1(\varphi) = e^{\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi + \int_0^\pi \omega_+^0 d\varphi}(1+o(1)).$$

8.3.10. *The asymptotics of* $c_1(\varphi)$. Now, we need to compute the asymptotics of $w(f_-^0,f_-^1)$ when $\varepsilon\to 0$.

Strip (I). In view of (8.20), we can use Lemma 8.1 to "continue" the asymptotic expansion for f_-^0 from K_0 to K_1 along a horizontal line (see Fig. 16). This allows us to get an asymptotic of c_1. We use formulae (8.20) – (8.23). The asymptotic expansion for f_-^0 on $[\varphi_b,\varphi_+]$ is

$$f_-^0(u,\varphi) = e^{-\frac{i}{\varepsilon}\int_0^\varphi \kappa_0 d\varphi}\cdot\left(q_0 e^{\int_0^\varphi \omega_-^0 d\varphi}(\psi_-^0+o(1))\right).$$

Here, we integrate along a curve going from K_0 to K_1 below the point φ_1; κ_0, ω_-^0 and q_0 are the analytic continuations of these functions along this curve. We use (8.20) and compute

$$\begin{aligned}\int_0^\varphi \kappa_0 d\varphi &= \int_0^{\varphi_1}\kappa_0 d\varphi - \int_{\varphi_1}^{\varphi}\kappa_1 d\varphi \\ &= \int_0^{\varphi_1}\kappa_0 d\varphi - \int_{\varphi_1}^{\pi}\kappa_1 d\varphi - \int_\pi^\varphi \kappa_1 d\varphi.\end{aligned} \tag{8.46}$$

Similarly, by (8.22), one obtains

$$\int_0^\varphi \omega_-^0 d\varphi = \int_0^{\varphi_1}\omega_-^0 d\varphi - \int_\pi^{\varphi_1}\omega_+^1 d\varphi + \int_\pi^\varphi \omega_+^1 d\varphi, \tag{8.47}$$

Using (8.46), (8.47), (8.21) and (8.23), we get

$$f_-^0(u,\varphi) = -ie^{-\frac{i}{\varepsilon}\int_0^{\varphi_1}\kappa_0 d\varphi + \frac{i}{\varepsilon}\int_{\varphi_1}^{\pi}\kappa_0 d\varphi + \int_0^{\varphi_1}\omega_-^0 - \int_\pi^{\varphi_1}\omega_+^0}\left[e^{\frac{i}{\varepsilon}\int_\pi^\varphi \kappa_1 d\varphi}(\Psi_+^1(u,\varphi)+o(1))\right].$$

On the other hand, we know that

$$f_-^1(u,\varphi) = e^{-\frac{i}{\varepsilon}\int_\pi^\varphi \kappa_1 d\varphi}(\Psi_-^1(u,\varphi)+o(1)).$$

Therefore, one has

$$w(f_-^0, f_-^1) = -i \cdot e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi} e^{2\frac{i}{\varepsilon}\int_{\varphi_1}^\pi \kappa_0 d\varphi} e^{\int_0^{\varphi_1} \omega_-^0 d\varphi - \int_\pi^{\varphi_1} \omega_+^0 d\varphi}(w_1 + o(1)),$$

that is

$$c_1(\varphi) = -i \cdot e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi} e^{2\frac{i}{\varepsilon}\int_{\varphi_1}^\pi \kappa_0 d\varphi} e^{\int_0^{\varphi_1} \omega_-^0 d\varphi - \int_\pi^{\varphi_1} \omega_+^0 d\varphi}(1 + o(1)). \tag{8.48}$$

This holds in the strip $\{-Y_0 + \delta \le \operatorname{Im}\varphi \le -\delta\}$, uniformly along horizontal lines.

Strip (II). As for b_1, we do not need any information on c_1 in this region.

Strip (III). We use Lemma 8.1 to compute the asymptotics of c_1. Let the segment $[\varphi_-, \varphi_+]$ be as in Fig. 17. In view of (8.29), for $\varphi \in [\varphi_-, \varphi_+]$, Lemma 8.1 implies that

$$f_-^0 = e^{-\frac{i}{\varepsilon}\int_0^\varphi \kappa_0 d\varphi}\left(q_0 e^{\int_0^\varphi \omega_-^0 d\varphi}\psi_-^0 + o(1)\right) \tag{8.49}$$

In (8.49), the integration contour first goes from 0 to φ_- in K_0, then to φ along $[\varphi_-, \varphi_+]$. Using (8.25), we rewrite the first integral in (8.49) in the form

$$\int_0^\varphi \kappa_0 d\varphi = \int_0^\pi \kappa_0 d\varphi + 2\int_\pi^{\varphi_2} \kappa_0 d\varphi + 2\pi(\varphi - \varphi_2) - \int_\pi^\varphi \kappa_1 d\varphi.$$

Also, by (8.27), we get

$$\int_0^\varphi \omega_-^0 d\varphi = \int_0^\pi \omega_-^0 d\varphi + \int_\pi^{\varphi_2} (\omega_-^0 - \omega_+^0) d\varphi + \int_\pi^\varphi \omega_+^0 \varphi.$$

Combining these formulae with (8.26) and (8.28), we obtain

$$f_-^0 = ie^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi - 2\frac{i}{\varepsilon}\int_\pi^{\varphi_2}(\kappa_0 - \pi)d\varphi} e^{-2\pi\frac{i}{\varepsilon}(\varphi - \pi)} e^{\int_0^{\varphi_2} \omega_-^0 d\varphi - \int_\pi^{\varphi_2} \omega_+^0 d\varphi}\left[e^{\frac{i}{\varepsilon}\int_\pi^\varphi \kappa_1 d\varphi}(\Psi_+^1 + o(1))\right].$$

Using this in conjunction with

$$f_-^1 = e^{-\frac{i}{\varepsilon}\int_\pi^\varphi \kappa_1 d\varphi}(\Psi_-^1 + o(1)),$$

we get

$$c_1(\varphi) = ie^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi - 2\frac{i}{\varepsilon}\int_\pi^{\varphi_2}(\kappa_0 - \pi)d\varphi} e^{-2\pi\frac{i}{\varepsilon}(\varphi - \pi)} e^{\int_0^{\varphi_2} \omega_-^0 d\varphi - \int_\pi^{\varphi_2} \omega_+^0 d\varphi}(1 + o(1)). \tag{8.50}$$

This asymptotic is valid for $\operatorname{Im}\varphi_2 + \delta \le \varphi \le Y_0 - \delta$ uniformly along horizontal lines.

Uniform asymptotics. To get global information on c_1, we let

$$c_1(\varphi) = \sum_{n\in\mathbb{Z}} \gamma_n e^{2i\pi n\frac{\varphi - \pi}{\varepsilon}}, \quad \gamma_n = \frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi_0 + \varepsilon} c_1(\varphi) e^{2i\pi n\frac{\pi - \varphi}{\varepsilon}} d\varphi. \tag{8.51}$$

To study γ_n with $n > 0$, we use (8.48) and (8.51) with $\operatorname{Im}\varphi_0 = -Y_0 + \delta$. This gives

$$|\gamma_n| \le C|e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}|e^{-\frac{2\pi}{\varepsilon}|n|Y_0},$$

In the case of $n = 0$, by (8.48) and (8.51) with $\operatorname{Im}\varphi_0 = 0$, we get

$$\gamma_0 = -ie^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi + 2\frac{i}{\varepsilon}\int_{\varphi_1}^\pi \kappa_0 d\varphi} e^{\int_0^{\varphi_1} \omega_-^0 d\varphi - \int_\pi^{\varphi_1} \omega_+^0 d\varphi}(1 + o(1)).$$

For $n < 0$, we use the asymptotic obtained in the strip (III) and (8.51) with $\operatorname{Im}\varphi_0 = Y_0 - \delta$. We obtain

$$\gamma_{-1} = \frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi_0 + \varepsilon} c_1(\varphi) e^{2i\pi\frac{\pi - \varphi}{\varepsilon}} d\varphi = ie^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi - 2\frac{i}{\varepsilon}\int_\pi^{\varphi_2}(\kappa_0 - \pi)d\varphi} e^{\int_0^{\varphi_2} \omega_-^0 d\varphi - \int_\pi^{\varphi_2} \omega_+^0 d\varphi}(1 + o(1))$$

and, for $n < -1$, one has

$$|\gamma_n| = \left|\frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi_0 + \varepsilon} c_1(\varphi) e^{2in\pi\frac{\pi - \varphi}{\varepsilon}} d\varphi\right| \le C|e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi - \frac{2i}{\varepsilon}\int_\pi^{\varphi_2}(\kappa_0 - \pi)d\varphi}|e^{-\frac{2\pi}{\varepsilon}|n+1|(Y_0 - \delta)}.$$

Putting all this together, we get for φ in the strip $\{|\operatorname{Im} z| \le Y\}$ (recall that $Y < Y_0 - \delta$):

$$\begin{aligned} c_1(\varphi) = -ie^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}\Big(& e^{2\frac{i}{\varepsilon}\int_{\varphi_1}^\pi \kappa_0 d\varphi} e^{\int_0^{\varphi_1} \omega_-^0 d\varphi - \int_\pi^{\varphi_1} \omega_+^0 d\varphi}(1 + o(1)) \\ & - e^{-2\frac{i}{\varepsilon}\int_\pi^{\varphi_2}(\kappa_0 - \pi)d\varphi} e^{\int_0^{\varphi_2} \omega_-^0 d\varphi - \int_\pi^{\varphi_2} \omega_+^0 d\varphi}\left[e^{-2\pi\frac{i}{\varepsilon}(\varphi - \pi)}\right](1 + o(1))\Big) \end{aligned}$$

where $o(1)$ is uniform in the strip $\{|\mathrm{Im} z| \le Y\}$. This ends the proof of Proposition 8.2.

8.4. **The asymptotics of T_2.** One has

$$a_2(\varphi) = \frac{1}{w_2} w(f_+^1, f_-^2), \quad b_2(\varphi) = -\frac{1}{w_2} w(f_+^1, f_+^2), \tag{8.52}$$

$$c_2(\varphi) = \frac{1}{w_2} w(f_-^1, f_-^2), \quad d_2(\varphi) = -\frac{1}{w_2} w(f_-^1, f_+^2), \tag{8.53}$$

$$w_2 = w(f_+^2, f_-^2) = f_-^2 \frac{d}{du} f_+^2 - f_+^2 \frac{d}{du} f_-^2. \tag{8.54}$$

These Wronskians are ε-periodic and, together with the solutions, analytic in φ.

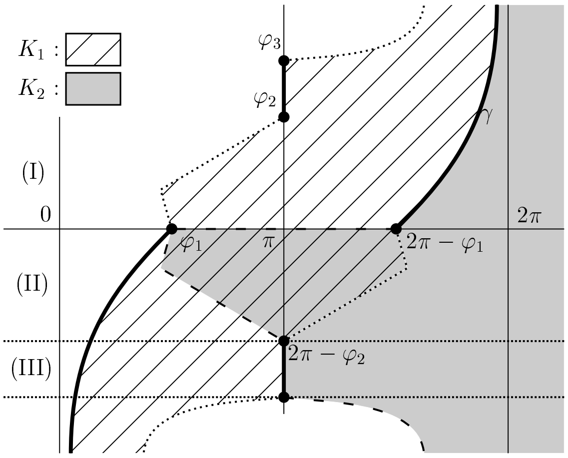

FIGURE 18. Going from K_1 to K_2

8.4.1. *Three substrips.* The coefficients of T_2 are computed in the strip

$$-Y_0 + \delta \le \mathrm{Im}\, \varphi \le Y_0 - \delta.$$

Here, Y_0 is the same as before, and δ is an arbitrarily small positive constant that is independent of ε. Again we divide this strip into three smaller strips denoted respectively by (I), (II) and (III), see Fig. 18.

8.4.2. *Properties of analytic objects.* Recall that the branches κ_2, $\psi_\pm^2$, $\omega_\pm^2$ and q_2 are defined as analytic continuations of κ_1, $\psi_\pm^1$, $\omega_\pm^1$ and q_1 from K_1 to K_2 trough thier common part (or along the periodic curve described in section 7.4.1).

In the strip (II), there is a non-empty intersection for K_1 and K_2. Here, we have

$$\kappa_1 = \kappa_2, \quad \psi_\pm^1 = \psi_\pm^2, \quad \omega_\pm^1 = \omega_\pm^2, \quad q_2 = q_1.$$

Consider the strip (I). Consider the common boundary of K_2 and K_1 in this strip (see Fig. 18). It is the Stokes lines beginning at $\varphi = 2\pi - \varphi_1$ and going upwards. Along this line, one has

$$\kappa_1(\varphi - 0) = -\kappa_2(\varphi + 0), \quad \psi_\pm^1(u, \varphi - 0) = \psi_\mp^2(u, \varphi + 0),$$
$$\omega_\pm^1(\varphi - 0) = \omega_\mp^2(\varphi + 0), \quad q_1(\varphi - 0) = i q_2(\varphi + 0),$$

and

$$\mathrm{Im}\, \kappa_2(\varphi + 0) = -\mathrm{Im}\, \kappa_1(\varphi - 0) > 0.$$

In the strip (III), the common boundary of K_2 and K_1 is a part of the interval $[\overline{\varphi_2}, \overline{\varphi_3}]$. This interval is the Stokes line joining $\overline{\varphi_2}$ to $\overline{\varphi_3}$. Along this line, one has

$$\kappa_1(\varphi - 0) = 2\pi - \kappa_2(\varphi + 0), \quad \psi_\pm^1(u, \varphi - 0) = \psi_\mp^2(u, \varphi + 0),$$
$$\omega_\pm^1(\varphi - 0) = \omega_\mp^2(\varphi + 0), \quad q_1(\varphi - 0) = -i q_2(\varphi + 0)$$

and

$$\mathrm{Im}\, \kappa_2(\varphi + 0) = -\mathrm{Im}\, \kappa_1(\varphi - 0) > 0.$$

8.4.3. *Auxiliary canonical domains.* When applying Lemma 8.1 to compute T_2, we use two auxiliary canonical domains: (1) the domain symmetric to $\tilde{K}_0$ with respect to the point π; (2) the domain $\overline{K_2}$ which is symmetric to K_2 with respect to the real line and symmetric to the auxiliary canonical domain $\overline{K_0}$ with respect to the point π.

8.4.4. *Partial results.* Now, we list the asymptotics of the coefficients of T_2 in the strips (I) – (III). As before, δ is an arbitrarily small, fixed, positive constant such that $\delta < Y_0 - Y$ and $Y > \operatorname{Im}\varphi_2$.

Coefficient a_2. As in the strip $\delta \le \operatorname{Im}\varphi \le Y_0 - \delta$ so in the strip $-Y_0 + \delta \le \operatorname{Im}\varphi \le -\operatorname{Im}\varphi_2 - \delta$, we get the estimate

$$|a_2(\varphi)| \le C|e^{\frac{i}{\varepsilon}\int_{2\pi-\varphi_1}^{2\pi}\kappa_2 d\varphi}|e^{C\delta/\varepsilon}.$$

It is uniform along horizontal lines. We get also the uniform asymptotics

$$a_2(\varphi) = e^{\frac{i}{\varepsilon}\int_\pi^{2\pi}\kappa_2 d\varphi}e^{\int_\pi^{2\pi}\omega_+^2 d\varphi}(1+o(1)), \quad -\operatorname{Im}\varphi_2 + \delta \le \operatorname{Im}\varphi \le \operatorname{Im}\delta.$$

Coefficient b_2. We get two asymptotic formulae

$$b_2(\varphi) = ie^{\frac{i}{\varepsilon}\int_\pi^{2\pi-\varphi_1}\kappa_1 d\varphi + \frac{i}{\varepsilon}\int_{2\pi}^{2\pi-\varphi_1}\kappa_2 d\varphi + \int_\pi^{2\pi-\varphi_1}\omega_+^1 d\varphi - \int_{2\pi}^{2\pi-\varphi_1}\omega_-^2 d\varphi}(1+o(1)),$$

as $\delta \le \operatorname{Im}\varphi \le Y_0 - \delta$, and

$$b_2(\varphi) = -ie^{\frac{2i}{\varepsilon}\int_\pi^{\overline{\varphi_2}}(\kappa_1-\pi)d\varphi - \frac{i}{\varepsilon}\int_\pi^{2\pi}\kappa_2 d\varphi + \int_\pi^{\overline{\varphi_2}}\omega_+^1 d\varphi + \int_{\overline{\varphi_2}}^{2\pi}\omega_-^2 d\varphi}\left[e^{\frac{2\pi i}{\varepsilon}(\varphi-\pi)}\right](1+o(1)),$$

as $-Y_0 + \delta < \operatorname{Im}\varphi < -\varphi_2 - \delta$. These asymptotics are uniform along horizontal lines.

Coefficient c_2. We get only estimates uniform along the horizontal lines. If $\delta \le \operatorname{Im}\varphi \le Y_0 - \delta$, we have

$$|c_2(\varphi)| \le Ce^{C\frac{\delta}{\varepsilon}}|e^{-\frac{i}{\varepsilon}\int_{2\pi}^{2\pi-\varphi_1}\kappa_2 d\varphi}|,$$

and, if $-Y_0 + \delta \le \operatorname{Im}\varphi \le -\operatorname{Im}\varphi_2 - \delta$, we have

$$|c_2(\varphi)| \le C|e^{-\frac{i}{\varepsilon}\int_{2\pi}^{2\pi-\varphi_1}\kappa_2 d\varphi}|e^{-\eta/\varepsilon}.$$

Here, η is a positive constant independent of ε and δ.

Coefficient d_2. We get the estimates

$$|d_2(\varphi)| \le C|e^{-\frac{i}{\varepsilon}\int_{2\pi-\varphi_1}^{2\pi}\kappa_2 d\varphi}|e^{C\delta/\varepsilon}.$$

They are uniform along horizontal lines as in the strip $\delta \le \operatorname{Im}\varphi \le Y_0 - \delta$ so in the strip $-Y_0 + \delta \le \operatorname{Im}\varphi \le -\operatorname{Im}\varphi_2 - \delta\}$. And, in the strip $-\operatorname{Im}\varphi_2 + \delta \le \operatorname{Im}\varphi \le -\delta$, we get the uniform asymptotics

$$d_2(\varphi) = e^{-\frac{i}{\varepsilon}\int_\pi^{2\pi}\kappa_2 d\varphi}e^{\int_\pi^{2\pi}\omega_-^2 d\varphi}(1+o(1)).$$

Uniform asymptotics for the coefficients of T_2 are obtained by analysing their Fourier series. This leads to the formulae announced in Proposition 8.3.

8.5. **The computation of $\tilde{M}$.** Let $K := K_0 \cup K_1 \cup K_2$. This domain is represented on Fig. 19. Note that it is cut along the Stokes lines shown by the bold lines. It is simply connected and regular. Continue analytically the main branch of the complex momentum to K. Denote the analytic continuation simply by κ. By definition, the functions κ_j, $j = 0, 1, 2$, are the restrictions of κ to K_j i.e.

$$\kappa|_{K_j} = \kappa_j.$$

Similarly, we introduce the functions $\omega_\pm$ single valued and analytic on K so that

$$\omega_\pm|_{K_j} = \omega_\pm^j.$$

These functions κ and $\omega_\pm$ have the following symmetry properties

$$\kappa(2\pi - \varphi) = \kappa(\varphi) \quad \text{and} \quad \omega_\pm(2\pi - \varphi) = -\omega_\pm(\varphi). \tag{8.55}$$

(8.55) follows from the analogous properties of κ_1 and $\omega_\pm^1$, see (7.9) and (7.12).

To get the asymptotics of the matrix $\tilde{M}$ described in Proposition 8.1, we proceed as follows

- in the asymptotic formulae for T_1 and T_2, we replace κ_j and $\omega_\pm^j$, $j = 1, 2$ by κ and ω respectively;
- we compute the product $T_1(\varphi + 2\pi)T_2(\varphi + 2\pi)$ (compare with (8.5));
- in the formulae thus obtained, we simplify the integrals by means of the relations (8.55).

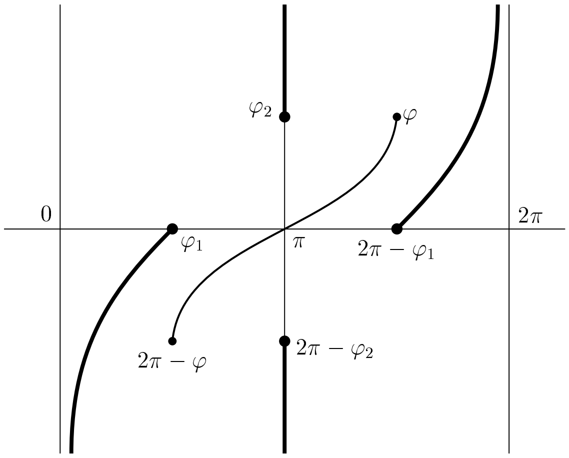

FIGURE 19. The domain K

In result, after a rather long but elementary calculation, we get

$$
\begin{aligned}
m_{11}(\varphi) &= \tilde{t}_h \cdot e^{\frac{i}{\varepsilon}\tilde{\Phi}} \left(1+o(1)+o\left(e^{-\frac{i}{\varepsilon}\int_{\varphi_1}^{2\pi-\varphi_1}\kappa d\varphi}\right)\right),\\
m_{12}(\varphi) &= iG\left[e^{\frac{i}{\varepsilon}\tilde{\Phi}}\left(1+o(1)+o\left(e^{-\frac{i}{\varepsilon}\int_{\varphi_1}^{\pi}\kappa d\varphi}\right)\right)-\tilde{t}_v e^{-i\theta+\frac{2\pi i}{\varepsilon}(\varphi+\pi)}(1+o(1))\right],\\
m_{21}(\varphi) &= -iG^{-1}\left[e^{\frac{i}{\varepsilon}\tilde{\Phi}}\left(1+o(1)+o\left(e^{-\frac{i}{\varepsilon}\int_{\varphi_1}^{\pi}\kappa d\varphi-\frac{i}{\varepsilon}\int_{\varphi_1}^{2\pi-\varphi_1}\kappa d\varphi}\right)\right)-\tilde{t}_v e^{i\theta-\frac{2\pi i}{\varepsilon}(\varphi+\pi)}(1+o(1))\right],\\
m_{22}(\varphi) &= \frac{1}{t_h}\left[e^{\frac{i}{\varepsilon}\tilde{\Phi}}(1+o(1))+e^{-\frac{i}{\varepsilon}\tilde{\Phi}}(1+o(1))+o\left(e^{-\frac{3i}{\varepsilon}\int_{\varphi_1}^{2\pi-\varphi_1}\kappa d\varphi}\right)\right]-\\
&\quad \frac{t_v}{t_h}\left[e^{-i\theta+\frac{2\pi i}{\varepsilon}(\varphi+\pi)}(1+o(1))+e^{i\theta-i\frac{2\pi}{\varepsilon}(\varphi+\pi)}(1+o(1))\right].
\end{aligned}
$$

Here, we have set

$$
\begin{aligned}
&\tilde{t}_h = e^{\frac{2i}{\varepsilon}\int_0^{\varphi_1}\kappa d\varphi}, \quad \tilde{\Phi} = \int_{\varphi_1}^{2\pi-\varphi_1}\kappa d\varphi, \quad \tilde{t}_v = e^{-\frac{i}{\varepsilon}\int_{\overline{\varphi_2}}^{\varphi_2}(\kappa-\pi)d\varphi},\\
&G = \exp\left(\int_0^{\varphi_1}(\omega_+-\omega_-)d\varphi\right), \quad \theta = i\int_{\varphi_1}^{\varphi_2}(\omega_+-\omega_-)d\varphi.
\end{aligned}
\tag{8.56}
$$

All the integrals are taken along curves in K. We note that κ is real along the interval $[\varphi_1, 2\pi-\varphi_1]$. So, all the terms of the form $o\left(e^{\cdots}\right)$ are, in fact, $o(1)$. This implies the asymptotic formulae of Proposition 8.1 if one checks that $\tilde{t}_h = t_h$, $\tilde{t}_v = t_v$ and $\tilde{\Phi} = \Phi$; this is done in section 9.1.1.

8.5.1. *Completing the proof of Proposition 8.1.* To finish the proof of Proposition 8.1, we check that $\varphi_{(0)}$ is real for $E \in \mathbb{R}$. Therefore, we prove that $\theta = i\int_{\varphi_1}^{\varphi_2}(\omega_+-\omega_-)d\varphi$ is real. One has

$$
\overline{\theta} = -i\overline{\int_{\varphi_1}^{\varphi_2}(\omega_+-\omega_-)d\varphi} = -i\int_{\varphi_1}^{\overline{\varphi_2}}\overline{(\omega_+-\omega_-)}d\bar{\varphi}.
$$

As $\overline{\omega_+(\varphi)} = \omega_-(\overline{\varphi})$ (see (7.12)), and $\int_{\varphi_2}^{\overline{\varphi_2}}\omega_\pm d\varphi = 0$ (see (8.55)), one has

$$
\overline{\theta} = i\int_{\varphi_1}^{\varphi_2}(\omega_+-\omega_-)d\varphi = \theta.
$$

This completes the proof of the Proposition 8.1. □

8.6. **Getting a monodromy matrix of the form** (2.14)**.** To get such a monodromy matrix, we change the consistent basis. Recall that, using Theorem 6.1, we have constructed the solution $f_+^0(u,\varphi)$ with the "standard" asymptotics in K_0. Denote it by f_0. Consider the function $f_0^*(u,\varphi) = \overline{f_0(u,\overline{\varphi})}$. It satisfies equation (7.1) and the consistency condition (6.3). First, we compute $T_0(\varphi)$, the matrix of (f_0, f_0^*) in the basis (f_+^0, f_-^0). Then, we check that the Wronskian of f_0 and f_0^* is a non-zero constant. So, the basis (f_0, f_0^*) is consistent; the monodromy matrix for this basis is the one described in Theorem 2.2. It is given by

$$M(\varphi) = T_0(\varphi + 2\pi)\tilde{M}(\varphi)T_0(\varphi)^{-1}, \tag{8.57}$$

where $\tilde{M}$ is the monodromy matrix for the basis (f_+^0, f_-^0). Finally, we use the asymptotics of T and $\tilde{M}$ to get the asymptotics of M.

As $f_0 = f_+^0$, the matrix T has the form

$$T_0(\varphi) = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}, \tag{8.58}$$

where a and b are the coefficients from the representation $f_0^* = af_+^0 + bf_-^0$ and, thus, are given by the formulae:

$$a = \frac{w(f_-^0, f_0^*)}{w(f_-^0, f_+^0)} \quad \text{and} \quad b = \frac{w(f_0^*, f_+^0)}{w(f_-^0, f_+^0)} = \frac{w(f_0^*, f_0)}{w(f_-^0, f_+^0)}. \tag{8.59}$$

Let us first compute a. First, note that, $K_0 \cap \overline{K_0}$ contains the rectangle $R_1 = \{\varphi \in K_0;\ |\mathrm{Im}\,\varphi| \leq \mathrm{Im}\,\varphi_3 - \delta,\ |\mathrm{Re}\,\varphi| \leq \delta\}$ for some $\delta > 0$. In this rectangle, we know the asymptotic of f_0. To get the one of f_0^*, we note that

$$\overline{\kappa(\overline{\varphi})} = -\kappa(\varphi), \quad \overline{\psi_+(u,\overline{\varphi})} = \psi_+(u,\varphi), \quad \overline{\omega_+(u,\overline{\varphi})} = \omega_+(u,\varphi), \quad \overline{q(\overline{\varphi})} = -iq(\varphi), \quad \varphi \in R_1. \tag{8.60}$$

Here, κ, $\psi_\pm$, $\omega_\pm$ and q coincide with κ_0, $\psi_\pm^0$, $\omega_\pm^0$ and q_0. The first of these relations follows as κ is purely imaginary on $(0, \varphi_1)$. The second is valid as the branches of the Bloch solution $\psi(x, \mathcal{E})$ are real for $\mathcal{E} < E_1$. The third one follows from the second one and the definitions of $\omega_\pm$. The last one follows from the choice of the branch of q_0: $q_0 \in e^{-i\pi/4}\mathbb{R}_+$ on $(0, \varphi_1)$, see section 7.4.1.

By (8.16), (8.17) and (8.60), we get

$$f_0^*(u,\varphi) = -ie^{\frac{i}{\varepsilon}\int_0^\varphi \kappa_0 d\varphi}(\Psi_+^0(u,\varphi) + o(1)). \tag{8.61}$$

Comparing this result with the asymptotics of $f_0 = f_+^0$, see (8.16), we obtain the uniform formula

$$a = -i + o(1), \quad \varphi \in R_1. \tag{8.62}$$

From the ε-periodicity and analyticity of a, we deduce that this asymptotics is valid and uniform in the whole strip $\{|\mathrm{Im}\,\varphi| \leq \mathrm{Im}\,\varphi_3 - \delta\}$.

We now turn to the asymptotics of b. Therefore, we compute the asymptotics of f_0 in $K_1 \cap \{\mathrm{Im}\,\varphi \leq -\delta\}$, for a fixed positive δ. By definition of T_1 and (8.6), we know that $f_0 = a_1 f_+^1 + b_1 f_-^1$. The asymptotics of a_1, b_1 and of (f_+^1, f_-^1) are known in K_1 (see (8.16) and Proposition 8.2). So, we get that

$$\begin{aligned} f_0(u,\varphi) &= e^{\frac{i}{\varepsilon}\int_0^\pi \kappa d\varphi} e^{\int_0^\pi \omega_+ d\varphi} e^{\frac{i}{\varepsilon}\int_\pi^\varphi \kappa d\varphi}(\Psi_+^1(u,\varphi) + o(1)) + o(1)e^{\frac{i}{\varepsilon}\int_0^\pi \kappa d\varphi} e^{-\frac{i}{\varepsilon}\int_\pi^\varphi \kappa d\varphi}(\Psi_-^1(u,\varphi) + o(1)) \\ &= e^{\frac{i}{\varepsilon}\int_0^\varphi \kappa d\varphi}\left(\Psi_+^0(u,\varphi) + o(1) + o(1)e^{-\frac{2i}{\varepsilon}\int_\pi^\varphi \kappa d\varphi}(\Psi_-^1(u,\varphi) + o(1))\right) \\ &= e^{\frac{i}{\varepsilon}\int_0^\varphi \kappa d\varphi}(\Psi_+^0(u,\varphi) + o(1)). \end{aligned} \tag{8.63}$$

Here, we have used the fact that the coefficient $c = e^{-\frac{2i}{\varepsilon}\int_\pi^\varphi \kappa d\varphi}$ is (exponentially) small. Indeed, the domain K_1 is canonical. So, there is a canonical curve connecting φ with the uppermost point of K_1. As $\mathrm{Im}\,\varphi < 0$, it intersects the segment $(\varphi_1, 2\pi - \varphi_1)$ which is a Stokes line. So, in the formula for c, we can integrate from φ to this segment along the canonical line, and, then, along this segment to π. Now, the definitions of the canonical lines and the Stokes lines imply the smallness of c.

In result of the computation (8.63), we see that the leading term of the asymptotics of f_0 in $K_1 \cap \{\mathrm{Im}\,\varphi < 0\}$ is obtained by analytic continuation from K_0 to $K_1 \cap \{\mathrm{Im}\,\varphi < 0\}$ across the common part of K_0 and K_1. Consider the "rectangle" $R_2 = \{\varphi_1 + \delta \leq \mathrm{Re}\,(\varphi) \leq \pi - \delta,\ \delta \leq |\mathrm{Im}\,\varphi| \leq \mathrm{Im}\,\varphi_3 - \delta\}$. We can assume that it is contained in $K_0 \cup K_1$. We have

$$f = e^{\frac{i}{\varepsilon}\int_0^\varphi \kappa d\varphi}(\Psi_+^0(u,\varphi) + o(1)), \quad \varphi \in R_2. \tag{8.64}$$

For the lower part of R_2, this follows from (8.63), and for $\mathrm{Im}\,\varphi > 0$, this is just the asymptotics from (8.16).

Now, we get from (8.64), the asymptotics of f_0^* in R_2. We note that

$$\overline{\kappa(\overline{\varphi})}=\kappa(\varphi),\quad \overline{\psi_-(u,\overline{\varphi})}=\psi_+(u,\varphi),\quad \overline{\omega_-(u,\overline{\varphi})}=\omega_+(u,\varphi),\quad \overline{q(\overline{\varphi})}=q(\varphi),\quad \varphi\in R_2. \tag{8.65}$$

Here, κ, $\psi_\pm$, $\omega_\pm$ and q are obtained by analytic continuation of κ_0, $\psi_\pm^0$, $\omega_\pm^0$ and q_0 from K_0 to R_2 through their common part. The first of these relations follows as κ is real on $(\varphi_1, 2\pi-\varphi_1)$. The second is valid as the branches of the Bloch solution $\psi(x,\mathcal{E})$ differ by the complex conjugation on the spectral bands. The third one follows from the second one and the definitions of $\omega_\pm$. The last one follows from the choice of the branch of q_0, see section 7.4.1.

The relations (8.65) and (8.60), and the asymptotics (8.64) imply that

$$\begin{aligned} f_0^*(u,\varphi)&=e^{\frac{i}{\varepsilon}\int_0^{\varphi_1}\kappa d\varphi}e^{-\frac{i}{\varepsilon}\int_{\varphi_1}^{\varphi}\kappa d\varphi}e^{\int_0^{\varphi_1}(\omega_+-\omega_-)d\varphi}(\Psi_-^0(u,\varphi)+o(1))\\ &=e^{\frac{2i}{\varepsilon}\int_0^{\varphi_1}\kappa d\varphi}e^{\int_0^{\varphi_1}(\omega_+-\omega_-)d\varphi}\left(e^{-\frac{i}{\varepsilon}\int_0^{\varphi}\kappa d\varphi}(\Psi_-^0(u,\varphi)+o(1))\right).\end{aligned}$$

The asymptotics is uniform for $\varphi\in R_2$. Now, using the asymptotics of f_0 and f_0^*, and the formula (9.5) for S_h, we get

$$w(f_0^*,f_0)=t_h e^{\int_0^{\varphi_1}(\omega_+-\omega_-)d\varphi}w(f_-^0,f_+^0)(1+o(1)), \tag{8.66}$$

As the Wronskian is periodic, this asymptotic is valid and uniform in the domain $\{\delta\le|\operatorname{Im}\varphi|\le\operatorname{Im}\varphi_3-\delta\}$.

We see that the leading term of the Wronskian is constant. The factor $1+o(1)$ is ε-periodic and real analytic. Recall that in order to define the monodromy matrix, we need that the Wronskian be constant. To "correct" the situation, we slightly modify f_0. Formula (8.66) shows that we can (and we do) divide f_0 by the real analytic and ε-periodic factor $\sqrt{1+o(1)}$ so that

$$w(f_0^*,f_0)=t_h e^{\int_0^{\varphi_1}(\omega_+-\omega_-)d\varphi}w(f_-^0,f_+^0).$$

The new functions f_0 and f_0^* form a consistent basis and still have the "old" asymptotics in $\{\delta\le|\operatorname{Im}\varphi|\le\operatorname{Im}\varphi_3-\delta\}$. So, for the new solutions we get the "old" formula for a and

$$b=t_h e^{\int_0^{\varphi_1}(\omega_+-\omega_-)d\varphi}. \tag{8.67}$$

Now, combining (8.58) with the asymptotics obtained for a and b, we get

$$T_0(\varphi)=\begin{pmatrix}1 & 0\\ -i+o(1) & t_h e^{\int_0^{\varphi_1}(\omega_+-\omega_-)d\varphi}\end{pmatrix},\quad |\operatorname{Im}(\varphi)|\le\operatorname{Im}(\varphi_3)-\delta.$$

Being obtained by means of the asymptotics of $f_\pm^0$ (constructed by Theorem 6.1) and the asymptotics of a_1 and b_1 given by Proposition 8.2, the asymptotics of T stay uniform in a constant neighborhood of $E=E_0\in J$.

As the monodromy matrix M is associated to the basis (f_0,f_0^*), it takes the the form

$$M(\varphi)=\begin{pmatrix}a(\varphi) & b(\varphi)\\ b^*(\varphi) & a^*(\varphi)\end{pmatrix}.$$

Using the asymptotics of T and of the monodromy matrix $\tilde{M}$ obtained in Theorem 8.1, one computes the coefficients a and b to obtain

$$\begin{aligned} a(\varphi)&=-\frac{1}{t_h}\left(e^{i\Phi/\varepsilon}(1+o(1))-t_v e^{\frac{2i\pi}{\varepsilon}(\varphi-\varphi_{(0)})}(1+o(1))\right),\\ b(\varphi)&=\frac{i}{t_h}\left(e^{i\Phi/\varepsilon}(1+o(1))-t_v e^{\frac{2i\pi}{\varepsilon}(\varphi-\varphi_{(0)})}(1+o(1))\right).\end{aligned} \tag{8.68}$$

The asymptotics are valid uniformly in the strip $|\operatorname{Im}(\varphi)|\le\operatorname{Im}(\varphi_3)-\delta$.
Now, to get the statement of Theorem 2.2, one passes to the variable $\phi=\varphi/\varepsilon$. The asymptotic representations for a_0, b_0, $\tilde{a}$ and $\tilde{b}$ described in Theorem 2.2 follow from (8.68) and standard estimates of the Fourier coefficients. ☐

9. Properties of the phase and actions integrals. The phase diagrams

First, we study properties of the phase integral Φ and the tunneling actions S_v and S_h. Then, we investigate the location of the asymptotic mobility edges and prove Theorem 2.6 and formula (2.25).

9.1. **Properties of the phase integrals.**

9.1.1. *Proof of Lemma 2.1.* We begin by discussing the phase integral Φ. The branch κ_* chosen in section 2.6 to define the phase integral is actually the branch κ_1 defined in section (7.3.2). Indeed,

1. κ_1 coincides with the main branch on $[\varphi_1, \pi]$;
2. the main branch satisfies $0 \leq \kappa_0 < \pi$ on the interval $[\varphi_1, \pi]$;
3. κ_1 satisfies (7.9).

Point 3 immediately implies the formula

$$\Phi = \frac{2}{\varepsilon} \int_{\varphi_1}^{\pi} \kappa_0 d\varphi. \tag{9.1}$$

Now, let us prove the properties of Φ described in Lemma 2.1. By (9.1), the positivity of Φ is obvious. Check that $\Phi'(E) > 0$ on J. Recall that $\kappa_0(\varphi) = k_0(E - \alpha\cos(\varphi))$. Therefore, (9.1) implies that

$$\Phi'(E) = \frac{2}{\varepsilon} \int_{\varphi_1}^{\pi} k_0'\,(E - \alpha\cos(\varphi)) d\varphi. \tag{9.2}$$

Note that, since $k_0(E)$ has square root branch points at the ends of the spectral zones, the integral in (9.2) converges. As $k_0'(E) > 0$ inside the first spectral zone, equation (9.2) implies the positivity of Φ' inside J. To prove the analyticity of Φ in E, we use the representation (2.13) of Φ by a contour integral taken along a closed curve γ_p corresponding to Fig. 3 This representation was announced in section (2.6); we prove it now.

We begin by noting that, as $\cos(2\pi - \varphi) = \cos\varphi$, on the integration contour in (2.11), the branch $\kappa_* = \kappa_1$ satisfies the relation $\kappa_1(2\pi - \varphi) = \kappa_1(\varphi)$, and, therefore,

$$\kappa_1(2\pi - \varphi_1) = \kappa_1(\varphi_1) = 0. \tag{9.3}$$

It suffices to prove (2.13) for the case where γ_p goes around the interval $[\varphi_1, 2\pi - \varphi_1]$ just along this interval. First, let us try to continue analytically κ_1 from the interval $[\varphi_1, 2\pi - \varphi_1]$ to the contour γ_p. We can continue κ_1 analytically on γ_p cut at one of its points. Check that κ_1 is continuous on the whole contour. Assume that γ_p is cut at $2\pi - \varphi_1 + 0$. We define κ_1 on γ_p so that the integral in (2.11) be the integral along the upper part of this contour. Check that the values of κ_1 at the "different" sides of the cut coincide. As κ_0 is purely imaginary on the interval $(-\varphi_1, \varphi_1)$, one has

$$\kappa_1(\overline{\varphi}) = -\overline{\kappa_1(\varphi)}, \quad \varphi \in \gamma_p. \tag{9.4}$$

This and (9.3) imply that the function κ_1 is continuous on the whole γ_p i. e. that $\kappa_1(2\pi - \varphi_1 + i0) = \kappa_1(2\pi - \varphi_1 - i0) = 0$. Now, the representation (2.13) for Φ immediately follows from (9.4).

The representation (2.13) implies the analyticity of Φ as

1. we can now deform γ_p to a closed curve going around $[\varphi_1, 2\pi - \varphi_1]$ and staying at a nonzero distance from this interval;
2. κ_1 is analytic in E on the integration contour when E is close enough to a given point in $J \subset \mathbb{R}$.

We have proved all the statements of Lemma 2.1 on Φ. The analysis of the tunneling actions S_v and S_h is done in the same way. In particular, one has

$$S_v(E) = 2i \int_{\pi}^{\varphi_2} (\kappa_0(\varphi) - \pi) d\varphi, \quad S_h(E) = -2i \int_{0}^{\varphi_1} \kappa_0(\varphi) d\varphi, \tag{9.5}$$

where, in the first integral, we integrate along the segment $[\pi, \varphi_2]$ of the line $\pi + i\mathbb{R}$. Point 4 of Lemma 2.1 follows from the representation for S_v from (9.5) as, on the integration contour, $0 < \kappa_0 < \pi$.

9.1.2. *Functions $\tilde{\Phi}$, $\tilde{t}_v$ and $\tilde{t}_h$.* In view of (9.1) and (9.5), the functions $\tilde{\Phi}$, $\tilde{t}_v$ and $\tilde{t}_h$ defined by (8.56) satisfy the relations

$$\tilde{\Phi} = \Phi, \quad \tilde{t}_v = t_v, \quad \tilde{t}_h = t_h,$$

needed to complete the computations of the monodromy matrix in section 8.5.

9.1.3. *Dependence of the action integrals on α.* We now study the actions S_h and S_v defined in (2.10) as functions of α and E. It will be convenient to change variables $(\alpha, E) \mapsto (\alpha, t)$ so that $E = E_1 + t\alpha$, and to keep the notations $S_h(\alpha, t)$ and $S_v(\alpha, t)$ for the functions obtained from $S_h(\alpha, E)$ and $S_v(\alpha, E)$ by this change of variables. Representations (9.5) imply that

$$S_v(\alpha,t) = 2i\int_\pi^{\varphi_2} (k_0(E_1+\alpha(t-\cos(\varphi)))-\pi)d\varphi, \quad S_h(\alpha,t) = -2i\int_0^{\varphi_1} k_0(E_1+\alpha(t-\cos(\varphi)))d\varphi, \tag{9.6}$$

where k_0 is the main branch of the Bloch quasi-momentum, see section 5.2.2. The domain $\mathcal{S}$ defined by (2.24) becomes

$$\mathcal{K} := \{(\alpha, t);\ -1 \le t \le 1,\ 0 < \alpha,\ (1+t)\alpha \le E_2 - E_1\}.$$

The formulae (9.6) show that the actions $S_h(\alpha, t)$ and $S_v(\alpha, t)$ are real analytic in (α, t) in $\operatorname{int}(\mathcal{K})$, the interior of $\mathcal{K}$, and continuous at its boundary. One proves

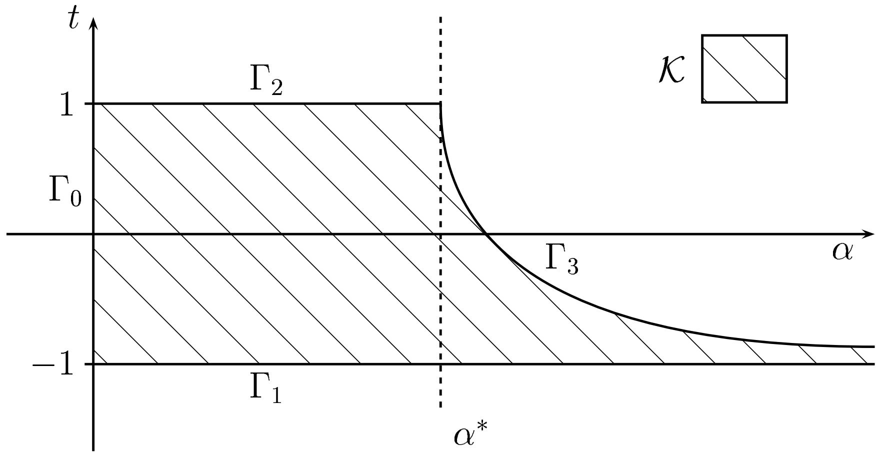

FIGURE 20. The domain $\mathcal{K}$

Lemma 9.1. *In* $\operatorname{int}(\mathcal{K})$, *one has*

$$\frac{\partial S_v}{\partial \alpha} < 0, \quad \frac{\partial S_h}{\partial \alpha} > 0, \ \text{and}\ \frac{\partial \Delta S}{\partial \alpha} = \frac{\partial (S_v - S_h)}{\partial \alpha} < 0 \tag{9.7}$$

Proof of Lemma 9.1. Taking into account (9.6), we compute

$$\partial_\alpha S_h = \frac{\partial S_h}{\partial \alpha} = -2i\int_0^{\varphi_1} k_0'(E_1+\alpha(t-\cos(\varphi)))(t-\cos(\varphi))d\varphi,$$
$$\partial_\alpha S_v = \frac{\partial S_v}{\partial \alpha} = 2i\int_\pi^{\varphi_2} k_0'(E_1+\alpha(t-\cos(\varphi)))(t-\cos(\varphi))d\varphi.$$

We notice that

- for $\varphi \in (0, \varphi_1) \subset \mathbb{R}$, we have $E - \alpha\cos(\varphi) < E_1$; hence, $-ik_0'(E - \alpha\cos(\varphi)) < 0$ (see Fig. 5). Furthermore, the inequality $E - \alpha\cos(\varphi) < E_1$ is equivalent to $t - \cos(\varphi) < 0$. Hence, $\partial_\alpha S_h \ge 0$. Moreover, $\partial_\alpha S_h = 0$ if and only if $\varphi_1 = 0$ i.e. if and only if $t = 1$.
- for $\varphi \in (\pi, \varphi_2) \subset W^{-1}(\mathbb{R})$, we have $E_1 < E - \alpha\cos(\varphi) < E_2$; hence, $k_0'(E - \alpha\cos(\varphi)) > 0$ (see Fig. 5). On the other hand, $E_1 < E - \alpha\cos(\varphi)$ means that $t - \cos(\varphi) > 0$. Hence, $\partial_\alpha S_v \le 0$. Moreover, $\partial_\alpha S_v = 0$ if and only if $\varphi_2 = \pi$ i.e. if and only if $(1+t)\alpha = E_2 - E_1$.

This completes the proof of Lemma 9.1. ☐

9.2. **The phase diagram.** Now, we study the asymptotic mobility edges: we prove Theorem 2.6 and formula (2.25). We use the notations of section 9.1.3

9.2.1. *The curve* $\Delta S = 0$. Lemma 9.1 implies that, for every $t \in [-1,1]$, there exists at most one α such that $(\alpha,t) \in \mathcal{K}$ and $\Delta S_h(\alpha,t) = 0$. So, to complete the proof of Theorem 2.6, we only are left with studying the behavior of ΔS on the boundary of $\mathcal{K}$. The boundary $\partial\mathcal{K}$ is made of four curves

1. $\Gamma_0 : \alpha = 0, -1 \le t \le 1$;
2. $\Gamma_1 : t = -1, 0 < \alpha$;
3. $\Gamma_2 : t = 1, 0 < \alpha < \alpha^*, \alpha^* = \frac{E_2 - E_1}{2}$;
4. $\Gamma_3 : \alpha^* \le \alpha, (1+t)\alpha = E_2 - E_1$.

Note that, in terms of α and t, the relations defining φ_1 and φ_2 have the form:

$$\cos\varphi_1 = t, \quad \cos\varphi_2 = t - \frac{E_2 - E_1}{\alpha}. \tag{9.8}$$

Now, we can study S_v and S_h on the boundary of $\mathcal{K}$.

- On Γ_3, one has $\varphi_2 = \pi$, $\varphi_1 \ge 0$, and $\varphi_1 = 0$ only at $(\alpha^*, 1)$; hence, $\Delta S < 0$ except at the point $(\alpha^*, 1)$, where $\Delta S = 0$.
- Near Γ_0 i.e. for α small and $t \in [-1,1]$, one has $\varphi_1 \in [0,\pi]$, and $k_0(E_1 + \alpha(t - \cos(\varphi)) \in [0,\pi]$. So, by (9.6), $S_h(\alpha,t)$ stays bounded. On the other hand, pick $0 < \beta < E_2 - E_1$. Then, on the interval $I_\beta = \{\varphi \in \pi + i\mathbb{R}; t - \beta/\alpha \le \cos\varphi\}$, one has $\kappa_0(\varphi) - \pi \le k_0(E_1 + \beta) - \pi < 0$. Hence, by (9.6), one has $S_v(t,\alpha) \ge 2(\pi - k_0(E_1+\beta))\mathrm{Im}\,\varphi_\beta$, where $\varphi_\beta \in \pi + i\mathbb{R}_+$ is defined by $\cos\varphi_\beta = t - \beta/\alpha$. This implies that, uniformly for $t \in [-1,1]$, one has $S_v(t,\alpha) \to +\infty$ as $\alpha \to 0$. Hence, for α sufficiently small, one has $\Delta S > 0$.
- The statements of Lemma 9.1 remain valid on Γ_1 and Γ_2.
- On Γ_1, one has $\varphi_1 = \pi$; and, when $\alpha \to +\infty$, one has $\varphi_2 \to \pi$, hence, $S_v \to 0$; on the other hand, $S_h \to +\infty$; so ΔS is negative for α large enough.
- On Γ_2, one has $\varphi_1 = 0$ and $\mathrm{Im}\,\varphi_2 > 0$. So, there, $\Delta S > 0$.

As a conclusion of this discussion, we see that, for every $t \in [-1,1]$, there exists a unique point $\theta(t)$ such that $\Delta(\alpha(t),t) = 0$ and $(\alpha(t),t) \in \mathcal{K}$. The analyticity of the curve $t \mapsto \alpha(t)$ is a consequence of the analyticity of ΔS, Lemma 9.1 and the local inversion Theorem. This completes the proof of Theorem 2.6. $\square$

9.2.2. *Mobility edges near the point* $(\alpha^*, 1)$. Consider the asymptotic mobility edge in a neighborhood of the point $(\alpha^*, 1)$. When proving the last theorem, we have seen that $S_h(\alpha^*, 1) = S_v((\alpha^*, 1) = 0$. Let the $(m_i)_{i=1,2}$ be the effective masses corresponding to $(E_i)_{i=1,2}$ the ends of the first spectral band of the periodic Schrödinger operator (see, section 5.2.2). One can obtain the Taylor formula for S_h and S_v at $(\alpha^*, 1)$:

$$S_h(\alpha,t) \sim \frac{\pi}{2}\sqrt{\frac{2m_2}{\alpha^*}}\,(2(\alpha^* - \alpha) + \alpha^*(t-1)) \text{ and } S_v(\alpha,t) \sim \frac{\pi}{2}\sqrt{2\alpha^* m_1}(1-t), \quad \alpha \sim \alpha^*, \quad t \sim 1,$$

where we have omitted the standard error terms. As $\Delta S = S_h - S_v$, this implies (2.25).

10. Positive Lyapunov exponent

Let $(M(\phi,\varepsilon))_{0<\varepsilon<1}$ be a family of $SL(2,\mathbb{C})$-valued, sufficiently regular, 1-periodic functions. Let h be a positive number. Define the matrix cocycle

$$P_N(\phi,\varepsilon) = M(\phi + Nh,\varepsilon) \cdot M(\phi + (N-1)h,\varepsilon) \cdots M(\phi + h,\varepsilon) \cdot M(\phi,\varepsilon).$$

Then, $\gamma(\phi)$, the Lyapunov exponent of this matrix cocycle at the point ϕ, is defined by

$$\gamma(\phi) = \lim_{N\to+\infty} \frac{1}{N} \log \|P_N(\phi,\varepsilon)\|. \tag{10.1}$$

It is well known (see [8, 37, 40] and references therein) that, if h is irrational, then this limit exists for almost all ϕ, is independent of ϕ, and satisfies

$$\gamma = \lim_{N\to+\infty} \frac{1}{N} \int_0^1 \log \|P_N(\theta,\varepsilon)\| d\theta. \tag{10.2}$$

We prove

Proposition 10.1. *Pick* $\varepsilon_0 > 0$. *Assume that there exist* y_0 *and* y_1 *satisfying the inequalities* $0 < y_0 < y_1 < \infty$ *and such that, for any* $\varepsilon \in (0,\varepsilon_0)$ *one has*

- *the function* $\phi \to M(\phi,\varepsilon)$ *is analytic in the strip* $S = \{\phi \in \mathbb{C};\ 0 \le \mathrm{Im}\,\phi \le y_1/\varepsilon\}$;

- *in the strip* $S_1 = \{\phi \in \mathbb{C};\ y_0/\varepsilon \le \operatorname{Im}\phi \le y_1/\varepsilon\} \subset S$, $M(\phi,\varepsilon)$ *admits the representation*
$$M(\phi,\varepsilon) = \lambda(\varepsilon) e^{i2\pi n_0 \phi} \cdot (M_0(\varepsilon) + M_1(\phi,\varepsilon)), \quad n_0 \in \mathbb{N}\setminus\{0\},$$
with a positive constant λ *and a matrix* M_0 *independent of* ϕ*;*
- $M_0(\varepsilon) = \begin{pmatrix} 1 & \beta(\varepsilon) \\ 0 & \alpha(\varepsilon) \end{pmatrix}$*;*
- *there exist constants* $C > 0$, $\beta > 0$ *and* $\alpha \in (0,1)$ *independent of* ε *and such that*
  1. $|\alpha(\varepsilon)| \le \alpha$ *and* $|\beta(\varepsilon)| \le \beta$,
  2. $\sup_{\phi \in S_1} \|M_1(\phi,\varepsilon)\| \le C\varepsilon$, $\|\cdot\|$ *is the matrix norm associated to* ℓ^1.

Then, there exit $C > 0$ *and* $\varepsilon_1 > 0$ *such that, if* $0 < \varepsilon < \varepsilon_1$*, one has*

$$\gamma(\phi) > \log\lambda(\varepsilon) - C\varepsilon. \tag{10.3}$$

Proof. To prove this result, we follow the ideas of Herman ([26]) as extended by Sorets and Spencer ([40]). One computes

$$M_0(\varepsilon) = U(\varepsilon) \cdot \begin{pmatrix} 1 & 0 \\ 0 & \alpha(\varepsilon) \end{pmatrix} \cdot U(\varepsilon)^{-1}, \quad \text{where} \quad U(\varepsilon) = \begin{pmatrix} 1 & \frac{-\beta(\varepsilon)}{1-\alpha(\varepsilon)} \\ 0 & 1 \end{pmatrix}.$$

Under our assumptions on M_0, the family $(U(\varepsilon))_{\varepsilon \in (0,\varepsilon_0)}$ is uniformly bounded.
Let n_0 be as in Proposition 10.1. Put

$$\tilde{M}(\phi,\varepsilon) = e^{-i2\pi n_0 \phi} \cdot U(\varepsilon)^{-1} \cdot M(\phi,\varepsilon) \cdot U(\varepsilon). \tag{10.4}$$

Note that, in S_1,

$$\tilde{M}(\phi,\varepsilon) = \lambda(\varepsilon) \left[\begin{pmatrix} 1 & 0 \\ 0 & \alpha(\varepsilon) \end{pmatrix} + \tilde{M}_1(\phi,\varepsilon) \right], \quad \sup_{\phi \in S_1} \|\tilde{M}_1\| \le \text{Const} \cdot \varepsilon. \tag{10.5}$$

Denote by $\tilde{P}_N(\phi,\varepsilon)$ the matrix cocycle associated to the pair $(\tilde{M}, h)$,

$$\tilde{P}_N(\phi,\varepsilon) = \tilde{M}(\phi + Nh) \cdots \tilde{M}(\phi + h, \varepsilon) \cdot \tilde{M}(\phi).$$

Hence, by (10.4) and (10.2), one has

$$\gamma(\phi,\varepsilon) = \lim_{N \to +\infty} \frac{1}{N} \int_0^1 \log \|\tilde{P}_N(\theta,\varepsilon)\| d\theta.$$

Define the function g_N by

$$g_N(\phi,\varepsilon) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tilde{P}_N(\phi,\varepsilon) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle,$$

where the angular brackets denote the scalar product in $\mathbb{R}^2$. One obtains the following lower bound on the Lyapunov exponent

$$\gamma(\phi,\varepsilon) \ge \lim_{N \to +\infty} \frac{1}{N} \int_0^1 \log |g_N(\theta,\varepsilon)| d\theta. \tag{10.6}$$

Let us now recall a version of Jensen's formula:

Lemma 10.1 ([40]). *Pick* $0 < \rho < 1$*. Let* f *be an analytic function in the annulus* $\mathcal{A} = \{z \in \mathbb{C};\ \rho < |z| < 1\}$ *and continuous on* $\overline{\mathcal{A}}$*. Let* $(r_j)_j$ *denote the roots of* f *in* $\mathcal{A}$*. Then, one has*

$$\int_0^1 \log|f(e^{i2\pi\theta})| d\theta = \int_0^1 \log|f(\rho e^{i2\pi\theta})| d\theta - \sum_{r_j \in \mathcal{A}} \log|r_j| - \frac{1}{2i\pi} \left[\int_{|z|=\rho} \frac{f'(z)}{f(z)} dz \right] \log\rho. \tag{10.7}$$

We let $z = e^{2\pi i \phi}$. As g_N is analytic and 1-periodic in $\phi \in S$, by the relation $f_N(z,\varepsilon) = g_N(\phi,\varepsilon)$, we can define a function analytic in the annulus $1 \ge |z| \ge e^{-2\pi y_2/\varepsilon}$. We take $\rho = e^{-2\pi y}$ with an y satisfying the estimates $y_0/\varepsilon < y < y_1/\varepsilon$, and apply Lemma 10.1 to $f_N(z,\varepsilon)$. Taking into account the fact that the contribution of the zeroes of f_N to (10.7) is non-positive, equation (10.6) tells us that

$$\int_0^1 \log|g_N(\theta,\varepsilon)| d\theta \ge \int_0^1 \log(f_N(\rho e^{i2\pi\theta}), \varepsilon) d\theta - \frac{|y_1|}{\varepsilon} \left| \int_{|z|=\rho} \frac{f_N'(\zeta,\varepsilon)}{f_N(\zeta,\varepsilon)} d\zeta \right|. \tag{10.8}$$

We are now left with estimating f_N and f_N' on the circles $C_y = \{|z| = e^{-2\pi y}\},\ y_0/\varepsilon < y < y_1/\varepsilon$.

Let $\tilde{M}_1$ be as in (10.5). We define inductively $\{a_k, b_k\}_{k=0}^{\infty}$ so that

$$a_0 = b_0 = 0, \quad (1 + b_k)\begin{pmatrix}1\\ a_k\end{pmatrix} = \left(\begin{pmatrix}1 & 0\\ 0 & \alpha(\varepsilon)\end{pmatrix} + \tilde{M}_1(z_k, \varepsilon)\right)\begin{pmatrix}1\\ a_{k-1}\end{pmatrix}, \quad z_k = e^{2\pi i(\phi + hk)}, \quad k \geq 1. \tag{10.9}$$

Then, one checks that $f_N(\phi) = \lambda(\varepsilon)^N (1 + b_N) \cdots (1 + b_0)$. We use

Lemma 10.2. *Let $m(\varepsilon) = \sup_{\phi \in S_1} \|\tilde{M}_1(\phi, \varepsilon)\|$. There exists $C > 0$ such that, for ε sufficiently small and $z \in C_y,\ y_0/\varepsilon < y < y_1/\varepsilon$, one has*

$$|a_k| \leq \sqrt{m(\varepsilon)} \text{ and } |b_k| \leq 2m(\varepsilon), \quad k \in \mathbb{N}. \tag{10.10}$$

Proof. Lemma 10.2 is proved inductively. Obviously, (10.10) holds for $k = 0$. Let us assume it holds up to rank k. Equation (10.9) implies

$$1 + b_{k+1} = 1 + \left\langle \begin{pmatrix}1\\ 0\end{pmatrix}, \tilde{M}_1(z_{k+1}, \varepsilon)\begin{pmatrix}1\\ a_k\end{pmatrix}\right\rangle$$

$$(1 + b_{k+1})a_{k+1} = \alpha(\varepsilon)a_k + \left\langle \begin{pmatrix}0\\ 1\end{pmatrix}, \tilde{M}_1(z_{k+1}, \varepsilon)\begin{pmatrix}1\\ a_k\end{pmatrix}\right\rangle.$$

Using assumptions 1. and 2. on M_1, $\alpha(\varepsilon)$ and $\beta(\varepsilon)$, one gets

$$|a_{k+1}| \leq \left|\frac{1}{1 + b_{k+1}}\right| [\alpha|a_k| + m(\varepsilon)(1 + |a_k|)] \text{ and } |b_{k+1}| \leq m(\varepsilon)(1 + |a_k|) \tag{10.11}$$

Using assumption 2. and the induction assumption, for ε sufficiently small, one gets

$$|b_{k+1}| \leq 2m(\varepsilon)$$

uniformly in k. Pluging this into the first inequality from (10.11), for ε sufficiently small, one obtains

$$|a_{k+1}| \leq \frac{1}{1 - 2m(\varepsilon)}\left[\alpha\sqrt{m(\varepsilon)} + m(\varepsilon)(1 + \sqrt{m(\varepsilon)})\right] \leq \sqrt{m(\varepsilon)},$$

uniformly in k. By induction, this completes the proof of Lemma 10.2. □

As an immediate consequence of Lemma 10.2, we get that, for any k, the functions a_k and b_k are analytic in the annulus $\mathcal{A} = \{z \in \mathbb{C};\ e^{-2\pi y_1/\varepsilon} \leq |z| \leq e^{-2\pi y_0/\varepsilon}\}$. Hence, by the Cauchy estimates, the derivative of b_k satisfies the estimate $|b_k'| \leq \text{Const} \cdot \varepsilon \cdot m(\varepsilon)$ in some smaller annulus $\tilde{\mathcal{A}} = \{z \in \mathbb{C};\ e^{-2\pi \tilde{y}_1/\varepsilon} \leq |z| \leq e^{-2\pi \tilde{y}_0/\varepsilon}\}$. So, there exists $C > 0$ such that, if $\rho = e^{-2\pi y}$ for $\tilde{y}_0/\varepsilon < y < \tilde{y}_1/\varepsilon$, one has

$$\frac{1}{\varepsilon}\left|\int_{|z|=\rho} \frac{f_N'(z)}{f_N(z)} dz\right| \leq \sum_{k=1}^{N} \frac{1}{\varepsilon}\left|\int_{|z|=\rho} \frac{b_k'(z)}{1 + b_k(z)} dz\right| \leq CN\rho m(\varepsilon).$$

and

$$\int_0^1 \log|f_N(\rho e^{i2\pi\theta})| d\theta = N \log\lambda(\varepsilon) + \sum_{k=1}^{N} \int_0^1 \log|1 + b_k(\rho e^{i2\pi\theta})| d\theta$$
$$\geq N\log\lambda(\varepsilon) - CNm(\varepsilon).$$

Pluging this into equation (10.8) and then into equation (10.6), one completes the proof of Proposition 10.1. □

11. A KAM theory construction

11.1. **The model equation.** Let $\phi \in \mathbb{C}$. Recall that a matrix function $M(\phi) \in GL(2, \mathbb{C})$ belongs to $\mathcal{M}$ if it is of the form (2.14)

We consider 1-periodic functions in $\mathcal{M}$ depending on a parameter η in $\mathcal{B}$, a Borel subset of $\mathbb{R}$. For functions $\mathcal{M}$ that are analytic in a strip $|\mathrm{Im}\,\phi| \leq r$ and that depend on $\eta \in \mathcal{B}$, we consider their Lipschitz norms

$$\|M\|_{r,\mathcal{B}} \equiv \sup_{|\mathrm{Im}\,\phi| \leq r, \eta \in \mathcal{B}} \|M(\phi, \eta)\| + \sup_{|\mathrm{Im}\,\phi| \leq r,\, \eta, \eta' \in \mathcal{B},\, \eta \neq \eta'} \frac{\|M(\phi, \eta) - M(\phi, \eta')\|}{|\eta - \eta'|}. \tag{11.1}$$

Here, $\|.\|$ is the matrix norm associated to the vector norm

$$\|v\| = |v_1| + |v_2|, \quad \text{for} \quad v = (v_1, v_2)^T \in \mathbb{C}^2.$$

If M does not depend on ϕ, we write $\|M\|_{\mathcal{B}}$ instead of $\|M\|_{r,\mathcal{B}}$.

Let I be a bounded real interval. Let D and A in $\mathcal{M}$ are two matrix-valued functions of $\phi \in \mathbb{C}$ such that

1. the matrices A and D depend on a parameter $\eta \in I$,

$$D = \operatorname{diag}(\exp(i\eta), \exp(-i\eta)), \quad A = A(\phi, \eta); \tag{11.2}$$

2. the matrix $A(\phi, \eta)$ is analytic in ϕ in a strip $|\operatorname{Im} \phi| \le r$ and $\|A\|_{r,I} \le 1$.

Let λ be parameter taking positive values, and let h be a fixed positive number. Consider the equation

$$\Psi(\phi + h) = (D + \lambda A)\Psi(\phi), \quad \phi \in \mathbb{R}. \tag{11.3}$$

Our aim is to study matrix solutions of equation (11.3) for small values of λ. Therefore, we use standard KAM theory ideas avoiding small neighborhoods of the KAM resonances (see [11, 2]). This allows us to construct solutions of (11.3) outside of some set of η of small measure.

As usual for KAM methods, we impose a Diophantine condition on the number h. We fix a $\sigma \in (0,1)$ and define

$$H_\sigma = \left\{ h \in (0,1);\ \min_{l \in \mathbb{N}} \left| h - \frac{l}{k} \right| \ge \frac{\lambda^\sigma}{k^3}, \quad k = 1, 2, 3 \dots \right\}. \tag{11.4}$$

We will assume that $h \in H_\sigma$.

Remark 11.1. Consider the complement of H_σ in $(0,1)$. For $\lambda < 1$, it is contained in the union of open intervals $I_{l,k}$,

$$\mathbb{R} \setminus H_\sigma \subset \bigcup_{k=1}^{\infty} \bigcup_{l=0}^{k} I_{k,l},$$

centered at the points l/k and of length $\frac{2\lambda^\sigma}{k^3}$. Hence, one gets

$$\operatorname{mes}\left((0,1) \setminus H_\sigma \right) \le \sum_{k=1}^{\infty} \frac{2\lambda^\sigma}{k^2} = \frac{\pi^2}{3} \lambda^\sigma. \tag{11.5}$$

In the sequel, C denotes a constant depending only on I, σ and r (but not on η, ϕ, λ or A). We prove

Proposition 11.1. *Let A and D be chosen as above; assume that* $\det(D + \lambda A) \equiv 1$. *Then, there exists $\lambda_0 = \lambda_0(r, \sigma, I)$ such that, for $0 < \lambda < \lambda_0$, there is a Borel set $\Phi_\infty \in I$, $\Phi_\infty(r, \sigma, I)$,*

$$\operatorname{mes} \Phi_\infty \le C\lambda^{\sigma/2}, \tag{11.6}$$

such that, for $\eta \in \mathcal{B}_\infty = I \setminus \Phi_\infty$, equation (11.3) has a solution of the form

$$\Psi(\phi, \eta) = U(\phi, \eta) \begin{pmatrix} e^{i\eta_\infty \cdot \phi/h} & 0 \\ 0 & e^{-i\eta_\infty \cdot \phi/h} \end{pmatrix},$$

where

- *U is defined and analytic for $|y| < r/2$ and satisfies $\|U - 1\|_{r/2, \mathcal{B}_\infty} \le C\lambda^{2-\sigma}$;*
- *η_∞ is a real valued Lipschitz function of η that satisfies $\|\eta_\infty(\eta) - \eta\|_{\mathcal{B}_\infty} \le 2\lambda$.*

11.2. **The proof of Proposition 11.1.** For a 1-periodic matrix $M(\phi) = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}$, we let $\operatorname{Diag} M = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0^* \end{pmatrix}$ where α_0 is the mean value of the function α. Let τ_h be the shift operator defined by $\tau_h f(\phi) = f(\phi + h)$.

11.2.1. *The induction procedure.* One proves Proposition 11.1 inductively. First, let us explain the heuristics guiding the KAM induction. The aim is to transform equation (11.3) with small matrix λA into an equation $\Psi_\infty(\phi+h) = D_\infty \Psi_\infty(\phi)$ with a constant diagonal matrix D_∞. Therefore, we search for a solution of (11.3) in the form $\Psi_0 = (I + U_1(\phi))\Psi_1$ where U_1 is 1-periodic and small. This leads to the equation

$$\Psi_1(\phi+h) = (I + \tau_h U_1(\phi))^{-1}(D + \lambda A)(I + U_1(\phi))\Psi_1(\phi).$$

As U_1 and λA are small, one has

$$(I + \tau_h U_1)^{-1}(D + A)(I + U_1) = D + (DU_1 - \tau_h U_1 D + A) + \dots,$$

where the dots denote lower order terms. So, if $DU_1 - \tau_h U_1 D + A = 0$, then (11.3) is replaced by a new equation of the same form, namely, $\Psi_1(\phi+h) = (D + A_1(\phi))\Psi_1(\phi)$. The new matrix A_1 is smaller then λA. We prove that one can find a periodic solution U of the equation $DU - \tau_h UD + A = 0$ only if $\operatorname{Diag} A = 0$. Let us turn to the induction.

First, we let

$$D_0 = D + \lambda \operatorname{Diag} A, \quad A_0 = \lambda A - \lambda \operatorname{Diag} A.$$

At the k-th step of the induction, we define two matrix-valued functions U_k and B_k so that

$$(I + \tau_h U_k(\phi))^{-1}(D_{k-1} + A_{k-1}(\phi))(I + U_k(\phi)) = D_{k-1} + B_k(\phi), \tag{11.7}$$

and set

$$D_k = D_{k-1} + \operatorname{Diag} B_k, \quad A_k(\phi) = B_k(\phi) - \operatorname{Diag} B_k. \tag{11.8}$$

As U_k we take a periodic solution of the *homological equation*

$$\tau_h U_k(\phi) D_{k-1} - D_{k-1} U_k(\phi) = A_{k-1}(\phi). \tag{11.9}$$

Then, by (11.7), one has

$$B_k(\phi) = (I + \tau_h U_k(\phi))^{-1} A_{k-1}(\phi)\, U_k(\phi). \tag{11.10}$$

We check that U_k and A_k quickly converge to zero and that the product

$$P(\phi) = \Pi_{k=1}^{\infty}(I + U_k(\phi)) = (I + U_1(\phi))\,(I + U_2(\phi))\,(I + U_3(\phi))\,\dots \tag{11.11}$$

converges to an invertible matrix. In result, we obtain

$$P^{-1}(\phi+h)(D + \lambda A(\phi))P(\phi) = D_\infty, \tag{11.12}$$

where D_∞ is a constant diagonal matrix.
This immediately implies that (11.3) has a solution of the form

$$\Psi(\phi) = P(\phi)\exp(\phi D_\infty / h). \tag{11.13}$$

We prove that this solution has all the properties announced in Proposition 11.1.

11.2.2. *Homological equation and small denominators.* Consider the homological equation (11.9). At each step of the induction procedure, A_k is a 1-periodic in ϕ. Therefore, we look for 1-periodic solutions of (11.9). This leads to a small denominator problem. We write the homological equation as

$$\tau_h U(\phi)\tilde D - \tilde D U(\phi) = \tilde A(\phi). \tag{11.14}$$

One proves

Lemma 11.1. *Let $\tilde A$ and $\tilde D$ belong to $\mathcal{M}$ such that*

- *$\tilde D$ be diagonal, $\tilde D = \operatorname{diag}(\tilde d, \tilde d^*)$ where d does not depend on ϕ;*
- $\operatorname{Diag} A \equiv 0$*;*
- *$\tilde A$ is analytic in a strip $|\operatorname{Im}\phi| \le \tilde r$;*
- *$\tilde A$ and $\tilde D$ be Lipschitz function of a parameter $\eta \in \mathcal{B}$ ($\mathcal{B}$, a Borel subset of $\mathbb{R}$) satisfying*

$$\|\tilde A\|_{\tilde r, \mathcal{B}} < \infty, \quad \|\tilde D\|_{\mathcal{B}} < \infty,$$

and $\inf_{\eta\in\mathcal{B}} |\tilde d| > 0$.

If the number h satisfy the Diophantine condition (11.4)*, and the parameter η is outside of the set*

$$\tilde{\Phi} = \bigcup_{k,l\in\mathbb{Z}} \left\{ \eta \in \mathcal{B} :\ |\alpha(\eta) - hk - l| < \frac{\lambda^{\sigma/2}}{\max^2\{1,k\}} \right\}, \quad \alpha(\eta) = \frac{1}{\pi} \arg d(\eta), \tag{11.15}$$

then, there exists $C_0 > 0$, a universal constant, and $U \in \mathcal{M}$, a solution of the homological equation (11.14) *analytic in the strip $|y| < \tilde{r}$ and satisfying the estimate*

$$\|U\|_{\tilde{r}-\rho,\mathcal{B}\setminus\tilde{\Phi}} \le C_0 \frac{\|\tilde{A}\|_{\tilde{r},\mathcal{B}} \|\tilde{D}\|_{\mathcal{B}}}{\lambda^\sigma\, \rho^5 \inf_{\eta\in\mathcal{B}} |\tilde{d}|^2} \left(\rho^2 + (1+\rho^5) \frac{\|\tilde{D}\|^2_{\mathcal{B}}}{\inf_{\eta\in\mathcal{B}} |\tilde{d}|^2} \right), \quad \forall \rho \in (0, \tilde{r}). \tag{11.16}$$

Proof. We are looking for $U = ((U_{ij}))_{1\le i,j\le 2} \in \mathcal{M}$. Let $u = U_{11}$ and $v = U_{12}$. Then, equation (11.14) is equivalent to the system of equations

$$\tau_h u - u = a/\tilde{d}, \tag{11.17}$$

$$e^{-2i\pi\alpha} \tau_h v - v = b/\tilde{d}, \tag{11.18}$$

where $a = A_{11}$, $b = A_{12}$ are the coefficients of $A = ((A_{ij}))_{1\le i,j\le 2}$. To solve the first equation, consider the Fourier series of a i.e. $a = \sum_{k\ne 0} a_k e^{2i\pi k\phi}$. The zeroth term vanishes as $\operatorname{Diag} A = 0$. The Lipschitz norm of a_k is estimated by

$$\|a_k\|_{\mathcal{B}} \le e^{-2\pi\tilde{r}|k|} \|\tilde{A}\|_{\tilde{r},\mathcal{B}}. \tag{11.19}$$

The 1-periodic solution of (11.17) is given by

$$u = \frac{1}{\tilde{d}} \sum_{k\ne 0} \frac{a_k e^{2\pi i k\phi}}{e^{2\pi i k h} - 1}.$$

This representation and (11.19) imply that

$$\|u\|_{\tilde{r}-\rho,\mathcal{B}} \le \left\| \frac{1}{\tilde{d}} \right\|_{\mathcal{B}} \|\tilde{A}\|_{\tilde{r},\mathcal{B}} \sum_{k\ne 0} \frac{e^{-2\pi\rho|k|}}{|e^{2\pi i k h} - 1|} \le \frac{\|\tilde{A}\|_{\tilde{r},\mathcal{B}} \|\tilde{D}\|_{\mathcal{B}}}{\inf_{\eta\in\mathcal{B}} |\tilde{d}|^2} \sum_{k\ne 0} \frac{e^{-2\pi\rho|k|}}{|e^{2\pi i k h} - 1|}. \tag{11.20}$$

As $\dfrac{1}{|e^{2\pi i k h} - 1|} \le C\lambda^{-\sigma} k^2$ and $\sum_{k\ne 0} k^2 e^{-2\pi\rho|k|} \le C/\rho^3$, $0 < \rho$, there exists $C > 0$, a universal constant, such that

$$\|u\|_{\tilde{r}-\rho,\mathcal{B}} \le C \frac{\|\tilde{A}\|_{\tilde{r},\mathcal{B}} \|\tilde{D}\|_{\mathcal{B}}}{\lambda^\sigma \rho^3 \inf_{\eta\in\mathcal{B}} |\tilde{d}|^2} \text{ where } 0 < \rho < \tilde{r}. \tag{11.21}$$

In terms of the Fourier coefficients of the function b, the 1-periodic solution of equation (11.18), can be written as

$$v = \frac{1}{\tilde{d}} \sum_{k\in\mathbb{Z}} \frac{b_k e^{2\pi i k\phi}}{e^{2\pi i (kh - \alpha(\eta))} - 1} \text{ where } \|b_k\|_{\mathcal{B}} \le e^{-2\pi\tilde{r}|k|} \|\tilde{A}\|_{\tilde{r},\mathcal{B}}.$$

The function v is estimated by

$$\|v\|_{\tilde{r}-\rho,\mathcal{B}\setminus\tilde{\Phi}} \le \left\| \frac{1}{\tilde{d}} \right\|_{\mathcal{B}} \|\tilde{A}\|_{\tilde{r},\mathcal{B}} \sum_{k\in\mathbb{Z}} e^{-2\pi\rho|k|} \left\| \frac{1}{e^{2\pi i (kh - \alpha(\eta))} - 1} \right\|_{\mathcal{B}\setminus\tilde{\Phi}}.$$

On the other hand, one has

$$\left\| \frac{1}{e^{2\pi i (kh - \alpha(\eta))} - 1} \right\|_{\mathcal{B}\setminus\tilde{\Phi}} \le \sup_{\eta\in\mathcal{B}\setminus\tilde{\Phi}} \left| \frac{1}{e^{2\pi i (kh - \alpha(\eta))} - 1} \right|^2 \|e^{2\pi i (hk - \alpha(\eta))} - 1\|_{\mathcal{B}},$$

and

$$\|e^{2\pi i (hk - \alpha(\eta))} - 1\|_{\mathcal{B}} = \|e^{2\pi i h k} \tilde{d}^*/\tilde{d} - 1\|_{\mathcal{B}} \le 1 + \|\tilde{d}^*/\tilde{d}\|_{\mathcal{B}} \le 1 + \frac{\|\tilde{D}\|^2_{\mathcal{B}}}{\inf_{\eta\in\mathcal{B}} |\tilde{d}|^2}.$$

By (11.15), one also has

$$\sup_{\eta\in\mathcal{B}\setminus\Phi}\left|\frac{1}{e^{2\pi i(kh-\alpha(\eta))}-1}\right|^2\le C\lambda^{-\sigma}\max\{1,k^4\},\quad\text{and}\quad\sum_{k>0}k^4e^{-2\pi\rho|k|}\le C/\rho^5,\ 0<\rho.$$

Finally, one obtains

$$\|v\|_{\tilde r-\rho,\mathcal{B}\setminus\Phi}\le C\frac{\|\tilde A\|_{\tilde r,\mathcal{B}}\|\tilde D\|_{\mathcal{B}}}{\lambda^\sigma\,\rho^5\,\inf_{\eta\in\mathcal{B}}|\tilde d|^2}\left(1+\frac{\|\tilde D\|^2_{\mathcal{B}}}{\inf_{\eta\in\mathcal{B}}|\tilde d|^2}\right)(1+\rho^5),\quad 0<\rho<\tilde r,\tag{11.22}$$

Estimates (11.21) and (11.22) imply (11.16) with a universal constant. This completes the proof of Lemma 11.1. □

11.2.3. *Induction procedure: estimates for A_k, U_k and D_k.* At each step of the induction procedure, we use Lemma 11.2 to construct U_k, a solution of the homological equation (11.9) for the matrices A_{k-1} and D_{k-1}. Then, (11.10) and (11.8) yield the matrices A_k and D_k. The price of the use of Lemma 11.2 is twofold.

- First, we can construct U_k only outside some set of values of η. Let $\mathcal{B}_{k-1}$ be the set used as $\mathcal{B}$ in Lemma 11.1. The set of "bad" values of η "thrown away" at the kth step is described by (11.15) with $\tilde d=(D_{k-1})_{11}$. Denote this set by Φ_k. Then, one has

$$\Phi_k=\bigcup_{l,m\in\mathbb{Z}}\left\{\eta\in\mathcal{B}_{k-1}:\ |\alpha_{k-1}(\eta)-hl-m|<\frac{\lambda^{\sigma/2}}{\max(l^2,1)}\right\},\quad \mathcal{B}_{k-1}=I\setminus\bigcup_{j=1}^{k-1}\Phi_j,\quad \mathcal{B}_0=I,\tag{11.23}$$

$$\text{where}\quad \alpha_k(\eta)=\frac{1}{\pi}\arg d_k(\eta).\tag{11.24}$$

  Note that, for all $k\ge0$, one has $\mathcal{B}_{k+1}\subset\mathcal{B}_k$.
- Second, by formula (11.16) we can control only $\|U\|_{\tilde r-\rho,\mathcal{B}\setminus\tilde\Phi}$, but not $\|U\|_{\tilde r,\mathcal{B}\setminus\tilde\Phi}$. So, if A_{k-1} was constructed in a strip $|\mathrm{Im}\,\phi|\le r_{k-1}$, then to get a good enough norm control, we have to consider U_k in a smaller strip $|\mathrm{Im}\,\phi|\le r_k,\ 0<r_k<r_{k-1}$. We choose

$$r_k=r-\sum_{l=1}^k\rho_l,\quad r_0=r,\quad \rho_l=\frac{r}{2^{l+1}}.$$

  Note that, one has

$$r_\infty\equiv r-\sum_{l=1}^\infty\rho_l=r/2.\tag{11.25}$$

We prove

Lemma 11.2. *Let $k\ge0$. Assume that $\mathcal{B}_{k+1}$ is nonempty. Let C_0 be the constant obtained in Lemma 11.1 and define*

$$S(k)=\sum_{l=0}^k\frac{2+l}{2^l},\quad S=\lim_{k\to\infty}S_k,\quad Q=2^5,\quad C(r)=24C_0(10+r^2+r^5)/r^5.$$

Assume

$$\lambda<1/4\ \textit{and}\ \lambda^{1-\sigma}\,2C(r)Q^S<1/2\tag{11.26}$$

Then, for $k\ge0$, one has

$$\|B_{k+1}\|_{r_{k+1},\mathcal{B}_{k+1}}\le\frac{1}{2C(r)}\lambda^\sigma Q^{2^kS(k)}\,F(k+1),\tag{11.27}$$

$$\|U_{k+1}\|_{r_{k+1},\mathcal{B}_{k+1}}\le\frac14 Q^{k/2+1+2^{k-1}S(k)}\,F(k).\tag{11.28}$$

where $F(k)=\left(2C(r)\lambda^{1-\sigma}\right)^{2^k}$.

Proof. Equations (11.27) and (11.28) are proved inductively. To simplify the notations, we write $\|A_k\|$, $\|D_k\|$, $\|U_k\|$ and $\|B_k\|$ instead of $\|A_k\|_{\mathcal{B}_k,r_k}$, $\|D_k\|_{\mathcal{B}_k}$, $\|U_k\|_{\mathcal{B}_k,r_k}$ and $\|B_k\|_{\mathcal{B}_k,r_k}$. Let us check (11.27) – (11.28) for $k=1$. One has

$$\|D_0\| \leq 1+\lambda < 5/4, \quad \inf_{\eta\in\mathcal{B}_0} |d_0| \geq 1-\lambda > 3/4, \quad \|A_0\| \leq 2\lambda.$$

Using this and Lemma 11.1 with $\tilde{A}=A_0$, $\tilde{D}=D_0$ and $\rho=\rho_1$, one gets a very rough estimate

(11.29)

$$\|U_1\| < \frac{20}{9} C_0 \frac{\|A_0\|}{\lambda^\sigma} \frac{r^5}{\rho_1^5} \left[\frac{\rho_1^2+\left(1+\frac{25}{9}\right)\left(1+\rho_1^5\right)}{r^5}\right] < \frac{20}{9} C_0 \|A_0\| \lambda^{-\sigma} Q^2 \left[\frac{r^2+10+r^5}{r^5}\right] < \frac{1}{2} Q^2 C(r) \lambda^{1-\sigma}.$$

So (11.28) is proved for $k=0$.

To estimate B_1, we compare (11.29) with the second condition on λ; this implies that $\|U_1\| < 1/8$ (since $Q^2 < Q^S$). Therefore, (11.10) implies that $\|B_1\| < \frac{8}{7}\|U_1\|\,\|A_0\|$. And using (11.29) and $\|A_0\| \leq 2\lambda$, we see that

$$\|B_1\| < 2C(r) Q^2 \lambda^{2-\sigma}.$$

This proves (11.27) for $k=0$.

Now, assume that (11.27) and (11.28) are valid for $1 \leq l \leq k$. We check that (11.27) and (11.28) then hold for $k+1$. Begin with checking that

$$\|D_k\| \leq 3/2, \quad \inf_{\eta\in\mathcal{B}_k} |d_k| \geq 1/2. \tag{11.30}$$

By (11.8), one has

$$\|D_k\| \leq \|D_k - D_{k-1}\| + \|D_{k-1} - D_{k-2}\| + \cdots + \|D_1 - D_0\| + \|D_0\| \leq \sum_{j=0}^{k-1} \|B_{j+1}\| + \|D_0\|.$$

Using (11.27) to estimate this last sum, we obtain

$$\sum_{j=0}^{k-1} \|B_{j+1}\| \leq \frac{1}{2C(r)} \lambda^\sigma \sum_{j=0}^{k-1} Q^{2^j S(j)} F(j+1). \tag{11.31}$$

By (11.26), the terms in the right hand sum above are super exponentially decaying. So, it is roughly of order of the first term, and we need only a very rough estimate:

$$Q^{2^j S(j)} F(j+1) \leq \frac{\left(2C(r)\lambda^{1-\sigma}Q^S\right)^{2^{j+1}}}{Q^{2^j S}} \leq \frac{\left(2C(r)\lambda^{1-\sigma}Q^S\right)^{2(j+1)}}{Q^S} \leq \frac{\left(2C(r)\lambda^{1-\sigma}Q^S\right)^2}{Q^S} \cdot \frac{1}{2^{2j}},$$

where at the last step, we have used (11.26). This and (11.31) imply that

$$\sum_{j=0}^{k-1} \|B_{j+1}\| < \frac{1}{C(r)Q^S} \lambda^\sigma \left(2Q^S C(r) \lambda^{1-\sigma}\right)^2 = 4Q^S C(r) \lambda^{2-\sigma}.$$

Since $D_0 \leq 1+\lambda$, we see that $\|D_k\| \leq 1+\lambda(1+4Q^S C(r)\lambda^{1-\sigma})$. This with (11.26) implies that $\|D_k\|$ satisfies (11.30).

The estimate for d_k is obtained in a similar way; one has

$$\begin{aligned} \|d_k - e^{i\eta}\| &\leq \|d_k - d_{k-1}\| + \cdots + \|d_1 - d_0\| + \|d_0 - e^{i\eta}\| \\ &\leq \|D_k - D_{k-1}\| + \cdots + \|D_1 - D_0\| + \|d_0 - e^{i\eta}\| \\ &\leq \sum_{j=0}^{k-1} \|B_{j+1}\| + \lambda < \lambda(1+4Q^S C(r)\lambda^{1-\sigma}). \end{aligned} \tag{11.32}$$

As $2Q^S C(r)\lambda^{1-\sigma} < 1/2$, $|d_k|$ satisfies (11.30).

Lemma 11.1 with $\tilde{D}=D_k$, $\tilde{A}=A_k$ and $\rho=\rho_k$, and the estimates (11.30) yield

$$\|U_{k+1}\| \leq \frac{6C_0(10+r^2+r^5)}{r^5}\,\lambda^{-\sigma} \frac{r^5}{\rho_{k+1}^5} \|A_k\| = \frac{1}{4} C(r) Q^{k+2} \|A_k\|\,\lambda^{-\sigma}.$$

By (11.8), one has $\|A_k\| \leq 2\|B_k\|$. Using (11.27) for $\|B_k\|$, we get

$$\|U_{k+1}\| \leq \frac{1}{4} Q^{k+2+2^{k-1}S(k-1)} F(k) = \frac{1}{4} Q^{k/2+1+2^{k-1}S(k)}\, F(k).$$

So, $\|U_{k+1}\|$ satisfies (11.28).
To prove that $\|B_{k+1}\|$ satisfies (11.27), we first get a very rough estimate $\|U_{k+1}\| \leq \frac{1}{2}$. It follows from

$$\|U_{k+1}\| \leq \frac{1}{4} Q^{k/2+1+2^{k-1}S(k)} F(k) = \frac{1}{4} Q^{2^{k-1}\left(\frac{k+2}{2^k}+S(k)\right)} F(k) \leq \frac{1}{4} Q^{2^k S} F(k) = \frac{1}{4}\left(2C(r)Q^S\lambda^{1-\sigma}\right)^{2^k}$$

and (11.26). The rough estimate and (11.10) give $\|B_{k+1}\| \leq 2\|A_k\|\|U_{k+1}\| \leq 4\|B_k\|\|U_{k+1}\|$. Using (11.27) for $\|B_k\|$ and (11.28) for U_{k+1}, we get

$$\|B_{k+1}\| \leq \frac{1}{2C(r)} \lambda^\sigma Q^{2^{k-1}S(k-1)+k/2+1+2^{k-1}S(k)}\, F(k+1).$$

Using the equality $2^{k-1}S(k-1) + k/2 + 1 + 2^{k-1}S(k) = 2^k S(k)$, we see that B_{k+1} satisfies (11.27). This completes the proof of Lemma 11.2. □

To complete the proof of Proposition 11.1, we also need the following rough estimates

Corollary 11.1. *Assume λ satisfies (11.26). Then, there exists $C > 0$ depending only on r such that, for $k \geq 0$, one has*

$$\|d_{k+1} - d_k\|_{\mathcal{B}_{k+1}} \leq C\lambda^\sigma (C\lambda^{1-\sigma})^{k+1}, \tag{11.33}$$

$$\|d_k - e^{i\eta}\|_{\mathcal{B}_k} \leq 2\lambda. \tag{11.34}$$

Proof. Estimate (11.33) follows immediately from (11.8).
To prove (11.34), we note that by (11.32)

$$\|d_k - e^{i\eta}\|_{\mathcal{B}_k} \leq \lambda(1 + 4Q^S C(r)\lambda^{1-\sigma}) \tag{11.35}$$

By the second estimate from (11.26), this implies that $\|d_k - e^{i\eta}\|_{\mathcal{B}_k} \leq 2\lambda$. This completes the proof of Corollary 11.1. □

11.2.4. *Construction of the solutions.* Let us for a while assume that the set $\mathcal{B}_\infty = \cap_k \mathcal{B}_k \neq \emptyset$; we prove it later. Then, as the sequences $(\mathcal{B}_k)_k$ and $(r_k)_k$ are decreasing, for $\eta \in \mathcal{B}_\infty$ and for all $k \geq 1$, B_k and U_k are analytic in the band $|\mathrm{Im}\,\phi| \leq r_\infty$, and one has

$$\|B_{k+1}\|_{r_\infty,\mathcal{B}_\infty} \leq \|B_{k+1}\|_{r_{k+1},\mathcal{B}_{k+1}} \leq \frac{1}{2C(r)}\lambda^\sigma \left(2C(r)Q^S\lambda^{1-\sigma}\right)^{2^{k+1}},$$

$$\|U_{k+1}\|_{r_\infty,\mathcal{B}_\infty} \leq \|U_{k+1}\|_{r_{k+1},\mathcal{B}_{k+1}} \leq \left(2C(r)Q^S\lambda^{1-\sigma}\right)^{2^k}.$$

Hence, if $\lambda < \min\{1/4, 2C(r)Q^S\lambda^{1-\sigma}\}$, the product $P(\phi)$ defined by (11.11) converges to a 1-periodic, $GL(2,\mathbb{C})$-valued function that is analytic in the strip $|\mathrm{Im}\,\phi| \leq r_\infty$. Moreover, one has $\|I - P(\phi)\|_{r_\infty,\mathcal{B}_\infty} \leq C\lambda^{1-\sigma}$.

As $\|A_k\|_{r_\infty,\mathcal{B}_\infty} \leq 2\|B_k\|_{r_\infty,\mathcal{B}_\infty} \to 0$, we get (11.12) where the constant diagonal matrix D_∞, satisfies the estimate $\|D_\infty\|_{\mathcal{B}} < \infty$. By construction, all the matrices U_k belong to $\mathcal{M}$; thus, (11.12) implies that D_∞ belongs to $\mathcal{M}$. Moreover, as we assumed $\det(D + \lambda A) \equiv 1$, (11.12) gives

$$\ln|\det D_\infty| = \int_0^1 d\phi(\ln|\det P(\phi)| - \ln|\det P(\phi+h)|) = 0.$$

These two observations imply that D_∞ has the form described in Proposition 11.1 with $\eta_\infty \in \mathbb{R}$. Letting in (11.34) $k \to +\infty$, we see that $\|d_\infty - d\|_{\mathcal{B}_\infty} \leq 2\lambda$. Hence, η_∞ can be defined as a Lipschitz function of η, and $\|\eta_\infty(\eta) - \eta\| \leq C\lambda$. So, we have proved that (11.13) is a solution with all the properties described in Proposition 11.1. To complete the proof of Proposition 11.1, we are left with estimating the measure of $\mathcal{B}_\infty$.

11.2.5. *The set $\mathcal{B}_\infty$.* To study the set $\mathcal{B}_\infty$, we study its complement and the sets Φ_k defined in (11.23); therefore, we first study the functions $\eta \mapsto \alpha_k(\eta)$ defined by (11.24).
Since d_k is continuous on $\mathcal{B}_k$, and, by (11.34), has no zeros, the function $\eta \mapsto \alpha_k(\eta)$ can be taken continuous. By Corollary 11.1, the function d_k is close to $e^{i\eta}$ for λ small; hence, the function α_k can be taken continuous with values close to η/π. More precisely, we get

Lemma 11.3. *There exists $C > 0$ (depending on r, σ, I) such that, for $0 < \lambda \le C$, one has*

$$\|\alpha_{k+1} - \alpha_k\|_{\mathcal{B}_{k+1}} \le \lambda^\sigma (C\lambda^{1-\sigma})^{k+1}, \quad k \ge 0, \tag{11.36}$$

$$\|\alpha_k - \eta/\pi\|_{\mathcal{B}_k} \le C\lambda, \quad k \ge 0. \tag{11.37}$$

Proof. We write $\|.\|$ instead of $\|.\|_{\mathcal{B}_{k+1}}$. Note that

$$\alpha_{k+1} - \alpha_k = \frac{1}{\pi}\arg(1 + w_k), \quad w_k = \frac{d_{k+1} - d_k}{d_k}. \tag{11.38}$$

By (11.30) and (11.33), for sufficiently small λ, we get that

$$|w_k(\eta)| \le 1/2, \quad \forall \eta \in \mathcal{B}_{k+1}.$$

This and (11.38) gives $\|\alpha_{k+1} - \alpha_k\| \le C\|w_k\|$ for some $C > 0$. Using (11.34) and (11.38), we get $\|\alpha_{k+1} - \alpha_k\| \le C\|d_{k+1} - d_k\|$. The bound (11.33) implies (11.36). Estimate (11.37) follows from (11.36). Indeed,

$$\begin{aligned} \|\alpha_k - \eta/\pi\| &\le \|\alpha_k - \alpha_{k-1}\| + \dots + \|\alpha_1 - \alpha_0\| + \|\alpha_0 - \eta/\pi\| \\ &\le \lambda^\sigma (C\lambda^{1-\sigma})^k + \dots + \lambda^\sigma (C\lambda^{1-\sigma}) + C\lambda \le C\lambda. \end{aligned} \tag{11.39}$$

This completes the proof of Lemma 11.3. □

The function α_k is defined only on $\mathcal{B}_k$. It is more convenient to work with functions defined on $I = \mathcal{B}_0$. Therefore, we extend α_k to I in the following inductive way. For α_0, we have nothing to do. Suppose that, for $0 \le l \le k$, α_l have been extended to I. Consider the function α_{k+1}. We have to define it on $I \setminus \mathcal{B}_{k+1}$ which is a countable set of open intervals. On each of these intervals, we define α_{k+1} so that the function $\alpha_{k+1} - \alpha_k$ be linear, and the function α_{k+1} remain continuous.
To keep notations short, the extended functions are still called α_k. Clearly, the continuation procedure does not change neither the Lipschitz norms of the differences $\alpha_{k+1} - \alpha_k$, i.e. $\|\alpha_{k+1} - \alpha_k\|_{\mathcal{B}_{k+1}} = \|\alpha_{k+1} - \alpha_k\|_{\mathcal{B}_0}$, nor the sets Φ_k (see (11.23)). Moreover, one checks that estimate (11.37) remains valid with $\|\,.\,\|_{\mathcal{B}_{k+1}}$ replaced by $\|\,.\,\|_{\mathcal{B}_0}$. This follows as we can change in (11.39) the sets $\mathcal{B}_{k+1}, \mathcal{B}_k, \dots, \mathcal{B}_1$ by $\mathcal{B}_0 = I$.

The idea of the estimate of Φ_∞ is following. As α_0 is close to η/π, the measure of Φ_0 is small. As long as α_k is close to α_0, Φ_k almost coincides with Φ_0. When these sets are quite different, the measure of Φ_k becomes very small. We prove

Lemma 11.4. *Let f_i, $i = 1, 2$, be two Lipschitz functions defined on an interval I such that, for some positive δ and δ_1, $0 < \delta_1 < 1$,*

$$|f_1(\eta) - f_2(\eta)| \le \delta \quad \forall \eta \in I, \quad \text{and} \quad \|f_{1,2} - Id\|_I \le \delta_1 < 1. \tag{11.40}$$

Fix $c \in I$. Let $\Delta > 0$. Define

$$F_{1,2} = \{\eta \in I : \ |f_{1,2}(\eta) - c| \le \Delta\}, \quad \text{and} \quad \delta_0 = 2\Delta\frac{1 - \delta_1}{1 + \delta_1}.$$

Then, one has $\operatorname{mes}(F_{1,2}) \le 2\Delta/(1 - \delta_1)$. *Moreover, if $\delta < \delta_0$, then* $\operatorname{mes}(F_2 \setminus F_1) \le 2\delta/(1 - \delta_1)$, *and if $\delta \ge \delta_0$, then* $\operatorname{mes}(F_2 \setminus F_1) \le 2\Delta/(1 - \delta_1)$.

Proof. Let us continue f_1 and f_2 continuously outside I so that they preserve their properties (11.40). It suffices to prove the statements for these "new" f_1 and f_2, and $I = \mathbb{R}$. First, we note that

$$(\eta - \eta')(1 - \delta_1) \le f_j(\eta) - f_j(\eta') \le (\eta - \eta')(1 + \delta_1), \quad j = 1, 2.$$

So, f_j are monotonous and F_j are intervals. Denote the left (resp. right) end of these intervals by $\eta_{1,2}^-$ (resp. $\eta_{1,2}^+$). Obviously, one has

$$\frac{2\Delta}{1 + \delta_1} \le \eta_{1,2}^+ - \eta_{1,2}^- \le \frac{2\Delta}{1 - \delta_1}, \quad f_1(\eta_1^\pm) = f_2(\eta_2^\pm) = c \pm \Delta.$$

So, one has

$$|\eta_1^\pm - \eta_2^\pm| \le \frac{|f_1(\eta_1^\pm) - f_1(\eta_2^\pm)|}{1-\delta_1} = \frac{|f_2(\eta_2^\pm) - f_1(\eta_2^\pm)|}{1-\delta_1} \le \frac{\delta}{1-\delta_1}.$$

Now, if $\delta < \delta_0$ then $2\Delta/(1+\delta_1) > \delta/(1-\delta_1)$; hence, the intersection of F_1 and F_2 is not empty, and $\text{mes}\,(F_2 \setminus F_1)$ can be estimated by the sum $|\eta_1^+ - \eta_2^+| + |\eta_1^- - \eta_2^-|$, that is, by $2\delta/(1-\delta_1)$. If $\delta \ge \delta_0$, we use the trivial estimate $\text{mes}\,(F_2 \setminus F_1) \le \text{mes}\,(F_2) \le 2\Delta/(1-\delta_1)$. This completes the proof of Lemma 11.4. □

Now, let us discuss the sets Φ_{k+1}. Each of these sets is contained in a countable union of open intervals $I_{l,m}(k)$, $l, m \in \mathbb{Z}$ defined by

$$I_{l,m}(k) = \left\{ \eta \in I:\ |\alpha_k(\eta) - hl - m| \le \frac{\lambda^{\sigma/2}}{\max(1, l^2)} \right\}.$$

In view of (11.37), if λ is sufficiently small, the length of the interval $I_{l,m}(k)$ can be bounded by $C\lambda^{\sigma/2} l^{-2}$ uniformly in m and k. Fix k and l. On any $I_{l,m}(k)$, one has $-1 - lh + \alpha_k \le m \le 1 - lh + \alpha_k$; so, the number of $I_{l,m}(k)$ that are not empty does not exceed $M = \max_{\eta \in \mathcal{B}_0} \alpha_k - \min_{\eta \in \mathcal{B}_0} \alpha_k + 3$. In view of (11.37), we see that M is bounded uniformly in k and l. Note that this implies in particular that, for some $C > 0$, one has

$$\text{mes}\,(\Phi_1) \le C\lambda^{\sigma/2}.$$

We can represent the set Φ_∞ as

$$\Phi_\infty = \Phi_1 + \bigcup_{k \ge 1} (\Phi_{k+1} \setminus \Phi_k).$$

Let us estimate the $\text{mes}\,(\Phi_{k+1} \setminus \Phi_k)$. One has

$$\text{mes}\,(\Phi_{k+1} \setminus \Phi_k) \le \sum_{l \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \text{mes}\ (I_{l,m}(k+1) \setminus I_{l,m}(k)).$$

To estimate $\text{mes}\ (I_{l,m}(k+1) \setminus I_{l,m}(k))$, we apply Lemma 11.4 with $f_1 = \alpha_k$ and $f_2 = \alpha_{k+1}$. In view of (11.36)–(11.37), for these functions, $\delta = \delta(k) = \lambda^\sigma (C\lambda^{1-\sigma})^{k+1}$, $\delta_1 = C\lambda$. We choose $\Delta = \lambda^{\sigma/2}/l^2$. Then $\delta_0(l) = 2\frac{1-C\lambda}{1+C\lambda}\frac{\lambda^{\sigma/2}}{l^2}$. For sufficiently small λ, one has

- $\text{mes}\ (I_{l,m}(k+1) \setminus I_{l,m}(k)) \le C\lambda^\sigma (C\lambda^{1-\sigma})^{k+1}$ if $\lambda^\sigma (C\lambda^{1-\sigma})^{k+1} < C\lambda^{\sigma/2}/l^2$, i.e. if $l \le L$, $L = C\lambda^{-\sigma/4}(C\lambda^{1-\sigma})^{-(k+1)/2}$,
- $\text{mes}\ (I_{l,m}(k+1) \setminus I_{l,m}(k)) \le 2\lambda^{\sigma/2} l^{-2}$ otherwise.

This implies

$$\text{mes}\,(\Phi_{k+1} \setminus \Phi_k) \le CL\lambda^\sigma (C\lambda^{1-\sigma})^{k+1} + \frac{C}{L}\lambda^{\sigma/2} \le C\lambda^{3\sigma/4}(C\lambda^{1-\sigma})^{(k+1)/2}.$$

Summing the exponentially convergent series in k, we see that Φ_∞ satisfies (11.6) and, hence, that $\mathcal{B}_\infty$ is not empty. This completes the proof of Proposition 11.1. □

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