A Hamiltonian model for linear friction in a homogeneous medium

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Abstract

We introduce and study rigorously a Hamiltonian model of a classical particle moving through a homogeneous dissipative medium at zero temperature in such a way that it experiences an effective *linear* friction force proportional to its velocity (at small speeds). The medium consists at each point in space of a vibration field modelling an obstacle with which the particle exchanges energy and momentum in such a way that total energy and momentum are conserved. We show that in the presence of a constant (not too large) external force, the particle reaches an asymptotic velocity proportional to this force. In a potential well, on the other hand, the particle comes exponentially fast to rest in the bottom of the well. The exponential rate is in both cases an explicit function of the model parameters and independent of the potential. We furthermore analyse in some detail the relation of our model to other models for dissipation that have been studied in the literature and derive in a simple manner a necessary condition for linear friction that has a straightforward physical interpretation.

1 Introduction

Many simple microscopic or macroscopic systems obey (at zero temperature) an effective equation of motion of the type

$$m\ddot{q}(t) + \tilde{\gamma}\dot{q}(t) = -\nabla V(q(t)), \qquad \tilde{\gamma} > 0. \tag{1.1}$$

Examples include the motion of electrons in a metal, or of a small particle in a viscous medium, but the coordinate q need not always be of geometrical nature (see [CL]). The energy loss due to the linear friction force $-\tilde{\gamma}\dot{q}$ (occurring at a rate $-\tilde{\gamma}\dot{q}^2$) implied by this equation leads to several well-known phenomena. First, for confining potentials V, the particle will come to a stop exponentially fast (with rate $\frac{\tilde{\gamma}}{2}$ if $\tilde{\gamma}$ is small enough) at one of the minima of the potential. If, on the other hand, $V(q) = -F\cdot q$, for some $F \in \mathbb{R}^d$, the particle will reach a limiting speed $v(F) = \frac{F}{\tilde{\gamma}}$ which is proportional to the applied field. This, in particular, is at the origin of Ohm's law. Again, the approach is exponential, but this time with rate $\tilde{\gamma}$. In particular, if $F = 0$, the particle comes exponentially fast to a full stop. The phenomenological friction force summarizes the reaction of the environment of the particle to its passage and the energy lost by the particle is transferred to the medium surrounding the particle by various processes (such as inelastic collisions, for example). A more fundamental, microscopic, treatment of the phenomena requires therefore considering the combined system consisting of the particle and the medium. This combined system should allow for a Hamiltonian treatment in which the total energy is conserved.

Our goal in this paper is to present and study a Hamiltonian model of a system composed of a particle and a homogeneous medium. We show rigorously that it has the behaviour described above and analyse the physical mechanisms at the origin of the observed phenomena. We stay within the context of classical mechanics and at zero temperature, hoping to come back to other points in the $\hbar - T$ plane at a later date. In particular, at positive temperature, a fluctuating force term is to be added to (1.1), transforming the equation to the Langevin equation. Such a term is indeed produced by our model, but is harder to analyse at positive temperatures.

The model we consider consists of one extended classical particle that is, on the one hand, coupled to "obstacles" represented by scalar vibration fields and on the other subjected to a time-independent external force $F = -\nabla V$. We are mostly interested in the more difficult case where F is constant (so $V = -F \cdot q$), but we will also deal with confining potentials.

More precisely, the equations of motion for the coupled system are:

$$\partial_t^2 \phi(x, y, t) - c^2 \Delta_y \phi(x, y, t) = -\rho(x - q(t), y) \tag{1.2}$$

$$\ddot{q}(t) = -\nabla V(q(t)) - \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} dy\ \rho(x - q(t), y)(\nabla_x \phi)(x, y, t). \tag{1.3}$$

Here ϕ is the vibration field and $q \in \mathbb{R}^d$ the position of the particle. The "form factor" $\rho(x, y)$ describes the extension of the particle and determines its coupling to the vibration field ϕ. We shall assume:

(H1) $\rho(x, y) = \rho_1(x)\rho_2(y) \not\equiv 0 \in C_0^\infty(\mathbb{R}^{d+n})$; $\rho_1, \rho_2 \geq 0$ and
$\rho_i(x) = \tilde{\rho}_i(|x|), i = 1, 2$ with $\rho_i(x) = 0$ if $|x| \geq R_i > 0$ $i = 1, 2$.

Note that the field ϕ plays the role of a potential for the particle. Indeed, the second term in (1.3) – which is the force $f(q)$ exerted by the environment on the particle – becomes $f(q) = -\nabla \phi(q, 0)$ if we consider a point particle so that $\rho(x, y) = \delta(x)\delta(y)$. To understand the model intuitively the following observations are helpful. First of all, the particle moves in x-space (or, more precisely, in the $y = 0$ subspace of $\mathbb{R}^{d+n}$). In fact, one can think of $\phi(x, \cdot)$ as representing, for each fixed value of x in the configuration space of the particle, an "obstacle", which has a large number of degrees of freedom, and is therefore modeled for simplicity by a vibration field $\phi(x, \cdot)$. The variables y should in other words not be interpreted geometrically and are in particular not spatial variables for the particle. To understand this, it is helpful to Fourier transform (1.2)-(1.3) in the y variable to obtain

$$\partial_t^2 \hat{\phi}(x, k, t) + c^2 |k|^2 \hat{\phi}(x, k, t) = -\rho_1(x - q(t))\hat{\rho}_2(k) \tag{1.4}$$

$$\ddot{q}(t) = -\nabla V(q(t)) + \int_{\mathbb{R}^n} dk f(q(t), k, t), \tag{1.5}$$

where

$$f(q,k,t) = -\int_{\mathbb{R}^d} dx \rho_1(x-q(t))\hat{\rho}_2(k)\nabla_x\hat{\phi}(x,k,t). \tag{1.6}$$

Clearly, $\hat{\phi}(x,k,t)$ is, for each value of x and k the amplitude of a driven oscillator of frequency $\omega(k) = c|k|$. All of these oscillators are decoupled and each of them contributes separately a force $-\rho_1(x-q(t))\hat{\rho}_2(k)\nabla_x\hat{\phi}(x,k,t)$ acting on the particle.

In (1.2)-(1.3) there is an "obstacle" at each point in x space. One could render the model more sophisticated by introducing a distribution $m(x) \geq 0$ of obstacles to obtain:

$$m(x)\left(\partial_t^2\phi(x,y,t) - c^2\Delta_y\right)\phi(x,y,t) = -m(x)\rho(x-q(t),y) \tag{1.7}$$

$$\ddot{q}(t) = -\nabla V(q(t)) - \int_{\mathbb{R}^d} m(x)dx \int_{\mathbb{R}^n} dy \rho(x-q(t),y)\nabla_x\phi(x,y,t). \tag{1.8}$$

This would allow to consider, for example, the case of an array of inelastic obstacles positioned at $x_i \in \mathbb{R}^d$, by putting

$$m(x) = \sum_i \delta(x - x_i)$$

(an inelastic Lorentz gas or pinball machine, if you wish), but would be considerably more difficult to treat mathematically. We shall only deal with the situation of (1.2)-(1.3) where $m(x) = 1$. One can also couple the obstacles at different x: we will make a brief comment on this in Section 6.

One way to get an intuitive understanding of why this model should exhibit dissipative behaviour is to imagine for a moment the particle is constrained to move in one dimension ($x \in \mathbb{R}$), and $y \in \mathbb{R}^2$, so that one can picture $\phi(x,y)$ as describing the vibrations of a membrane positioned at x, perpendicular to the axis on which the particle moves. As the particle hits the successive membranes, it creates a wake, much like a boat ploughing the surface of a lake (Figure 1). Taking for the moment $V(q) = 0$ in (1.3) (so that there is no external field: $F = 0$) one can imagine launching the particle with an initial speed v_0, with all membranes initially at rest. In that case the intuition predicts that the particle should loose all its kinetic energy into the membranes and come to a full stop. We shall prove that this intuition is correct and that the particle stops exponentially fast for arbitrary values of d, but with $n = 3$ (Theorem 2) and for c large enough. The physical origin of this restriction to the case $n = 3$ and c large is explained in Sections 2 and 6.

Another situation of interest is the case where V is confining. Then techniques similar to the ones used in [KKS1] allow to show the particle comes to rest on one of the equilibrium positions of the potential V. We furthermore show this approach is exponential with the expected rate (Theorem 3) provided the particle comes to rest on a non-degenerate minimum of the potential.

Our main interest is in the case where $V(q) = -F \cdot q$. In that case, we show that for a suitable class of initial conditions (and for c large enough) the particle approaches asymptotically a constant speed $v(F)$ (Theorem 2) which is linear in F for small F, as in an ohmic medium.

The model proposed above may at first sight look rather special, but we will show in Section 6 that it isn't. Indeed, a large class of models purporting

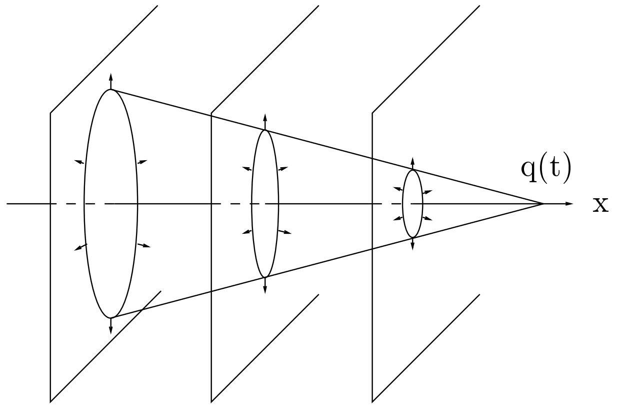

Figure 1: Waves created by the passage of the particle through the successive membranes.

to provide a Hamiltonian treatment of dissipation (through linear friction or otherwise) was proposed in [CL]. These models involve the coupling of the particle to an oscillator bath which, under suitable conditions on the frequency spectrum of the bath and on the coupling constants is argued (but not proven) to lead to a linear friction term of the type $-\tilde{\gamma}\dot{q}$ and therefore to the behaviour recalled above. However, among the models of this large class that have actually been studied in the literature, none describes a homogeneous medium, as we explain in Section 6.

After introducing a suitably generalized version of the class presented in [CL], we derive conditions on the model parameters for it to describe a homogeneous medium, and present a large class of such models, to which ours belongs. We then propose a mathematically simple and physically transparent derivation of necessary conditions on the parameters of this class for the models to describe linear friction (at low speeds) in a homogeneous medium. We show it is equivalent to a weak form of the Caldeira-Leggett conditions and of course, that it is satisfied in our model. We will finally argue that our model has some additional structure that may be indispensable for the above phenomena to hold and that are at any rate crucial to our proofs.

Not all dissipation is due to linear friction. Another important source of dissipation is radiation damping. We therefore briefly discuss the differences between our work on dissipation through linear friction and related recent work on dissipation through radiation damping [KS] [KKS1].

The rest of the paper is organized as follows. In Section 2 we study in detail the friction force exercised by the medium on the particle. This allows us to discuss in some detail the intuition behind the model. The rather boring but essential question of existence and uniqueness of the solutions of (1.2)-(1.3) is settled in Section 3. In Section 4 we study the long time asymptotics of the particle behaviour for the case when $V(q) = -F \cdot q$, whereas Section 5 is devoted to the confined case.

2 The friction force

Crucial for understanding the model and for the proofs of our results is a detailed study of the force $f(q)$. It is in particular instructive to study this reaction force of the medium when it is supposed that the particle executes a uniform rectilinear motion with some fixed velocity $v \in \mathbb{R}^d$. It is easy to see that the model admits solutions $(q(t) = q_0 + vt, \phi(t), p(t) = v, \pi(t))$ in which the particle moves at constant speed v provided the right time and space independent external force $F(v) = -\nabla V$ is applied. They are found as follows. We suppose the particle moves at constant velocity v (*i.e.* $q(t) = q_0 + vt$) and compute the force $f(v, q_0, t)$ resulting from the back-reaction of the field on the particle. It will turn out that $f(v, q_0, t) = f(v)$ does not depend on q_0 or t so that we can conclude by putting $F = -f(v)$. To see this, we consider the following particular solution $\phi(x, y, t)$ of (1.4) with $q(t) = q_0 + vt$:

$$\hat{\phi}(x, k, t) = -\int_{-\infty}^{t} ds\, \rho_1(x - vs - q_0)\hat{\rho}_2(k)\frac{\sin(c|k|(t-s))}{c|k|}. \tag{2.1}$$

This is the so-called retarded solution, describing the waves created in the "membranes" by the passage of the particle. Note that it has zero initial conditions at $t = -\infty$ in the sense that, for all $x, y \in \mathbb{R}^{d+n}$, there exists T so that $\phi(x, y, t) = 0$ for all $t \leq T$ (Figure 1).

This wave $\phi(x, y, t)$ induces a force $f(v, q_0, t)$ on the particle that is easily computed from (2.1) using a change of variables in the integration ($x \to x + vt + q_0, \quad \tau = t - s$):

$$\begin{aligned} f(v, q_0, t) &= -\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} dk \int_0^{+\infty} d\tau \nabla\rho_1(x)\rho_1(x + v\tau) \\ &\qquad\qquad \times |\hat{\rho}_2(k)|^2 \frac{\sin(c|k|\tau)}{c|k|} \\ &= f(v), \end{aligned} \tag{2.2}$$

which is clearly independent of q_0 and t, as promised. As a result, $\phi(x, y, t)$ in (2.1) and $q(t) = vt + q_0$ will satisfy (1.3) provided $V(q) = f(v) \cdot q$. One can also write $f(v)$ using the y variable as follows:

$$f(v) = \frac{-1}{4\pi c^2} \iiint dx\, dy\, dz\, \frac{\rho_2(y-z)\rho_2(y)}{|z|}\rho_1(x + v\frac{|z|}{c})\nabla\rho_1(x). \tag{2.3}$$

Our main result will say that, given "any" initial condition and force F not too large, the particle asymptotically converges to a constant velocity trajectory (Theorem 2). It is important for the proof of this result to understand the behaviour of the function $f(v)$ rather well, a task we now turn to. It is clear that $f \in C^\infty(\mathbb{R}^d)$. Furthermore, it is easy to see that

$$f(v) = f_r(\|v\|)\frac{v}{\|v\|}, \quad f_r(\|v\|) < 0, \tag{2.4}$$

so that the reaction force of the medium on the particle is directed opposite to the particle velocity as required for a friction force. To prove this, first note that the rotational invariance of ρ_1 implies that

$$\forall R \in O(d), \quad R[f(v)] = f(Rv).$$

Now, if $v = |v|e_1$, one finds, after a few changes of variables ($\lambda = |v|\tau$ and $\tilde{k} = \frac{k}{|v|}$):

$$f(v) = -|v|^{n-2} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} d\tilde{k} \int_0^{+\infty} d\lambda \nabla\rho_1(x)\rho_1(x + \lambda e_1)|\hat{\rho}_2(|v|\tilde{k})|^2 \frac{\sin(c\lambda|\tilde{k}|)}{c|\tilde{k}|}.$$

The rotational invariance of ρ_1 now implies that $f_i(|v|e_1) = 0$ for $i = 2, ..., d$, so that $f(v)$ has the direction of e_1 and so in the general case ($v \neq 0$) one has indeed

$$f(v) = f_r(|v|)\frac{v}{|v|}.$$

We need to study the asymptotique behaviour of $f(v)$ as $|v|$ goes to 0 and as $|v|$ goes to $+\infty$. For that purpose, we write

$$f(v) = \int_{\mathbb{R}^n} dk\ f(v, k),$$

with (after some manipulations)

$$f(v, k) = f_r(|v|, |k|)\frac{v}{|v|} \tag{2.5}$$

$$f_r(|v|, |k|) = -\frac{1}{|v|^2}|\hat{\rho}_2(k)|^2 h(\frac{c}{|v|}k) \tag{2.6}$$

$$h(\xi) = \int_0^{+\infty} d\lambda \int_{\mathbb{R}^d} dx\ \partial_1\rho_1(x)\rho_1(x_1 + \lambda, x_\perp)\frac{\sin\lambda|\xi|}{|\xi|} \tag{2.7}$$

$$= \pi \int_{\mathbb{R}^{d-1}} dx_\perp |\hat{\rho}_1(|\xi|, x_\perp)|^2. \tag{2.8}$$

It follows immediately from the above that $f_r(|v|) < 0$ and that, given $\ell \in \mathbb{N}$, there exists a constant $C_\ell > 0$ so that

$$|f(v, k)| \leq C_\ell \frac{1}{|k|^2}\left(\frac{|v|}{c|k|}\right)^\ell.$$

In other words, for fixed k, $f(v, k)$ vanishes to all orders in $|v|$ as $v \to 0$. So, as $v \to 0$, the force on the particle due to one of the oscillators of frequency $\omega(k) = c|k|$ present at x, decreases faster than any power of $|v|$ for small v (*i.e.* when $|v| << c|k|R_1$). Roughly speaking, the coupling of the particle to such an oscillator is extremely weak when $|v|$ is much smaller than $c|k|R_1$. This corresponds to a well-known piece of physical intuition: if the particle has speed v, it interacts during a time of order $\frac{R_1}{|v|}$ with any given oscillator. For the energy transfer between the particle and the oscillator to be efficient, this interaction time has to be comparable to the period of the oscillator as an explicit computation easily confirms. Indeed, the total energy transfer ΔE to a driven oscillator of frequency ω

$$\ddot{u}(t) + \omega^2 u(t) = \sigma(t)$$

is easily computed to be $\Delta E = \pi|\hat{\sigma}(\omega)|^2$. Applying this to (1.4) with $q(t) = vt$, one finds $\Delta E = \frac{\pi}{|v|^2}|\hat{\rho}_2(k)|^2|\hat{\rho}_1(c|k|/|v|, 0)|^2$ which vanishes again to all orders in $|v|$ as $|v| \to 0$.

In particular, it is clear from this observation that, when coupling the particle to a family of oscillators, all of the same fixed frequency (as in a pinball machine where each circular obstacles would be mounted on a spring), no ohmic behaviour can be expected since the friction force is not linear in v at small v in that case. As the particle slows down, it couples less and less effectively to such oscillators, leading to a friction force vanishing to all orders in $|v|$. To remedy this effect, one has to couple the particle to a family of sufficiently many oscillators of arbitrarily low-frequency. As the particle slows down, it will then transfer energy to those oscillators with which it is in resonance. In the model above, the number of low-frequency oscillators present at the point x depends on the dimension n of the y variables through the volume element $dk = |k|^{n-1}d|k|d\Omega$. Because of the factor $|k|^{n-1}$, the higher the dimension n, the fewer such oscillators are present. This reflects itself immediately in the low v behaviour of the force $f(v)$:

$$\begin{aligned} f_r(|v|) &= -\frac{1}{c^2}\left(\frac{|v|}{c}\right)^{n-2}\int_{\mathbb{R}^d}|\hat{\rho}_2(\frac{|v|}{c}\xi)|^2 h(\xi)d\xi \qquad (2.9) \\ &= -\frac{1}{c^2}\left(\frac{|v|}{c}\right)^{n-2}|\hat{\rho}_2(0)|^2\int_{\mathbb{R}^d}h(\xi)d\xi \\ &\qquad +\mathrm{o}(\left(\frac{|v|}{c}\right)^{n-2}). \end{aligned}$$

One notices indeed that for small $|v|$, f_r is smaller if n is higher. A friction force proportional to the velocity (and hence ohmic behaviour) is obtained only when $n = 3$! This explains the restriction to $n = 3$ in the theorems below.

We now turn to the behaviour of $f_r(|v|)$ for large values of $|v|$. It is easy to see from (2.9) that $\lim_{|v|\to+\infty} f_r(|v|) = 0$. In other words, at high speeds as well, the friction force exerted by the medium on the particle is small. As one can see in equations (1.4) and (2.9), this is mostly due to the fact that for high $\omega(k) = c|k|$, the oscillators are only very weakly coupled to the particle due to the presence of the smooth form factor $\hat{\rho}_2$. A further explanation is given in Section 6. The model therefore can only be expected to display dissipative behaviour when v is not too large.

Equation (2.9) implies that

$$f_r(|v|) = \frac{1}{c^2}\tilde{f}_r(|w|)$$

with $v = cw$ and $\tilde{f}_r$ not depending on c and that $| f_r'(0) |> 0$. Defining w_M to be the smallest zero of $\tilde{f}_r'$ it is then clear that $| f_r |$ is strictly increasing for $| v |\leq w_M c$. As a result, we can define, for each $F \in \mathbb{R}^d$ with $|F| \leq F_M(c) =| f_r(w_M c) |$, $v(F) \in \mathbb{R}^d$ by

$$f(v(F)) = F, \quad |v(F)| \leq w_M c. \qquad (2.10)$$

Note that $F_M(c) = \frac{\tilde{F}_M}{c^2}$. Finally, using (2.9), we can extend $f_r(|v|)$ to a differentiable function at $v = 0$. If we note Γ_v the differential of the function $f(v)$, one can see that in the basis $(e_1, \ldots, e_d)$ we have

$$\Gamma_v = \begin{pmatrix} f_r'(|v|) & 0 & \cdots & 0 \\ 0 & \frac{f_r(|v|)}{|v|} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{f_r(|v|)}{|v|} \end{pmatrix}$$

for $v \neq 0$ and

$$\Gamma_0 = f_r'(0) Id.$$

We can see that for $n = 3$, $f_r'(0)$, $f_r'(|v|)$ and $\frac{f_r(|v|)}{|v|}$ are of order c^{-3} and are strictly negative provided $|v| < w_M c$. We define

$$\theta = \gamma =| f_r'(0) | c^3 \tag{2.11}$$

if $v = 0$ and

$$\theta = c^3 \max\left(| f_r'(|v|) |, \frac{| f_r(|v|) |}{|v|} \right), \quad \gamma = c^3 \min\left(| f_r'(|v|) |, \frac{| f_r(|v|) |}{|v|} \right) \tag{2.12}$$

if $v \neq 0$. The profile for $f(v)$ when ρ is a Gaussian is given in Figure 2.

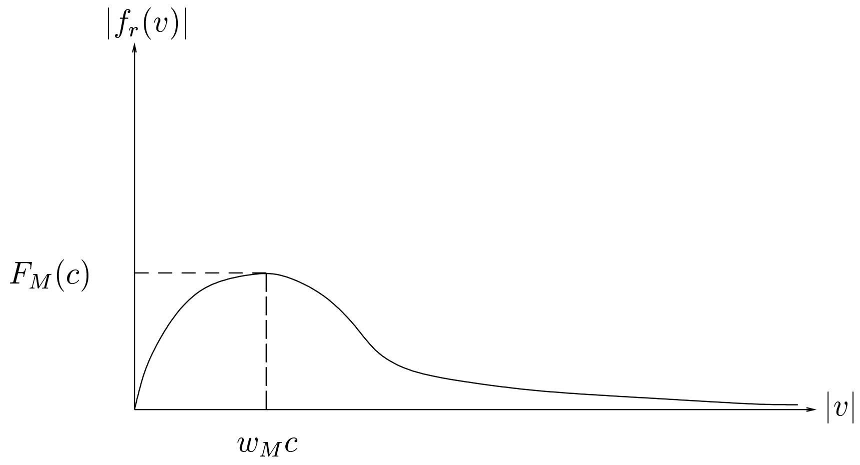

Figure 2: Profile of $f_r(|v|)$

A final word is in order on the role of the speed c. Most of our results are only proven for c sufficiently large. Although it is not totally clear whether this is a deficiency of the proofs, it does seem like a reasonable restriction. Indeed, it guarantees that the energy transferred by the particle to a given membrane at x, is rapidly evacuated by this membrane (away from $y = 0$) so that the particle can not recuperate this energy if at some later time it comes back to the same membrane (see also Section 6).

3 Existence of solutions

The assumptions on the potential are

(H2) $V \in C^1(\mathbb{R}^d)$ and ∇V is lipschitz. Moreover, one of the two following assumptions holds: either ∇V is bounded (such as when $V(q) = -F \cdot q$) or V is bounded from below.

We are now ready to introduce the phase space $\mathcal{E}$ of the model. Let $\|\cdot\|_2$ denote the usual norm on $L^2(\mathbb{R}^{d+n}, dxdy)$. On $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^n)$, $\|\phi\| = \|\nabla_y \phi\|_2$ defines a norm. Let E be the completion of $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^n)$ with this norm. Actually, as a consequence of the Sobolev imbedding theorems ([B],chapter 9), E is the space $L^2(\mathbb{R}^d, D, dx)$ where

$$D = \{\phi \in L^{\frac{2n}{n-2}}(\mathbb{R}^n, dy) | \nabla_y \phi \in L^2(\mathbb{R}^n, dy)\}.$$

We then define

$$\mathcal{E} = E \times \mathbb{R}^d \times L^2(\mathbb{R}^{d+n}) \times \mathbb{R}^d$$

with the norm:

$$|Y|_{\mathcal{E}} = (\|\phi\|^2 + |q|^2 + \|\pi\|_2^2 + |p|^2)^{\frac{1}{2}} \text{ for } Y = (\phi, q, \pi, p).$$

With this norm, $\mathcal{E}$ is a Hilbert space.

We now write the problem (1.2) in a more convenient way, so as to prove existence and uniqueness of a solution.

$$\begin{cases} \dot{Y}(t) = G(Y(t)) \\ Y(0) = Y_0 \in \mathcal{E} \end{cases} \tag{3.1}$$

where

$$\begin{aligned} G : (\phi, q, \pi, p) \quad &\to \quad (\pi, p, c^2 \Delta_y \phi - \rho(x - q, y), -\nabla V(q) + \\ &\qquad \int_{\mathbb{R}^{d+n}} dx\, dy\, \nabla_x \rho(x - q, y) \phi(x, y)). \end{aligned} \tag{3.2}$$

By solution, we mean that:

$$Y(t) = Y_0 + \int_0^t G(Y(s)) ds$$

in the sense of the distributions.

Theorem 1. *Let $n \geq 3$. Under the assumptions (H1) and (H2), we have:*

1. *For each Y_0 in $\mathcal{E}$, the problem (3.1) has a unique solution $Y(t)$ in $C^0(\mathbb{R}, \mathcal{E})$.*
2. *For every $t \in \mathbb{R}$, the map $W^t : Y_0 \to Y(t)$ is continuous on $\mathcal{E}$.*
3. *For every $t \in \mathbb{R}$, $H(Y(t)) = H(Y_0)$ where*

$$\begin{aligned} H(Y) &= \frac{p^2}{2} + V(q) + \frac{1}{2} \int_{\mathbb{R}^{d+n}} dx\, dy\, (c^2 |\nabla_y \phi(x, y)|^2 + |\pi(x, y)|^2) \\ &\qquad + \int_{\mathbb{R}^{d+n}} dx\, dy\, \rho(x - q, y) \phi(x, y) \\ &= H_0(Y) + V(q) + \int_{\mathbb{R}^{d+n}} dx\, dy\, \rho(x - q, y) \phi(x, y) \end{aligned} \tag{3.3}$$

is a continuous function on $\mathcal{E}$.

Remark 1. *Using the first part of the theorem and* (1.3), *one sees that* $q(t) \in C^2(\mathbb{R}, \mathbb{R}^3)$.

Note that the densely defined bilinear anti-symmetric form

$$\omega(Y_1, Y_2) = q_1 p_2 - p_1 q_2 + \int_{\mathbb{R}^{d+n}} dxdy\, (\phi_1 \pi_2 - \pi_1 \phi_2)$$

makes $\mathcal{E}$ into a symplectic vector space. The equations of motion (1.2)-(1.3) are of course the Hamiltonian equations for the Hamiltonian H in (3.3), which is the total energy of the system. Note that the latter is not bounded from below when V is not, such as when $V = -F \cdot q$. This makes for a slight complication in the existence proof, which is otherwise standard and largely follows [KKS1].
Proof: We start by showing there is a local solution. Then we will use conservation of energy to show the solution is global.

We first look at the problem

$$\begin{cases} \dot{Y}(t) = G_0(Y(t)) \\ Y(0) = Y_0 \end{cases} \tag{3.4}$$

with

$$G_0 Y = (\pi; 0; c^2 \Delta_y \phi; 0). \tag{3.5}$$

This problem is just the free wave equation in $\mathbb{R}^n$ with a parameter $x \in \mathbb{R}^d$. It admits a unique solution: $t \in \mathbb{R} \to Y(t) \in \mathcal{E}$. Moreover, let denotes by $W_0^t : Y_0 \to Y(t)$ the corresponding continuous group, W_0^t turns out to be a linear isometry with the norm $|\ |_{\mathcal{E}}$. It is also continuous on $\mathbb{R} \times \mathcal{E}$ ([LM],chapter 3).

Now define $Z(t) = W_0^{-t} Y(t)$ or $Y(t) = W_0^t Z(t)$. In particular, $Z(0) = Y(0) = Y_0$. We have $\dot{Y}(t) = G_0 Y(t) + W_0^t \dot{Z}(t)$. $Y(t)$ is a solution of the problem (3.1) if and only if $Z(t)$ satisfies

$$\begin{cases} \dot{Z}(t) = W_0^{-t} G_1(W_0^t Z(t)) \\ Z(0) = Y_0 \end{cases} \tag{3.6}$$

where

$$\begin{aligned} G_1 : (\phi; q; \pi; p) \in \mathcal{E} \quad \to \quad & \Big(0; p; -\rho(x - q, y); -\nabla V(q) + \\ & \int_{\mathbb{R}^6} dx\, dy\, \nabla_x \rho(x - q, y) \phi(x, y)\Big) \in \mathcal{E}. \end{aligned} \tag{3.7}$$

Introducing

$$\tilde{G} : (t, Z) \in \mathbb{R} \times \mathcal{E} \to W_0^{-t} G_1(W_0^t Z) \in \mathcal{E}$$

it is clear that $\tilde{G}$ is continuous on $\mathbb{R} \times \mathcal{E}$ and lipschitz on $\mathcal{E}$ because W_0^t is an isometry and G_1 is lipschitz. This problem satisfies all the conditions of the Cauchy-Lipschitz theorem ([ZQ],chapter 10), so it has a unique solution which is defined on an open interval. More precisely, there exists an open interval J such that $0 \in J$ and there exists a unique function $Z : t \in J \to Z(t) \in \mathcal{E}$ satisfying (3.6). Moreover, $\tilde{W}^t : Z_0 \to Z(t)$ is continuous on $\mathcal{E}$ for every $t \in J$ and so we have the same results for

$$W^t : Y_0 \in \mathcal{E} \to Y(t) = W_0^t \tilde{W}^t Y_0 \in \mathcal{E}.$$

In order to prove global existence, we now prove conservation of energy. We first prove the result for smooth initial data (*i.e.* $\phi_0, \pi_0 \in C^\infty(\mathbb{R}^{d+n})$). Let $Y_0 = (\phi_0, q_0, \pi_0, p_0)$ with $\phi_0, \pi_0 \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. Then $W_0^t Y_0$ is smooth ([CH],chapter 6) and by the integral representation:

$$Y(t) = W_0^t Y_0 + \int_0^t ds\, W_0^{t-s} G_1(Y(s)),$$

it is clear that $\phi(t), \pi(t)$ are smooth as well (in x and y). Note that $\phi(x, y, t)$ and $\pi(x, y, t)$ are also smooth in t ([LM], chapter 3). For such initial data a simple computation then yields:

$$\frac{d}{dt} H(Y(t)) = 0,$$

so that, for smooth initial data, $H(Y(t))$ is a constant for all t in J. We now prove that H is continuous on $\mathcal{E}$. The continuity of W^t on $\mathcal{E}$ and the fact that smooth initial data are dense in $\mathcal{E}$ will then imply the result for all initial data. Since V is continuous, it only remains to show the interaction term in H is continuous.

$$\begin{aligned}
& \mid \textstyle\int_{\mathbb{R}^{d+n}} dx\, dy\, \rho(x-q, y)\phi(x, y) - \int_{\mathbb{R}^{d+n}} dx\, dy\, \rho(x-q, y)\psi(x, y) \mid \\
= & \mid \textstyle\int_{\mathbb{R}^{d+n}} dx\, dy\, \rho(x-q, y)(\phi(x, y) - \psi(x, y)) \mid \\
= & \mid \textstyle\int_{\mathbb{R}^{d+n}} dx\, dk\, \bar{\hat{\rho}}(x-q, k)(\hat{\phi}(x, k) - \hat{\psi}(x, k)) \mid \\
= & \mid \textstyle\int_{\mathbb{R}^{d+n}} dx\, dk\, \frac{\bar{\hat{\rho}}(x-q,k)}{|k|}(|k|\hat{\phi}(x, k) - |k|\hat{\psi}(x, k)) \mid \\
\leq & \| \tfrac{\bar{\hat{\rho}}(x-q,k)}{|k|} \|_{L^2} \times \| \; |k|(\hat{\phi} - \hat{\psi}) \; \|_{L^2} \\
\leq & \| \tfrac{\hat{\rho}(x-q,k)}{|k|} \|_{L^2} \times \|\phi - \psi\|.
\end{aligned}$$

Because ρ has compact support and $n \geq 3$ the first factor of the right-hand side is finite and so H is continuous.

We will furthermore need the following obvious inequality:

$$\mid \int_{\mathbb{R}^{d+n}} dx\, dy\, \rho(x - q(t), y)\phi(x, y, t) \mid \leq \frac{c^2}{4} \parallel \phi \parallel^2 - \frac{1}{c^2}\langle \rho; \Delta_y^{-1} \rho \rangle. \tag{3.8}$$

Hence:

$$H(Y(t)) \geq \frac{1}{2}p(t)^2 + V(q(t)) + \frac{c^2}{4}\|\phi(t)\|^2 + \frac{1}{2}\|\pi(t)\|^2 + \frac{1}{c^2}\langle \rho; \Delta_y^{-1} \rho \rangle. \tag{3.9}$$

We are now ready to prove that $J = \mathbb{R}$. We know that J can be written $]a; b[$ with $-\infty \leq a < 0$ and $0 < b \leq +\infty$. We will show by contradiction that $b = +\infty$ (the same can be done for $a = -\infty$). If $b < +\infty$, we know by the theory of differential equations ([ZQ], chapter 10) that

$$\lim_{t \to b} |Z(t)|_{\mathcal{E}} = +\infty$$

and the same holds for $|Y(t)|_{\mathcal{E}}$ because

$$|Y(t)|_{\mathcal{E}} = |W_0^t Z(t)|_{\mathcal{E}} = |Z(t)|_{\mathcal{E}}.$$

We consider first the (harder) case where ∇V is bounded (but V is not necessarily bounded below). We have

$$\dot{p}(t) = -\nabla V(q(t)) + \int_{\mathbb{R}^{d+n}} dx\, dy\, \nabla_x \rho(x - q(t), y)\phi(x, y, t).$$

For $t > 0$, we can write ϕ as:

$$\phi = \phi^r + \phi^0 \tag{3.10}$$

where ϕ^r is the solution of the wave equation with initial data equal to 0 and ϕ^0 is the solution of the homogeneous wave equation with initial data ϕ_0 and π_0 [CH] [J]. Consequently

$$\begin{aligned}\dot{p}(t) \quad = \quad & -\nabla V(q(t)) + \int_{\mathbb{R}^{d+n}} dx\, dy\, \nabla_x \rho(x - q(t), y)\phi^r(x, y, t) \\ & + \int_{\mathbb{R}^{d+n}} dx\, dy\, \nabla_x \rho(x - q(t), y)\phi^0(x, y, t).\end{aligned}$$

The first term is bounded by hypothesis. The second one is bounded using the Cauchy-Schwartz inequality and the exact form of ϕ^r given in (4.2). Using (3.8) with $\nabla_x \rho$ instead of ρ, we have

$$\begin{aligned}\left| \int_{\mathbb{R}^{d+n}} dx\, dy\, \nabla_x \rho(x - q(t))\phi^0(x, y, t) \right| \quad \leq \quad & \frac{c^2}{2} \parallel \nabla_y \phi^0(t) \parallel_2^2 \\ & + \frac{1}{2c^2} \parallel \nabla_y^{-1} \nabla_x \rho(x - q(t)) \parallel_2^2 .\end{aligned}$$

But ϕ^0 is a solution of the free wave equation with initial conditions ϕ_0 and π_0, so

$$c^2 \parallel \nabla_y \phi^0(t) \parallel_2^2 + \parallel \frac{d}{dt}\phi^0(t) \parallel_2^2 = c^2 \parallel \nabla_y \phi_0 \parallel_2^2 + \parallel \pi_0 \parallel_2^2 \tag{3.11}$$

and so $c^2 \parallel \nabla_y \phi^0(t) \parallel_2^2$ is bounded too.

Finally, $\dot{p}(t)$ is bounded on J: there exists $C > 0$ such that

$$\forall t \in J, t > 0 \quad | \dot{p}(t) | \leq C. \tag{3.12}$$

We have supposed b to be finite, so $p(t)$ and $q(t)$ are also bounded for $t > 0, t \in J$.

By energy conservation (3.3) implies:

$$H(Y_0) \geq \frac{1}{2}p(t)^2 + V(q(t)) + \frac{c^2}{4}\|\phi(t)\|^2 + \frac{1}{2}\|\pi(t)\|_2^2 + \frac{1}{c^2}\langle \rho; \Delta_y^{-1} \rho \rangle$$

and so $\|\phi(t)\|$ and $\|\pi(t)\|_2$ are bounded and then $|Y(t)|_{\mathcal{E}}$ too which is a contradiction with the fact that b is finite.

We finally deal with the second (easier) case, where V is bounded from below. There exists $V_0 \in \mathbb{R}$ such that for every $q \in \mathbb{R}^d$, $V(q) \geq V_0$.

(3.3) implies

$$H(Y_0) \geq \frac{1}{2}p(t)^2 + V_0 + \frac{c^2}{4}\|\phi(t)\|^2 + \frac{1}{2}\|\pi(t)\|_2^2 + \frac{1}{c^2}\langle \rho; \Delta_y^{-1} \rho \rangle. \tag{3.13}$$

So $p(t), \|\phi(t)\|$ and $\|\pi(t)\|_2$ are bounded on J and because b is supposed to be finite, $q(t)$ is also bounded which is again a contradiction. $\square$

4 Behaviour of the solutions: constant force

Definition 1. *: Let $\mathcal{D}$ be the set of all states Y_0 in $\mathcal{E}$ such that*

$$\sup_{x\in\mathbb{R}^d} [|\phi_0(x,y)| + |y|(|\nabla_y\phi_0(x,y)| + |\pi_0(x,y)|)] \le \kappa(1+|y|)^{-\nu} \tag{4.1}$$

for some $\nu > 2$ and $\kappa > 0$.

Recall that $v(F)$ is defined in (2.10) and γ in (2.11) and (2.12).

Theorem 2. *Let ρ satisfy (H1) and consider* (3.1) *with $V(q) = -F\cdot q,\ F\in\mathbb{R}^d$ and $n=3$.*

For all $F_0, K, R, \varepsilon, \eta > 0$ there exists $c_0(\rho,\varepsilon,\eta,F_0,K,R) > 0$ such that for any $c > c_0$, $|F| < F_0c^{-2-\varepsilon}$ and $Y_0 \in \mathcal{E}$ with compact support in $B_R \subset \mathbb{R}^{d+3}$ satisfying $H_0(Y_0) < Kc^{2-2\varepsilon}$, there exists $q_\infty(F,Y_0)\in\mathbb{R}^d, K' > 0$ such that

$$q(t) = q_\infty + v(-F)t + \epsilon(t)$$

with $|\epsilon(t)| \le K'e^{-\frac{\gamma(1-\eta)}{c^3}t}$. If $Y_0\in\mathcal{D}$ with $\kappa < Kc^{2-\varepsilon}$ and $H_0(Y_0) < Kc^{2-2\varepsilon}$ but is not necessarily of compact support, one has the same result with $\epsilon(t) \le K'|t|^{2-\nu}$.

Remark 1. *Throughout the proof, the different constants will only depend on $\rho,\eta,\varepsilon,F_0,K,R$ and not on the initial conditions.*

Proof: We first fix c large enough so that $F_0c^{-2-\varepsilon} < F_M(c)$ and we consider (3.1) for some $F\in\mathbb{R}^d, |F| \le F_0c^{-2-\varepsilon}$ and $Y_0\in\mathcal{D}$. We will write $v = v(-F)$. Solving (1.2) yields, according to (3.10),

$$\phi(x,y,t) = \phi^r(x,y,t) + \phi^0(x,y,t)$$

where in the 3-dimensional case:

$$\phi^r(x,y,t) = \frac{-1}{4\pi c^2}\int_{|z|\le ct} dz\, \frac{\rho_2(y-z)}{|z|}\rho_1\left(x - q(t-\frac{|z|}{c})\right) \tag{4.2}$$

$$\phi^0(x,y,t) = \frac{1}{4\pi c^2t^2}\int_{S_{ct}(y)} [\phi_0(x,\sigma) + \sigma\cdot\nabla_y\phi_0(x,\sigma) + t\pi_0(x,\sigma)]d\sigma \tag{4.3}$$

and $S_{ct}(y)$ is the sphere of radius ct centered on y ([J], chapter 3). Inserting this in (1.3) leads to the following integro-differential equation for $q(t)$:

$$\begin{aligned}\ddot{q}(t) \quad = \quad & F - \frac{1}{4\pi c^2}\iiint_{|z|\le ct} dx\,dy\,dz\, \frac{\rho_2(y-z)\rho_2(y)}{|z|}\rho_1(x - q(t-\frac{|z|}{c})) \\ & \qquad \times\nabla\rho_1(x-q(t)) \\ & + \frac{1}{4\pi c^2t^2}\iint dx\,dy \int_{S_{ct}(y)} d\sigma\, [\phi_0(x,\sigma) + \sigma\cdot\nabla_y\phi_0(x,\sigma) \\ & \qquad + t\pi_0(x,\sigma)]\rho_2(y)\nabla\rho_1(x-q(t)).\end{aligned} \tag{4.4}$$

We denote the last term of the right-hand side by $A(t)$. Then we write $F = -f(v)$ and use (2.3) to obtain

$$\begin{aligned}\ddot{q}(t) &= \frac{1}{4\pi c^2}\iiint dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\rho_1(x+v\frac{|z|}{c})\nabla\rho_1(x) \\ &-\frac{1}{4\pi c^2}\iiint_{|z|\leq ct} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\rho_1(x-q(t-\frac{|z|}{c}))\nabla\rho_1(x-q(t)) \\ &+A(t).\end{aligned}$$

We now divide the first integral in two parts:

$$\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dz = \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \int_{|z|\leq ct} dz + \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \int_{|z|\geq ct} dz$$

We denote by $\tilde{f}(t)$ the second one of these two terms. Now, remark that for $|z| \geq 2R_2,\ \rho_2(y)\rho_2(y-z) = 0$ because

$$|y-z| \geq |z| - |y| \geq 2R_2 - |y|$$

and $\rho_2(y) = 0$ for $|y| \geq R_2$. So $\tilde{f}(t)$ vanishes if $t \geq \frac{2R_2}{c}$. Finally, let $\tilde{A}(t) = A(t) + \tilde{f}(t)$ This leads to:

$$\begin{aligned}\ddot{q}(t) &= \frac{1}{4\pi c^2}\iiint_{|z|\leq ct} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\Big[-\rho_1\left(x-q(t-\frac{|z|}{c})\right) \\ &\nabla\rho_1(x-q(t)) + \rho_1\left(x+v\frac{|z|}{c}\right)\nabla\rho_1(x)\Big] + \tilde{A}(t).\end{aligned} \tag{4.5}$$

Inserting

$$q(t-\frac{|z|}{c}) = q(t) - \int_{t-\frac{|z|}{c}}^{t} \dot{q}(s)ds$$

in (4.5) and using translation invariance, we find:

$$\begin{aligned}\ddot{q}(t) &= \frac{1}{4\pi c^2}\iiint_{|z|\leq ct} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\Big[\rho_1\left(x+v\frac{|z|}{c}\right) \\ &-\rho_1\left(x+\int_{t-\frac{|z|}{c}}^{t}\dot{q}(s)ds\right)\Big]\nabla\rho_1(x) + \tilde{A}(t).\end{aligned} \tag{4.6}$$

Since we expect to prove that $\dot{q}(t) \to v$, it is convenient to introduce $h(t) = q(t) - vt$, in terms of which (4.6) becomes

$$\begin{aligned}\ddot{h}(t) &= \frac{1}{4\pi c^2}\iiint_{|z|\leq ct} dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\Big[\rho_1\left(x+v\frac{|z|}{c}\right) \\ &-\rho_1\left(x+v\frac{|z|}{c}+\int_{t-\frac{|z|}{c}}^{t}\dot{h}(s)ds\right)\Big]\nabla\rho_1(x) + \tilde{A}(t).\end{aligned}$$

We can now write

$$\begin{aligned}
&\rho_1\left(x+v\frac{|z|}{c}+\int_{t-\frac{|z|}{c}}^{t}\dot h(s)ds\right)-\rho_1\left(x+v\frac{|z|}{c}\right)\\
=\;&\int_{t-\frac{|z|}{c}}^{t}\dot h(s)\cdot\nabla\rho_1\left(x+v\frac{|z|}{c}\right)ds\\
&\qquad+\frac{1}{2}\Big\langle \mathrm{Hess}\rho_1(\tilde x_{t,|z|})\int_{t-\frac{|z|}{c}}^{t}\dot h(s)ds;\int_{t-\frac{|z|}{c}}^{t}\dot h(s)ds\Big\rangle
\end{aligned}$$

for some $\tilde x_{t,|z|}\in[x+v\frac{|z|}{c};x+v\frac{|z|}{c}+\int_{t-\frac{|z|}{c}}^{t}\dot h(s)ds]$. In addition, an integration by parts yields:

$$\int_{t-\frac{|z|}{c}}^{t}\dot h(s)ds=\frac{|z|}{c}\dot h(t)+\int_{t-\frac{|z|}{c}}^{t}(t-\frac{|z|}{c}-s)\ddot h(s)ds.$$

As a result

$$\begin{aligned}
\ddot h(t)\;=\;&-\frac{1}{4\pi c^3}\iiint_{|z|\le ct}dx\,dy\,dz\,\rho_2(y-z)\rho_2(y)\big[\dot h(t)\cdot\nabla\rho_1(x+v\frac{|z|}{c})\big]\nabla\rho_1(x)\\
&-\frac{1}{4\pi c^2}\iiint_{|z|\le ct}dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\
&\qquad\times\Big[\Big(\int_{t-\frac{|z|}{c}}^{t}(t-\frac{|z|}{c}-s)\ddot h(s)ds\Big)\cdot\nabla\rho_1(x+v\frac{|z|}{c})\Big]\nabla\rho_1(x)\\
&-\frac{1}{4\pi c^2}\iiint_{|z|\le ct}dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\
&\qquad\times\frac{1}{2}\Big\langle \mathrm{Hess}\rho_1(\tilde x_{t,|z|})\int_{t-\frac{|z|}{c}}^{t}\dot h(s)ds;\int_{t-\frac{|z|}{c}}^{t}\dot h(s)ds\Big\rangle\nabla\rho_1(x)\\
&+\tilde A(t).
\end{aligned}$$

Once again we rewrite the first integral

$$\int_{|z|\le ct}=\int_{\mathbb{R}^3}-\int_{|z|\ge ct}.$$

It is easily seen from (2.3) that the first term equals $\Gamma_v\cdot\dot h(t)$ whereas the second one is once again vanishing for $t\ge\frac{2R_2}{c}$, so that we can put it in the $\tilde A$ term without changing its asymptotic properties. We finally obtain the following convenient form of the integro-differential equation for $\dot h(t)$:

$$\begin{aligned}
\ddot h(t)\;=\;&\Gamma_v\cdot\dot h(t)-\frac{1}{4\pi c^2}\iiint_{|z|\le ct}dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\
&\qquad\times\Big[\Big(\int_{t-\frac{|z|}{c}}^{t}(t-\frac{|z|}{c}-s)\ddot h(s)ds\Big)\cdot\nabla\rho_1(x+v\frac{|z|}{c})\Big]\nabla\rho_1(x)\\
&-\frac{1}{4\pi c^2}\iiint_{|z|\le ct}dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\
&\qquad\times\frac{1}{2}\Big\langle \mathrm{Hess}\rho_1(\tilde x_{t,|z|})\int_{t-\frac{|z|}{c}}^{t}\dot h(s)ds;\int_{t-\frac{|z|}{c}}^{t}\dot h(s)ds\Big\rangle\nabla\rho_1(x)\\
&+\tilde A(t).
\end{aligned}\tag{4.7}$$

One recognizes here, in the first two terms, the equation (1.1) with $V = -F \cdot q$.

Our goal is to prove $\dot{h}(t)$ tends to zero. For that purpose, we will first show that $\dot{h}(t)$ is bounded (see (4.14)). Note that in the case considered here ($V = -F \cdot q$) the Hamiltonian is not bounded below (unless $F = 0$), so that there is no a priori reason why this should be true. We start by controlling $\ddot{h}(t)$. Using (4.4) and $\ddot{h}(t) = \ddot{q}(t)$ we have:

$$|\ddot{h}(t)| \leq |F| + \frac{K}{c^2} + |A(t)|.$$

But

$$\begin{aligned} |A(t)| &= |\int\int dx\, dy\, \rho_2(y)\nabla\rho_1(x - q(t))\phi^0(x, y, t)| \\ &\leq c^2\|\nabla_y\phi^0(t)\|_2 \times \frac{1}{c^2}\|\nabla^{-1}\rho_2(y)\nabla\rho_1(x - q(t))\|_2 \end{aligned}$$

and using (3.11) together with the hypothesis $H_0(Y_0) \leq Kc^{2-2\varepsilon}$ we have

$$|A(t)| \leq Ac^{-1-\varepsilon}. \tag{4.8}$$

Then remember that $|F| \leq F_0 c^{-2-\varepsilon} \leq F_M(c) = \frac{\tilde{F}_M}{c^2}$, so finally:

$$|\ddot{h}(t)| \leq K_0 c^{-1-\varepsilon}. \tag{4.9}$$

We now bound $\dot{h}(t)$. Multiplying (4.7) by $e^{-\Gamma_v t}$ and integrating between 0 and T, we obtain after some rewriting:

$$\begin{aligned} \dot{h}(T) &= e^{\Gamma_v T}\dot{h}(0) - \frac{1}{4\pi c^2}\int_0^T dt \iiint_{|z|\leq ct} dx\, dy\, dz\, \frac{\rho_2(y-z)\rho_2(y)}{|z|} \\ &\quad \times \Big[\Big(\int_{t-\frac{|z|}{c}}^t (t - \frac{|z|}{c} - s)\ddot{h}(s)ds\Big) \cdot \nabla\rho_1(x + v\frac{|z|}{c})\Big] e^{-\Gamma_v(t-T)}\nabla\rho_1(x) \\ &\quad -\frac{1}{4\pi c^2}\int_0^T dt \iiint_{|z|\leq ct} dx\, dy\, dz\, \frac{\rho_2(y-z)\rho_2(y)}{|z|} \\ &\quad \times \frac{1}{2}\Big\langle \mathrm{Hess}\rho_1(\tilde{x}_{t,|z|}) \int_{t-\frac{|z|}{c}}^t \dot{h}(s)ds; \int_{t-\frac{|z|}{c}}^t \dot{h}(s)ds \Big\rangle e^{-\Gamma_v(t-T)}\nabla\rho_1(x) \\ &\quad + \int_0^T dt\, e^{-\Gamma_v(t-T)}\tilde{A}(t). \end{aligned}$$

Defining $B(t) = \sup_{0\leq s\leq t}|\dot{h}(s)|$ and using (4.9), we find for all $t \geq 0$ and for some $K_1, K_2 > 0$:

$$\begin{aligned} |\dot{h}(t)| &\leq |\dot{h}(0)| + \frac{K_1}{c^{5+\varepsilon}}\int_0^t \|e^{-\Gamma_v(s-t)}\|ds + \frac{K_2}{c^4}\int_0^t \|e^{-\Gamma_v(s-t)}\|B^2(s)ds \\ &\quad + \int_0^t \|e^{-\Gamma_v(s-t)}\tilde{A}(s)\|ds. \end{aligned}$$

Then, using the notations of Section 2,

$$\begin{aligned} |\dot{h}(t)| &\leq |\dot{h}(0)| + \frac{K_1}{c^{5+\varepsilon}}\int_0^t e^{\frac{\gamma}{c^3}(s-t)}ds + \frac{K_2}{c^4}B^2(t)\int_0^t e^{\frac{\gamma}{c^3}(s-t)}ds \\ &\quad + \int_0^t e^{\frac{\gamma}{c^3}(s-t)}|\tilde{A}(s)|ds. \end{aligned}$$

If ϕ_0 and π_0 have compact support, so do $A(s)$ and $\tilde{A}(s)$. One should remember that $\tilde{A}$ differs from A by terms which have compact support. Moreover one of these terms is bounded by $\frac{1}{c^2}$ and the other by $\frac{|\dot{h}(t)|}{c^3}$. Therefore, using (4.8), the last integral can be bounded the following way:

$$\begin{aligned}
\int_0^t e^{\frac{\gamma}{c^3}(s-t)}|\tilde{A}(s)|ds &\leq \int_0^\alpha e^{\frac{\gamma}{c^3}(s-t)}|\tilde{A}(s)|ds \\
&\leq (Ac^{-1-\varepsilon}+kB(t)c^{-3})\int_0^\alpha e^{\frac{\gamma}{c^3}(s-t)}ds \\
&\leq \frac{Ac^{2-\varepsilon}+kB(t)}{\gamma}[e^{\frac{\gamma}{c^3}\alpha}-1] \\
&\leq A'\alpha c^{-1-\varepsilon}+k'B(t)\alpha c^{-3} \qquad (4.10)
\end{aligned}$$

where α is such that $\tilde{A}(s)=0$ for $s>\alpha$ (note that $\alpha<\sup\{\frac{2R_2}{c},\frac{R+R_2}{c}\}$) and provided c is large enough. In the case when ϕ_0 and π_0 do not have compact support, we need a different bound on $\tilde{A}(t)$ than (4.8). For $|\,y\,|\geq R_2$, $\rho_2(y)=0$, so

$$\begin{aligned}
A(t) &= \frac{1}{4\pi c^2t^2}\int dx\int_{|y|\leq R_2}dy\int_{S_{ct}(y)}d\sigma\,[\phi_0(x,\sigma)+\sigma\cdot\nabla_y\phi_0(x,\sigma) \\
&\qquad +t\pi_0(x,\sigma)]\rho_2(y)\nabla\rho_1(x-q(t)).
\end{aligned}$$

If $|\,y\,|\leq R_2$, then $|\,\sigma\,|\geq|\,ct-R_2\,|$. According to (4.1), and the hypothesis on ϕ_0, π_0

$$|\,\phi_0(x,\sigma)\,|\leq Kc^{2-\varepsilon}\,|\,ct-R_2\,|^{-\nu}$$
$$|\,\sigma\cdot\nabla_y\phi_0(x,\sigma)\,|\leq Kc^{2-\varepsilon}\,|\,ct-R_2\,|^{-\nu}$$
$$|\,t\pi_0(x,\sigma)\,|\leq Kc^{2-\varepsilon}t\,|\,ct-R_2\,|^{-\nu-1}$$

uniformly in the x variable. So we have

$$|\,A(t)\,|\leq\frac{Ac^{2-\varepsilon}}{(1+ct)^\nu}. \qquad (4.11)$$

This bound is also valid for $\tilde{A}(t)$ (except the part containing $\dot{h}(t)$ which has compact support and will be controlled by $k'B(t)c^{-3}$ as in (4.10)), so

$$\begin{aligned}
\int_0^t e^{\frac{\gamma}{c^3}(s-t)}|\tilde{A}(s)|ds &= Ac^{2-\varepsilon}\int_0^t\frac{e^{\frac{\gamma}{c^3}(s-t)}}{(1+cs)^\nu}ds \\
&= Ac^{2-\varepsilon}\int_0^{\frac{1}{c}}\frac{e^{\frac{\gamma}{c^3}(s-t)}}{(1+cs)^\nu}ds+Ac^{2-\varepsilon}\int_{\frac{1}{c}}^t\frac{e^{\frac{\gamma}{c^3}(s-t)}}{(1+cs)^\nu}ds \\
&\leq Ac^{2-\varepsilon}\frac{c^3}{\gamma}(e^{\frac{\gamma}{c^4}}-1)e^{-\frac{\gamma}{c^3}t}+\frac{Ac^{2-\varepsilon}}{c(1-\nu)}[\frac{1}{(1+ct)^{\nu-1}}-\frac{1}{2^{\nu-1}}] \\
&\leq A"c^{1-\varepsilon}
\end{aligned}$$

because $\nu>2$. Finally we have for all $t\geq 0$

$$|\dot{h}(t)|\leq|\dot{h}(0)|+K_3c^{1-\varepsilon}+\frac{K_2}{\gamma c}B^2(t)+k'B(t)c^{-3}.$$

Since $B(t)$ is increasing, we have, for all $0 \le t \le T$,

$$|\dot{h}(t)| \le |\dot{h}(0)| + K_3 c^{1-\varepsilon} + \frac{K_2}{\gamma c} B^2(T) + k' B(T) c^{-3}.$$

So, taking the supremum over t, we have the following inequality for all $T \ge 0$:

$$B(T) - k' B(T) c^{-3} \le |\dot{h}(0)| + K_3 c^{1-\varepsilon} + \frac{K_2}{\gamma c} B^2(T). \tag{4.12}$$

Using the hypothesis on F and $H_0(Y_0)$, one has $|\dot{h}(0)| < K c^{1-\varepsilon}$, so

$$B(T)(1 - k' c^{-3}) \le (K + K_3) c^{1-\varepsilon} + \frac{K_2}{\gamma c} B^2(T). \tag{4.13}$$

An easy computation and the continuity of $B(t)$ tell us that:

$$B(t) \le B_- \qquad \forall t \ge 0$$

or

$$B(t) \ge B_+ \qquad \forall t \ge 0$$

with

$$B_\pm = \frac{\gamma c}{2K_2} \left(1 - k' c^{-3} \pm \sqrt{(1 - k' c^{-3})^2 - \frac{4K_2(K + K_3)}{\gamma c^\varepsilon}} \right).$$

We will now take c large enough so that there exists two constants β and β' such that:

$$B_- \le \beta c^{1-\varepsilon},$$
$$B_+ \ge \beta' c > K c^{1-\varepsilon}.$$

Note now that

$$B(0) = | \dot{h}(0) | \le K c^{1-\varepsilon} < B_+$$

and so we have finally the following bound for $\dot{h}(t)$:

$$|\dot{h}(t)| \le B(t) \le B_- \le \beta c^{1-\varepsilon} \qquad \forall t \ge 0. \tag{4.14}$$

We can now show $\dot{h}(t) \to 0$ and control the rate of convergence. We first define $g(t) = e^{\frac{\theta}{c^3} t} \dot{h}(t)$. We have

$$\dot{h}(t) = e^{-\frac{\theta}{c^3} t} g(t) \quad \ddot{h}(t) = -\frac{\theta}{c^3} e^{-\frac{\theta}{c^3} t} g(t) + e^{-\frac{\theta}{c^3} t} \dot{g}(t) = -\frac{\theta}{c^3} \dot{h}(t) + e^{-\frac{\theta}{c^3} t} \dot{g}(t),$$

so that (4.7) becomes

$$\begin{aligned}
\dot{g}(t) &= [\frac{\theta}{c^3} Id + \Gamma_v] \cdot g(t) \\
&\quad + \frac{\theta}{4\pi c^5} \iiint_{|z|\le ct} dx\, dy\, dz\, \frac{\rho_2(y-z)\rho_2(y)}{|z|} \\
&\qquad \times \Big[\Big(\int_{t-\frac{|z|}{c}}^{t} (t - \frac{|z|}{c} - s) e^{\frac{\theta}{c^3}(t-s)} g(s) ds \Big) \cdot \nabla\rho_1(x + v\frac{|z|}{c}) \Big] \nabla\rho_1(x) \\
&\quad - \frac{1}{4\pi c^2} \iiint_{|z|\le ct} dx\, dy\, dz\, \frac{\rho_2(y-z)\rho_2(y)}{|z|} \nabla\rho_1(x) \\
&\qquad\qquad \times \frac{1}{2} \Big\langle \mathrm{Hess}\rho_1(\tilde{x}_{t,|z|}) \int_{t-\frac{|z|}{c}}^{t} e^{\frac{\theta}{c^3}(t-s)} g(s) ds; \int_{t-\frac{|z|}{c}}^{t} \dot{h}(s) ds \Big\rangle \\
&\quad - \frac{1}{4\pi c^2} \iiint_{|z|\le ct} dx\, dy\, dz\, \frac{\rho_2(y-z)\rho_2(y)}{|z|} \\
&\qquad \times \Big[\Big(\int_{t-\frac{|z|}{c}}^{t} (t - \frac{|z|}{c} - s) e^{\frac{\theta}{c^3}(t-s)} \dot{g}(s) ds \Big) \cdot \nabla\rho_1(x + v\frac{|z|}{c}) \Big] \nabla\rho_1(x) \\
&\quad + e^{\frac{\theta}{c^3}t} \tilde{A}(t). \qquad (4.15)
\end{aligned}$$

Note that in the third term of the right-hand side we only replaced one factor $\dot{h}(s)$ by $e^{-\frac{\theta}{c^3}s} g(s)$. We will use (4.14) to control the other one. We define

$$M(t) = \sup_{0\le s\le t} |g(s)|$$

and

$$N(t) = \sup_{0\le s\le t} |\dot{g}(s)|.$$

Writing $R(t)$ for the right-hand side of (4.15) and using (4.14) to control its third term, we find

$$\begin{aligned}
|R(t)| &\le (\frac{\theta}{c^3} - \frac{\gamma}{c^3}) M(t) + \frac{\theta}{c^7} K_4 e^{2\theta \frac{R_2}{c^4}} M(t) + \frac{K_5 e^{2\theta\frac{R_2}{c^4}}}{c^{3+\varepsilon}} M(t) \\
&\qquad + \frac{K_6}{c^4} e^{2R_2 \frac{\theta}{c^4}} N(t) + e^{\frac{\theta}{c^3}t} |\tilde{A}(t)|.
\end{aligned}$$

So we have, for all $s \ge 0$,

$$\begin{aligned}
|\dot{g}(s)| &\le \Big(\frac{\theta}{c^3} - \frac{\gamma}{c^3} + \frac{K_5 e^{2\theta\frac{R_2}{c^4}}}{c^{3+\varepsilon}} + \frac{\theta}{c^7} K_4 e^{2\theta\frac{R_2}{c^4}} \Big) M(s) \\
&\qquad + N(s) \frac{K_6}{c^4} e^{2R_2\frac{\theta}{c^4}} + e^{\frac{\theta}{c^3}s} |\tilde{A}(s)|. \qquad (4.16)
\end{aligned}$$

Taking the supremum over all $s \in [0,t]$, first in the right-hand side and then in the left-hand side of this inequality, we obtain

$$\begin{aligned}
N(t)[1 - \frac{K_6}{c^4} e^{2R_2\frac{\theta}{c^4}}] &\le \Big(\frac{\theta}{c^3} - \frac{\gamma}{c^3} + \frac{K_5 e^{2\theta\frac{R_2}{c^4}}}{c^{3+\varepsilon}} + \frac{\theta}{c^7} K_4 e^{2\theta\frac{R_2}{c^4}} \Big) \\
&\qquad \times M(t) + \sup_{0\le s\le t} (e^{\frac{\theta}{c^3}s} |\tilde{A}(s)|). \qquad (4.17)
\end{aligned}$$

We denote by k_c the inverse of the factor of $N(t)$ in (4.17). Remark that $k_c \sim 1 + \frac{k}{c^4}$. Then, using (4.11), we have

$$N(t) \leq k_c\Big(\frac{\theta}{c^3} - \frac{\gamma}{c^3} + \frac{\theta}{c^7} K_4 e^{2\theta\frac{R_2}{c^4}} + \frac{K_5}{c^{3+\varepsilon}} e^{2\theta\frac{R_2}{c^4}}\Big) M(t) + k_c A c^{2-\varepsilon} \sup_{0\leq s\leq t} [\frac{e^{\frac{\theta}{c^3}s}}{(1+cs)^\nu}].$$

A simple computation shows there exists $s_* > 0$ so that

$$\begin{aligned}
\sup_{0\leq s\leq t}[\frac{e^{\frac{\theta}{c^3}s}}{(1+cs)^\nu}] &= \frac{e^{\frac{\theta}{c^3}t}}{(1+ct)^\nu} \qquad \forall t \geq t_* = \frac{s_*}{c}, \\
&= 1 \qquad \forall t \leq t_* = \frac{s_*}{c}.
\end{aligned}$$

Now we have, for all $0 \leq s \leq t$,

$$\begin{aligned}
|g(s)| &\leq |g(0)| + \int_0^s |\dot{g}(u)| du \\
&\leq |g(0)| + \int_0^s N(u) du \\
&\leq |g(0)| + \int_0^t N(u) du,
\end{aligned}$$

so that

$$\begin{aligned}
M(t) &\leq |M(0)| + \int_0^t N(u) du \\
&\leq |g(0)| + k_c\Big(\frac{\theta}{c^3} - \frac{\gamma}{c^3} + \frac{\theta}{c^7} K_4 e^{2\theta\frac{R_2}{c^4}} + \frac{K_5}{c^{3+\varepsilon}} e^{2\theta\frac{R_2}{c^4}}\Big) \int_0^t M(u) du \\
&\qquad + k_c A c^{2-\varepsilon} \int_0^t \frac{e^{\frac{\theta}{c^3}u}}{(1+cu)^\nu} du + k_c A c^{2-\varepsilon} t_*.
\end{aligned}$$

It is now easy to see that for all $\eta > 0$, there exists $c_0(\rho, K, F_0, \epsilon, \eta)$ so that, for all $c > c_0$ one has

$$k_c\Big(\frac{\theta}{c^3} - \frac{\gamma}{c^3} + \frac{\theta}{c^7} K_4 e^{2\theta\frac{R_2}{c^4}} + \frac{K_5}{c^{3+\varepsilon}} e^{2\theta\frac{R_2}{c^4}}\Big) \leq (\theta - \gamma(1-\eta))\frac{1}{c^3}.$$

Using the Gronwall lemma [ZQ], we have

$$\begin{aligned}
|g(t)| \leq M(t) &\leq (|g(0)| + k_c A c^{2-\varepsilon} t_*) e^{(\theta - \gamma(1-\eta))\frac{t}{c^3}} \\
&\qquad + k_c A c^{2-\varepsilon} \Big(\int_0^t \frac{e^{\frac{\gamma}{c^3}(1-\eta)s}}{(1+cs)^\nu} ds\Big) e^{(\theta-\gamma(1-\eta))\frac{t}{c^3}}. \quad (4.18)
\end{aligned}$$

Remembering that $\dot h(t) = e^{-\frac{\theta}{c^3}t}g(t)$, this yields

$$\begin{aligned}
|\dot h(t)| &\leq (|\dot h(0)| + k_c A c^{2-\varepsilon} t_*) e^{-\frac{\gamma}{c^3}(1-\eta)t} + k_c A c^{2-\varepsilon}\Big(\int_0^t \frac{e^{\frac{\gamma}{c^3}(1-\eta)s}}{(1+cs)^\nu} ds\Big) e^{-\frac{\gamma}{c^3}(1-\eta)t} \\
&\leq (|\dot h(0)| + k_c A c^{2-\varepsilon} t_*) e^{-\frac{\gamma}{c^3}(1-\eta)t} + k_c A c^{2-\varepsilon}\Big(\int_0^{\frac{t}{2}} \frac{e^{\frac{\gamma}{c^3}(1-\eta)s}}{(1+cs)^\nu} ds\Big) e^{-\frac{\gamma}{c^3}(1-\eta)t} \\
&\qquad + k_c A c^{2-\varepsilon}\Big(\int_{\frac{t}{2}}^t \frac{e^{\frac{\gamma}{c^3}(1-\eta)s}}{(1+cs)^\nu} ds\Big) e^{-\frac{\gamma}{c^3}(1-\eta)t} \\
&\leq (|\dot h(0)| + k_c A c^{2-\varepsilon} t_*) e^{-\frac{\gamma}{c^3}(1-\eta)t} + k_c A c^{2-\varepsilon} \frac{e^{-\frac{\gamma}{c^3}(1-\eta)t}}{\frac{\gamma}{c^3}(1-\eta)} [e^{\frac{\gamma}{c^3}(1-\eta)\frac{t}{2}} - 1] \\
&\qquad + k_c A c^{2-\varepsilon} \int_{\frac{t}{2}}^t \frac{ds}{(1+cs)^\nu} \\
&\leq (|\dot h(0)| + k_c A c^{2-\varepsilon} t_*) e^{-\frac{\gamma}{c^3}(1-\eta)t} + \frac{k_c}{\gamma(1-\eta)} A c^{5-\varepsilon} e^{-\frac{\gamma}{c^3}(1-\eta)\frac{t}{2}} \\
&\qquad + \frac{k_c A c^{1-\varepsilon}}{(\nu-1)(1+c\frac{t}{2})^{\nu-1}}.
\end{aligned}$$

Consequently

$$\dot h(t) = O(t^{1-\nu}),$$

so that we can conclude that there exists $q_\infty(Y_0, K, F, \varepsilon) \in \mathbb{R}^d$ with the property that

$$q(t) = q_\infty + v(-F)t + O(t^{2-\nu}),$$

which proves the first part of the theorem.

If ϕ_0 and π_0 are compactly supported, so is $\tilde A(t)$. In that case, (4.17) becomes

$$N(t) \leq k_c\Big(\frac{\theta}{c^3} - \frac{\gamma}{c^3} + \frac{\theta}{c^7}K_4 e^{2\theta\frac{R_2}{c^4}} + \frac{K_5}{c^{3+\varepsilon}} e^{2\theta\frac{R_2}{c^4}}\Big) M(t) + \tilde N$$

where $\tilde N$ is a number which depends on everything except t, yielding in stead of (4.18)

$$|g(t)| \leq |g(0)| e^{(\theta-\gamma(1-\eta))\frac{t}{c^3}} + \tilde N e^{(\theta-\gamma(1-\eta))\frac{t}{c^3}}$$

and hence

$$|\dot h(t)| \leq (|\dot h(0)| + \tilde N) e^{-\frac{\gamma}{c^3}(1-\eta)t}.$$

from which the announced behaviour of $q(t)$ follows again.

□

5 The confined case

We turn onto the case of a confining potential. We make the following assumptions on V and ρ_2:

(C) $\lim_{|q|\to+\infty} |V(q)| = +\infty$

(W) $\hat\rho_2(k) \neq 0$ for all $k \in \mathbb{R}^3$

Let $S = \{q^* \in \mathbb{R}^d | \nabla V(q^*) = 0\}$ be the set of critical points of V. We suppose that S is discrete. For all $q \in \mathbb{R}^d$, we denote by ϕ_q the solution of (1.2). Therefore, $\{(\phi_q, q, 0, 0) | q \in S\}$ is the set of the equilibrium points for the dynamics.

Theorem 3. *Suppose that* (H_1), (H_2), (C) *and* (W) *are satisfied and* $n = 3$. *Denote by* $Y(t) = (\phi(t); q(t); \pi(t); p(t))$ *the solution of* (3.1). *For all* $Y_0 \in \mathcal{D}$, *there exists* $q^* \in S$ *such that:*

$$\lim_{t \to +\infty} q(t) = q^*$$

and

$$\lim_{t \to +\infty} \dot{q}(t) = 0.$$

If moreover, q^* *is a non-degenerate minimum for* V, *then there exists for all* $\eta < 1$ *a* $c_0 > 0$ *such that for any* $c > c_0$ *and for all* ϕ_0 *and* π_0 *with compact support, we have:*

$$q(t) = q^* + \varepsilon(t)$$

with $|\varepsilon(t)| \leq K e^{\frac{-\delta}{c^3}}$ *and* $\eta \frac{\gamma}{2} < \delta < \frac{\gamma}{2}$. *A similar result holds for* $\dot{q}(t)$.

One could also, as in [KKS1], study the convergence of $\phi(t)$ to ϕ_{q^*}, but we shall not do this here. Note that the first part of the theorem does not require c to be large. In fact, using the linearization method of [KKS1], one could prove the convergence is exponential for all c as well. In this way, we do not, however, obtain a very explicit expression for the exponential rate of decay. Our method here show it to be equal to $\frac{\gamma}{2c^3}$, confirming the solutions in this model behave very much like those of the phenomenological equation (1.1).

Proof: In order to prove the convergence of $q(t)$ and $\dot{q}(t)$, we follow the method of [KKS1]. The exponential rate is then obtained by the same techniques as in the case $V(q) = -F \cdot q$.

Using the conservation of energy and the hypothesis on V, one knows that $q(t)$, $\dot{q}(t)$ and $\ddot{q}(t)$ are bounded. Let $B_R \subset \mathbb{R}^3$ be the ball of radius R centered at 0. We define:

$$\begin{aligned} E_R(t) \quad = \quad & \frac{p^2(t)}{2} + V(q(t)) + \frac{1}{2} \int_{\mathbb{R}^d} dx \int_{B_R} dy \, (c^2 |\nabla_y \phi(x,y,t)|^2 + |\pi(x,y,t)|^2) \\ & + \int_{\mathbb{R}^{d+n}} dx \, dy \, \rho(x - q(t), y) \phi(x,y,t). \end{aligned} \tag{5.1}$$

We take $R \geq R_2$. Using (3.10), we can write ϕ as $\phi^r + \phi^0$ and in a similar way, $\pi = \pi^r + \pi^0$ where $\pi^r(x,y,t) = \dot{\phi}^r(x,y,t)$ and $\pi^0(x,y,t) = \dot{\phi}^0(x,y,t)$. Using this decomposition and the regularity of ϕ_0 and π_0, we see that $\phi(x,y,t)$ and $\pi(x,y,t)$ are differentiable. Let us write $n(y) = \frac{y}{|y|}$ and let $d\sigma$ be the surface area element of ∂B_R. Then differentiating (5.1), we have

$$\begin{aligned} \frac{d}{dt} E_R(t) \quad &= \quad \frac{d}{dt} \Big(H(Y(t)) - \frac{1}{2} \int_{\mathbb{R}^d} dx \int_{|y| > R} dy \, (c^2 |\nabla_y \phi(x,y,t)|^2 + |\pi(x,y,t)|^2) \Big) \\ &= \quad c^2 \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y) \, n(y) \cdot \nabla_y \phi(x,y,t) \pi(x,y,t) \\ &= \quad c^2 \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y) \, n(y) \cdot (\nabla_y \phi^r(x,y,t) \pi^r(x,y,t) + \nabla_y \phi^r(x,y,t) \\ & \qquad \times \pi^0(x,y,t) + \nabla_y \phi^0(x,y,t) \pi^r(x,y,t) + \nabla_y \phi^0(x,y,t) \pi^0(x,y,t)). \end{aligned}$$

We bound the three last terms by the Young inequality, and we then integrate in t. Hence, for all $T > 0$,

$$\begin{aligned}
&E_R(T+\frac{R}{c}) - E_R(\frac{R+R_2}{c}) \\
\leq\ & c^2 \int_{\frac{R+R_2}{c}}^{T+R} dt \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y) \Big(n(y)\cdot\nabla_y \phi^r(x,y,t)\pi^r(x,y,t) \\
&+\frac{c^2}{4}(|\nabla_y\phi^r(x,y,t)|^2 + |\pi^r(x,y,t)|^2) + 2(|\nabla_y\phi^0(x,y,t)|^2 + |\pi^0(x,y,t)|^2)\Big).
\end{aligned}$$

We know that $E_R(\frac{R+R_2}{c}) \leq H(Y(\frac{R+R_2}{c})) = H(Y_0)$, and the hypothesis on the potential and (3.13) tell us that:

$$\begin{aligned}
E_R(t) &\geq H(Y(t)) - \frac{1}{2}(\|\pi\|_2^2 + c^2\|\phi\|^2) \\
&\geq -H(Y_0) + 2V_0 + \frac{2}{c^2}\langle \rho; \Delta_y^{-1}\rho\rangle
\end{aligned}$$

where V_0 is the infimum of V. So we have

$$E_R(T+\frac{R}{c}) - E_R(\frac{R+R_2}{c}) \geq -C$$

where C is a constant not depending on R, T. Hence,

$$-c^2 \int_{\frac{R+R_2}{c}}^{T+\frac{R}{c}} dt \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y) \Big(n(y)\cdot\nabla_y\phi^r(x,y,t)\pi^r(x,y,t) + \frac{1}{4}(|\nabla_y\phi^r|^2 + |\pi^r|^2)\Big)$$

$$\leq C + 2c^2 \int_{\frac{R+R_2}{c}}^{T+\frac{R}{c}} dt \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y)(|\nabla_y\phi^0(x,y,t)|^2 + |\pi^0(x,y,t)|^2). \quad (5.2)$$

We first need to bound the right-hand side. This is Lemma 3.3 of [KKS1]:

$$\int_{\frac{R+R_2}{c}}^{T+\frac{R}{c}} dt \int_{\mathbb{R}^d} dx \int_{\partial B_R} d\sigma(y)(|\nabla_y\phi^0(x,y,t)|^2 + |\pi^0(x,y,t)|^2) \leq I_0$$

uniformly in R and T.

Then using (4.2), we have

$$\phi^r(x,y,t) = \frac{-1}{4\pi c^2} \int_{|y-z|\leq ct} dz\, \frac{\rho_2(z)}{|y-z|}\rho_1\left(x - q(t - \frac{|y-z|}{c})\right).$$

If $|y| = R$, because $\rho_2(z) = 0$ for $|z| \geq R_2$, we have for $t > \frac{R+R_2}{c}$

$$\phi^r(x,y,t) = \frac{-1}{4\pi c^2} \int_{|z|<R_2} dz\, \frac{\rho_2(z)}{|y-z|}\rho_1\left(x - q(t - \frac{|y-z|}{c})\right).$$

Consequently, still for $|y| = R$ and $t > \frac{R+R_2}{c}$

$$\pi^r(x,y,t) = \dot{\phi}^r(x,y,t) = \frac{-1}{4\pi c^2} \int_{|z|<R_2} dz\, \frac{\rho_2(z)}{|y-z|}\frac{\partial}{\partial t}\rho_1\left(x - q(t - \frac{|y-z|}{c})\right) \quad (5.3)$$

and

$$
\begin{aligned}
\nabla_y \phi^r(x,y,t) \quad = \quad & \frac{1}{4\pi c^3}\int_{|z|<R_2} dz\, \frac{\rho_2(z)}{|y-z|}\frac{\partial}{\partial t}\rho_1\left(x-q(t-\frac{|y-z|}{c})\right) n(y) \\
& +\frac{1}{4\pi c^2}\int_{|z|<R_2} dz\, \frac{\rho_2(z)}{|y-z|^2}\rho_1\left(x-q(t-\frac{|y-z|}{c})\right) n(y-z) \\
& +\frac{1}{4\pi c^3}\int_{|z|<R_2} dz\, \frac{\rho_2(z)}{|y-z|}\frac{\partial}{\partial t}\rho_1\left(x-q(t-\frac{|y-z|}{c})\right) \\
& \qquad\qquad\qquad\qquad \times(n(y-z)-n(y)).
\end{aligned}
$$

The last two integrals are bounded by KR^{-2} because $\dot q$ is bounded. Hence

$$
\nabla_y \phi^r(x,y,t) = -\frac{1}{c}\pi^r(x,y,t)n(y) + \mathcal{O}(|y|^{-2}).
$$

Since we know that $q(t)$ is bounded by some constant Q_0, for $|x|>Q_0+R_1=\tilde R_1$ we have $\phi^r(x,y,t)=\pi^r(x,y,t)=0$. So (5.2) becomes

$$
\int_{\frac{R+R_2}{c}}^{T+\frac{R}{c}} dt \int_{|x|<Q_0+R_1} dx \int_{\partial B_R} d\sigma(y)\, |\pi^r(x,y,t)|^2 \le K + T\mathcal{O}(R^{-2})
$$

and using once again (5.3) we have

$$
\begin{aligned}
&\int_{\frac{R+R_2}{c}}^{T+\frac{R}{c}} dt \int_{B_{\tilde R_1}} dx \int_{\partial B_R} d\sigma(y)\Big| \int_{B_{R_2}} dz \frac{\rho_2(z)}{|y-z|}\frac{\partial}{\partial t}\rho_1(x-q(t-\frac{|y-z|}{c}))\Big|^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \le K + T\mathcal{O}(R^{-2}).
\end{aligned}
$$

But $|y-z|\sim R$ and $t+\frac{R}{c}-\frac{|y-z|}{c} = t+n(y)\cdot\frac{z}{c}+\mathcal{O}(R^{-1})$, so

$$
\begin{aligned}
&\int_{\frac{R_2}{c}}^{T} dt \int_{B_{\tilde R_1}} dx \int_{\partial B_R} d\sigma(y)\,\Big|\int_{B_{R_2}} dz\, \frac{\rho_2(z)}{|y-z|}\frac{\partial}{\partial t}\rho_1\Big(x-q(t+n(y)\cdot\frac{z}{c}+\mathcal{O}(R^{-1}))\Big)\Big|^2. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \le K + T\mathcal{O}(R^{-2})
\end{aligned}
$$

We now take the limit $R\to+\infty$ and so

$$
\int_{\frac{R_2}{c}}^{T} dt \int_{B_{\tilde R_1}} dx \int_{S^2} d\sigma(y)\Big|\int_{B_{R_2}} dz\, \rho_2(z)\frac{\partial}{\partial t}\rho_1\Big(x-q(t+n(y)\cdot\frac{z}{c})\Big)\Big|^2 \le K.
$$

This bounds holds for all T and so

$$
\begin{aligned}
&\int_0^{+\infty} dt\Big[\int_{B_{\tilde R_1}} dx \int_{S^2} d\sigma(y)\Big|\int_{B_{R_2}} dz\rho_2(z)\nabla\rho_1\Big(x-q(t+n(y)\cdot\frac{z}{c})\Big) \qquad (5.4)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot\dot q(t+n(y)\cdot\frac{z}{c})\Big|^2\Big] \quad < \quad +\infty.
\end{aligned}
$$

We define the function

$$
I(x,\sigma,t) = \Big|\int_{B_{R_2}} dz\,\rho_2(z)\nabla\rho_1\Big(x-q(t+\sigma\cdot\frac{z}{c})\Big)\cdot\dot q(t+\sigma\cdot\frac{z}{c})\Big|^2
$$

which is differentiable in x, σ and uniformly Lipschitz in t because $\dot q$ and $\ddot q$ are bounded. As a result

$$\lim_{t\to+\infty} I(x,\sigma,t)=0 \tag{5.5}$$

uniformly in $x\in B_{\tilde R_1}$ and $\sigma\in S^2$. We fix σ and x. We take a basis of $\mathbb{R}^3$ such that $\sigma=e_1$, and we define

$$\tilde\rho_2(z_1)=\int dz_2\,dz_3\rho_2(z_1,z_2,z_3)$$

and $s=\sigma\cdot z$. Then we have

$$\begin{aligned}
I(x,\sigma,t) &= \Big|\int ds\,\tilde\rho_2(s)\nabla\rho_1\Big(x-q(t+\frac{s}{c})\Big)\Big)\cdot\dot q(t+\frac{s}{c})\Big|^2\\
&= c^2\Big|\int ds\,\tilde\rho_2(c(t-s))\nabla\rho_1\Big(x-q(s)\Big)\Big)\cdot\dot q(s)\Big|^2\\
&= c^2|\bar\rho_2\star(\nabla\rho_1(x-q)\cdot\dot q)(t)|^2
\end{aligned}$$

with $\bar\rho_2(t)=\tilde\rho(ct)$. Then (5.5) leads to

$$\lim_{t\to+\infty}\bar\rho_2\star(\nabla\rho_1(x-q)\cdot\dot q)(t)=0.$$

Hence (W) and Pitt's extension to Wiener's Tauberian theorem [R] implies

$$\lim_{t\to+\infty}\nabla\rho_1(x-q(t))\cdot\dot q(t)=0$$

uniformly in x,σ. So we have

$$\begin{aligned}
0 &= \lim_{t\to+\infty}\sup_{x\in B_{\tilde R_1}}\nabla\rho_1(x-q(t))\cdot\dot q(t)\\
&= \lim_{t\to+\infty}\sup_{x\in\mathbb{R}^d}\nabla\rho_1(x-q(t))\cdot\dot q(t)
\end{aligned}$$

because $\nabla\rho_1(x-q(t))=0$ if $|x|>\tilde R_1$. Hence

$$\lim_{t\to+\infty}\nabla\rho_1(x)\cdot\dot q(t)=0\quad\forall x\in\mathbb{R}^d$$

which proves that $\dot q(t)$ tends to zero. It remains to show that $q(t)$ converges to some q^* which satisfies $\nabla V(q^*)=0$. Remember that ϕ_q is the solution of (1.2) corresponding to $q(t)=q$. Let $\mathcal{A}=\{Y_q=(\phi_q,q,0,0)\quad q\in\mathbb{R}^d,\ |q|\le Q_0\}$. $\mathcal{A}$ is compact in $\mathcal{E}$. Finally, we note by $\|.\|_R$ the L^2 norm restricted to the ball of radius R and $|Y|^2_{\mathcal{E},R}=\|\nabla_y\phi\|^2_R+|q|^2+\|\pi\|^2_R+|p|^2$. We first prove

$$\begin{aligned}
&\inf_{Y_q\in\mathcal{A}}|Y(t)-Y_q|^2_{\mathcal{E},R} &(5.6)\\
=\;& |p(t)|^2+\|\pi(t)\|^2_R+\inf_{|q|\le Q_0}(\|\nabla_y(\phi(t)-\phi_q)\|^2_R+|q(t)-q|^2\to_{t\to+\infty}0.
\end{aligned}$$

We know that $|p(t)|\to0$ as $t\to+\infty$. Then (5.3) implies that $\|\pi^r(t)\|_R\to0$ as $t\to+\infty$. The bound on $\|\pi^0(t)\|_R$ (see Lemma 3.3 of [KKS1]) then shows that the same result holds for $\|\pi(t)\|_R$. To estimate the infimum over q in (5.6) we take $q(t)$ for q. Then the last term vanishes and we have to control

$$\begin{aligned}\nabla_y(\phi^r(x,y,t)-\phi_{q(t)}(x,y)) &= \nabla_y\Big[\frac{-1}{4\pi c^2}\int dz\frac{\rho_2(y-z)}{|z|}\Big(\rho_1(x-q(t-\frac{|z|}{c}))\\ &\qquad -\rho_1(x-q(t))\Big)\Big]\end{aligned}$$

for $|y|\le R$, the term $\nabla_y\phi^0$ being controlled using once again Lemma 3.3 of [KKS1]. The difference $\rho_1(x-q(t-\frac{|z|}{c}))-\rho_1(x-q(t))$ can be written using an integral depending only on $\dot q(s)$ for $s\in[t-\frac{R+R_2}{c},t]$ which tends to zero as t goes to infinity uniformly in $(x,y)\in B_R$. All this proves (5.6).

Given a solution $Y(t)$ of (3.1), we call $\mathcal{B}$ the set of all $\bar Y\in\mathcal{E}$ such that there exists some sequence $t_n\to+\infty$ with $Y(t_n)\to\bar Y$ in the norm $|.|_{\mathcal{E},R}$ for all R. The continuity of W^t tells us that $\mathcal{B}$ is an invariant set. Then, (5.6) tells us that $\mathcal{B}\subset\mathcal{A}$. So, for $\bar Y\in\mathcal{B}$ there exists a C^2 curve $t\to\tilde q(t)\in\mathbb{R}^d$ such that $W^t\bar Y=Y_{\tilde q(t)}$. But $W^t\bar Y$ is a solution of (3.1) so we must have $\dot{\tilde q}(t)=0$, hence $\tilde q(t)=q^*$ with $\nabla V(q^*)=0$ and $q^*\in S$. Therefore $\bar Y=Y_{q^*}$ and $\mathcal{B}\subset\{Y_q,\,q\in S\}$.

We now prove that $q(t)\to q^*$. Suppose there exists $R,\epsilon>0$ and a sequence $t_n\to+\infty$ such that

$$\inf_{q\in S}|Y(t_n)-Y_q|_{\mathcal{E},R}\ge\epsilon. \tag{5.7}$$

But (5.6) and the compactness of $\mathcal{A}$ imply that there exists $\bar Y\in\mathcal{A}$ and a subsequence t'_n such that $Y(t'_n)\to\bar Y$ in the norm $|.|_{\mathcal{E},R}$ for all R, where $\bar Y\in\mathcal{A}$. Then, by definition, $\bar Y\in\mathcal{B}$. But (5.7) is then a contradiction to $\mathcal{B}\subset\{Y_q,\,q\in S\}$. So

$$\inf_{q\in S}|q(t)-q|\to 0$$

and because S is discrete, there exists $q^*\in S$ such that $q(t)\to q^*$.

We now suppose that q^* is a non-degenerate minimum for V. Because of the translation invariance of the interaction term, we can suppose that $q^*=0$. Hence, the same computation as in Section 4 leads to:

$$\begin{aligned}\ddot q(t) &= -\nabla V(q(t))-\frac{\gamma}{c^3}\dot q(t)-\frac{1}{4\pi c^2}\iiint_{|z|\le ct}dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\ &\qquad\times\Big[\Big(\int_{t-\frac{|z|}{c}}^t(t-\frac{|z|}{c}-s)\ddot q(s)ds\Big)\cdot\nabla\rho_1(x)\Big]\nabla\rho_1(x)\\ &\quad -\frac{1}{4\pi c^2}\iiint_{|z|\le ct}dx\,dy\,dz\,\frac{\rho_2(y-z)\rho_2(y)}{|z|}\\ &\qquad\times\frac12\Big\langle\mathrm{Hess}\rho_1(\tilde x_{t,|z|})\int_{t-\frac{|z|}{c}}^t\dot q(s)ds;\int_{t-\frac{|z|}{c}}^t\dot q(s)ds\Big\rangle\nabla\rho_1(x)\\ &\quad +\tilde A(t).\end{aligned} \tag{5.8}$$

Moreover, $\nabla V(q(t))=W\cdot q(t)+f(q(t))$ where W is the Hessian matrix of V at $q=0$ and $f(q)=o(|q|)$. Remark that $\tilde A(t)$ has compact support. We now define $Q(t)=(q(t),\dot q(t))\in\mathbb{R}^{2d}$ and $\tilde W$ the $2d\times 2d$ matrix:

$$\tilde W=\begin{pmatrix} O & I\\ -W & -\frac{\gamma}{c^3}I\end{pmatrix}.$$

Since 0 is a non-degenerate minimum for V, W is a diagonalizable positive definite matrix. One should remark that $\tilde{W}$ is diagonalizable as well with eigenvalues $\lambda_k = -\frac{\gamma}{2c^3} + i\alpha_k$, and so for all t in $\mathbb{R}$, $\|e^{\tilde{W}t}\| = e^{-\frac{\gamma}{2c^3}t}$. We rewrite (5.8) as

$$\dot{Q}(t) = \tilde{W}Q(t) + \psi(Q, \dot{Q})$$

where ψ is a function we will control in terms of $|\ddot{q}(t)|$, $|\dot{q}(t)|$ and $|q(t)|$. Defining $X(t) = e^{-\tilde{W}t}Q(t) = (x_1(t), x_2(t))$ we have:

$$\dot{X}(t) = e^{-\tilde{W}t}\psi(e^{\tilde{W}t}X(t), \tilde{W}e^{\tilde{W}t}X(t) + e^{\tilde{W}t}\dot{X}(t))$$

$$\begin{aligned}
|\dot{X}(t)| \quad \leq \quad & \frac{1}{4\pi c^2}\iiint_{|z|\leq ct} dx\, dy\, dz\, \frac{\rho_2(y-z)\rho_2(y)}{|z|}\Big(\int_{t-\frac{|z|}{c}}^{t}(t-\frac{|z|}{c}-s) \\
& \qquad \times e^{\frac{\gamma}{2c^3}(t-s)}(|\tilde{W}X(s)| + |\dot{X}(s)|)ds\Big)|\nabla\rho_1(x)|^2 \\
& + \frac{1}{4\pi c^2}\iiint_{|z|\leq ct} dx\, dy\, dz\, \frac{\rho_2(y-z)\rho_2(y)}{|z|} \times \frac{1}{2}|\mathrm{Hess}\rho_1(\tilde{x}_{t,|z|})| \\
& \times\Big(\int_{t-\frac{|z|}{c}}^{t} e^{\frac{\gamma}{2c^3}(t-s)}|X(s)|ds\Big)\Big(\int_{t-\frac{|z|}{c}}^{t}|Q(s)|ds\Big)|\nabla\rho_1(x)| \\
& + e^{\frac{\gamma}{2c^3}t}|\tilde{A}(t)| + e^{\frac{\gamma}{2c^3}t}|f(q(t))|. \qquad (5.9)
\end{aligned}$$

Let $\varepsilon > 0$, $f(q) = o(|q|)$ so there exists $\delta > 0$ such that for all $|q| < \delta$, $|f(q)| < \varepsilon|q| < \varepsilon|Q|$. We define $\eta = \min(\delta, \varepsilon)$. Moreover, we already know that $Q(t) \to 0$, so there exists T such that for all $t \geq T$, $|Q(t)| \leq \eta$. Using the fact that the evolution of the solution $Y(t)$ of (3.1) is given by a continuous linear group and that $Y(t)$ satisfies the same conditions as Y_0, we can suppose $T = 0$. So we have

$$\forall t \geq 0, |Q(t)| < \eta < \varepsilon \qquad \text{and} \qquad |f(q)| < \varepsilon|Q|. \qquad (5.10)$$

As in Section 4, we define

$$M(t) = \sup_{0\leq s\leq t}|Y(s)|$$

and

$$N(t) = \sup_{0\leq s\leq t}|\dot{Y}(s)|$$

Using (5.9) and (5.10), we have for all $0 \leq s \leq t$

$$\begin{aligned}
|\dot{X}(s)| \quad \leq \quad & \frac{K_1}{c^4}e^{\frac{\gamma R_2}{c^4}}\|\tilde{W}\|M(t) + \frac{\varepsilon K_2}{c^4}e^{\frac{\gamma R_2}{c^4}}M(t) + \frac{K_1}{c^4}e^{\frac{\gamma R_2}{c^4}}N(t) \\
& + \sup_{0\leq s\leq t}(\varepsilon e^{\frac{\gamma}{2c^3}s}|Q(s)|) + \sup_{0\leq s\leq t}(e^{\frac{\gamma}{2c^3}s}|\tilde{A}(s)|).
\end{aligned}$$

Remark that $e^{\frac{\gamma}{2c^3}s}|Q(s)| = |e^{-\tilde{W}s}Q(s)| = |X(s)|$ and $\sup_{0\leq s\leq t}(e^{\frac{\gamma}{2c^3}s}|\tilde{A}(s)|) \leq K_3$, so taking the supremum over all $s \in [0,t]$ in the left hand side, we have

$$N(t) \leq \Big(\frac{K_1}{c^4}\|\tilde{W}\|e^{\frac{\gamma R_2}{c^4}} + \frac{\varepsilon K_2}{c^4}e^{\frac{\gamma R_2}{c^4}} + \varepsilon\Big)M(t) + \frac{K_1}{c^4}e^{\frac{\gamma R_2}{c^4}}N(t) + K_3$$

and so

$$(1-\frac{K_1}{c^4}e^{\frac{\gamma R_2}{c^4}})N(t) \le \Big(\frac{K_1}{c^4}\|\tilde{W}\|e^{\frac{\gamma R_2}{c^4}} + \frac{\varepsilon K_2}{c^4}e^{\frac{\gamma R_2}{c^4}} + \varepsilon\Big)M(t) + K_3.$$

We call $(k_c')^{-1}$ the factor of $N(t)$. We can chose ε as small as we want, so the factor of $M(t)$ can be bounded by $\frac{K'}{c^4}$ Then, the same computation as in Section 4 leads to

$$M(t) \le \Big(M(0) + \frac{K_3 c^4}{K'}\Big)e^{\frac{k_c K'}{c^4}t}$$

and finally we have

$$|Q(t)| \le \Big(M(0) + \frac{K_3 c^4}{K'}\Big)e^{(\frac{k_c K'}{c^4} - \frac{\gamma}{2c^3})t}$$

which is the annouced result. □

6 Other models

Various Hamiltonian models for dissipation in general and for linear friction in particular have previously been proposed in the physics literature, mostly with the purpose of deriving the classical or quantum Langevin equation for a particle moving through some medium with which it can exchange energy and momentum (see [CL] and [FLO1] for further references). As in the model we propose here, they all involve the coupling of a particle to a family of oscillators representing the degrees of freedom of the environment and playing the role of a heath bath. We wish to show in this section how these models are related to the one we present in this work and briefly analyse their merits and limitations. We will in particular argue that, unlike the model we presented here, none of the models of the heath bath and of its coupling to the particle considered up to now can correctly describe the behaviour of a particle moving through a homogeneous medium in the presence of each of the three most commonly studied potentials: $V = 0, V = -F \cdot q$ or V confining.

With an eye towards possible generalizations of our work, we will describe a large class of translationally invariant models (to which ours belongs) and derive in a very simple manner physically transparent conditions on the model parameters for it to describe linear friction at low speeds.

It was argued in [MS][CL] that the most general Hamiltonian describing the interaction of a particle with a dissipative environment is of the form

$$H = \frac{p^2}{2m} + V(q) + \sum_{\alpha=1}^{N}\Big[\frac{p_\alpha^2}{2m_\alpha} + \frac{1}{2}m_\alpha\omega_\alpha^2 q_\alpha^2\Big] + \sum_{\alpha=1}^{N}\sigma_\alpha(q)q_\alpha + \sum_{\alpha=1}^{N}W_\alpha(q) \quad (6.1)$$

with

$$W_\alpha(q) = \frac{\sigma_\alpha(q)^2}{2m_\alpha\omega_\alpha^2}. \quad (6.2)$$

The last term, which is just an additional potential for the particle, is referred to as the “counterterm”. Writing

$$V_{eff} = V + W, \qquad W(q) = \sum_{\alpha=1}^{N} W_\alpha(q),$$

one can think of W as a potential due to the presence of the bath, but which is independent of the state of the latter. We'll explain its role below. Of course, for any of these models to describe dissipation, the limit $N \to +\infty$ has to be taken at some point in the analysis in such a way that the frequency spectrum becomes continuous. We find it therefore convenient to write down a slightly more abstract form of the above class of models, incorporating this possibility from the outset. It will also make it easier to show that our model is of the above type and to analyse the conditions that have to be imposed on the many parameters of the Hamiltonian for it to describe dissipation through linear friction.

Let (A, μ) be a measure space and consider the Hamiltonian

$$\begin{aligned} H(q,p,\phi,\pi) &= \frac{p^2}{2m} + V(q) + \int_A d\mu(\alpha) \left[\frac{|\pi_\alpha|^2}{2\rho_\alpha} + \frac{1}{2}\rho_\alpha \omega_\alpha^2 |\phi_\alpha|^2 \right] \\ &\quad + \int_A d\mu(\alpha) \sigma_\alpha(q)\phi_\alpha + \int_A d\mu(\alpha) W_\alpha(q) \end{aligned} \tag{6.3}$$

with the corresponding second order equations of motion

$$\rho_\alpha \ddot{\phi}_\alpha(t) + \rho_\alpha \omega_\alpha^2 \phi_\alpha(t) = -\sigma_\alpha(q(t)),$$

and

$$m\ddot{q}(t) = -\nabla V_{eff}(q(t)) + f(t),$$

where

$$f(t) = -\int \sigma'_\alpha(q)\phi_\alpha \; d\mu(\alpha)$$

is the force exerted on the particle by the oscillator bath. Note that the equations for the "field" variables ϕ_α are linear, reflecting the fact that the Hamiltonian is quadratic in ϕ, π, its only non-linearity being in q. This is of course what makes this type of model tractable.

Needless to say, taking for A the set of N elements and for μ the counting measure, one obtains the class of Hamiltonians proposed by Caldeira-Leggett if we set $\pi_\alpha = p_\alpha, \phi_\alpha = q_\alpha$, *etc.*. Taking, on the other hand, $A = \mathbb{R}^{d+n}$, μ the Lebesgue measure, $\alpha = (x,k), \phi_\alpha = \hat{\phi}(x,k)$, *etc.*, and, most importantly,

$$\sigma_\alpha(q) = \rho_1(x-q)\hat{\rho}_2(k)$$

it is easy to see that the model considered in this paper is also of the above type (with the counterterm identically zero).

Not all choices of the parameters used in (6.3) to describe the heath bath and its coupling to the particle lead to physically reasonable models. We shall derive a certain number of conditions for that purpose. A first obvious requirement is that one and the same choice of these parameters should describe the system correctly for different choices of V. In particular, the model should make sense when $V = 0$. This immediately leads to the following observations, that do not seem to have been made before. When the external potential V vanishes identically, one expects the particle to be able to "sit still" at any point in $\mathbb{R}^d$, in the surrounding medium, itself at rest, without experiencing any net force. In other words, the Hamiltonian must in that case allow for a stationary state with the particle at rest anywhere in space. This is only possible, as is easily checked, if one chooses the counterterm W so that:

$$\nabla W(q) = \nabla U(q), \text{ where } U(q) = \int \frac{\sigma_\alpha(q)^2}{2\rho_\alpha \omega_\alpha^2}\, d\mu(\alpha). \tag{6.4}$$

This first requirement explains therefore very simply the form of the counterterm in (6.2) (which tends to be motivated in a different manner [CL] [CT] [FLO1]). One notices in addition immediately that the counterterm in (6.2) is simply the potential energy of the environment when the system is in its stationary state with the particle at position q! We therefore naturally require

$$U(q) < +\infty \tag{6.5}$$

and note that with $W(q) = U(q)$, one has

$$\begin{aligned} H(q,p,\phi,\pi) \quad &= \quad \frac{p^2}{2m} + V(q) + \\ &\int_A d\mu(\alpha) \left[\frac{|\pi_\alpha|^2}{2\rho_\alpha} + \frac{1}{2}\rho_\alpha\omega_\alpha^2 \left(\phi_\alpha + \frac{\sigma_\alpha(q)}{\rho_\alpha \omega_\alpha^2} \right)^2 \right] \end{aligned} \tag{6.6}$$

so that the resulting Hamiltonian is now clearly bounded below.

As a result, it is clear that in translationally invariant systems (homogeneous medium), the counterterm is constant and can therefore be dropped from the Hamiltonian. Indeed, in that case, this potential energy has to be independent of q so that one can take W identically equal to zero. In conclusion, the Hamiltonian (6.3) can describe the coupling of a particle to a homogeneous dissipative medium only if

$$W(q) = U(q), U(q) = U_* < +\infty, \ \forall q \in \mathbb{R}^d. \tag{6.7}$$

The only model of the class (6.3)-(6.4) that seems to have been actively studied in the literature is the one with linear coupling, *i.e.* $\sigma_\alpha(q) = -\lambda_\alpha q$. (See [CL] [CT] [FLO1] [FLO2] [JP1] and references therein.) The Hamiltonian can then be rewritten

$$H_{CL} = \frac{p^2}{2m} + V(q) + \int_A d\mu(\alpha) \left[\frac{\pi_\alpha^2}{2\rho_\alpha} + \frac{1}{2}\rho_\alpha\omega_\alpha^2 (\phi_\alpha - \frac{\lambda_\alpha}{\rho_\alpha\omega_\alpha^2} q)^2 \right]. \tag{6.8}$$

In the case A is a discrete set with N elements it is easy to see through a simple scaling of the variables that, in one dimension, this model is equivalent to the "independent oscillator model" [FLO1][FLO2] in which the particle is attached with independent springs of spring constant $m'_\alpha \omega'_\alpha$ to N particles of mass m'_α. The presence of a non-trivial counterterm clearly shows this model is not translationally invariant in the above sense, and is therefore not suitable for describing the motion of a particle through a homogeneous medium in the absence of an external potential V. Since there seems to be some confusion about this, it is perhaps useful to point out that the model is nevertheless translationally invariant in a more formal sense. Indeed, this Hamiltonian is clearly invariant under the canonical transformation $q \to q + a, p \to p, q_\alpha \to q_\alpha + \frac{\lambda_\alpha}{m_\alpha \omega_\alpha^2} a, p_\alpha \to p_\alpha$ and can in this sense be claimed to be translationally invariant; it is however not translationally invariant in the physical sense in which that term is used above.

The model (6.8) is nevertheless applicable as an approximation when studying the motion of a particle through a dissipative medium in the presence of a confining potential V. Indeed, supposing the Hamiltonian (6.3)-(6.4) has a fixed point at $q = q_*, p = 0, \phi_\alpha = \phi^*_\alpha, \pi_\alpha = 0$, one sees that q_* is a critical point of V. Correctly linearizing the problem around this fixed point one recovers the Hamiltonian H_{CL} in the variables $q - q_*, \phi_\alpha - \phi^*_\alpha$. This linearized model is therefore a reasonable approximation if one wishes to study the system in a confining potential, with q close to q_* at all times, but not in absence of a confining potential. In particular, it is not suitable for studying the case where a constant external force F is present or when the particle is free ($V = 0$), a fact that does not always seem to be realized [HA].

The only work we are aware of where a non-linear coupling (*i.e.* non-linear σ_α and non-linear V) are considered is [JP2], but the models considered there are not translationally invariant. A large class of translationally invariant models, to which ours belongs, is obtained as follows. One takes

$$A = \mathbb{R}^d \times B, \quad \alpha = (x, \beta) \in \mathbb{R}^d \times B, \quad d\mu = dx d\nu(\beta), \quad \omega_\alpha = \omega_\beta, \rho_\alpha = \rho_\beta \tag{6.9}$$

and, most importantly, the coupling of the form

$$\sigma_\alpha(q) = \sigma_1(x - q)\sigma_2(\beta). \tag{6.10}$$

The idea here is that at each point $x \in \mathbb{R}^d$, there is a family of oscillators indexed by $\beta \in B$ and of frequency ω_β. The particle interacts with the oscillators at x whenever q is such that $\sigma_1(x - q) \neq 0$. In other words, the oscillators form obstacles with which the particle undergoes inelastic collisions. Oscillators at different points in space do not interact.

We will assume from now on that σ_1 is spherically symmetric, smooth and rapidly decreasing and furthermore (with some abuse of notation) that $\sigma_2(\beta) = \sigma_2(\omega_\beta)$, $\rho_\beta = \rho(\omega)$ so that in particular the coupling of the particle to the oscillator β depends only on its frequency. We introduce some handy notations. Let

$$N(\omega) = \int_{\omega_\beta \leq \omega} d\nu(\beta)$$

be the integrated density of frequencies for the model and write $dN = n(\omega)d\omega$. We can then write

$$U_* = \int_0^{+\infty} u(\omega)d\omega, \text{ with } u(\omega) = \| \sigma_1 \|^2 \frac{\sigma_2(\omega)^2}{2\rho(\omega)\omega^2} n(\omega). \tag{6.11}$$

We also introduce

$$K(\lambda) = -\int_{\mathbb{R}^d} dx \ (\partial_1\sigma_1)(x_1, x^\perp)(\partial_1\sigma_1)(x_1 + \lambda, x^\perp), \tag{6.12}$$

and notice that

$$\hat{K}(\xi) = \sqrt{2\pi} \int_{\mathbb{R}^{d-1}} dx^\perp \left[\partial_1 \hat{\sigma}_1(\xi, x^\perp)\right]^2 \leq 0. \tag{6.13}$$

It is then straightforward to apply the reasoning of Section 2 to analyse under what conditions on σ_1, σ_2, ρ and n such a model can describe linear friction. For that purpose we compute in precisely the same manner as in Section 2 the friction force exerted by the medium on the particle if the latter moves at constant velocity v. The result is, after some computation,

$$f(v) = f_r(|v|)\frac{v}{|v|},$$

with

$$f_r(|v|) = \frac{\sqrt{2\pi}}{\| \sigma_1 \|^2}\left[\int_0^{+\infty} d\omega\ u(\omega)\hat{K}(\frac{\omega}{|v|})\right].$$

Since u is integrable, since $\hat{K}(0) = 0$ (because $\partial\sigma_1$ is odd in x_1) and since $\hat{K}$ is rapidly decreasing, we immediately see that $f_r(0) = 0 = \lim_{|v|\to+\infty} f_r(|v|)$, so that it is not possible that $f(v)$ depends linearly on v for all v in any of the models of the above class. Since however f_r is a smooth function of $|v|$, it is reasonable to impose a weaker condition, namely that f_r should vanish linearly as $|v|$ tends to zero, *i.e.* that

$$\tilde{\gamma} = -\lim_{|v|\to 0}\frac{f_r(|v|)}{|v|} = -u(0)\frac{\sqrt{2\pi}}{\| \sigma_1 \|^2}\int_0^{+\infty} d\xi\ \hat{K}(\xi) > 0 \tag{6.14}$$

or, equivalently, that

$$u(0) \neq 0. \tag{6.15}$$

This can be very simply assured by imposing

$$0 < \sigma_2(\omega), \rho(\omega) < +\infty, \quad n(\omega) \sim n_0\omega^2\ (\omega \to 0). \tag{6.16}$$

In other words, if we assume the particle couples non-trivially to the low-frequency oscillators, the condition $u(0) \neq 0$ imposes a very simple condition on the frequency density $n(\omega)$, namely that $n(\omega) \sim \omega^2$ for low frequencies. Of course, in that case, the energy dissipated per unit time, given by $f(v)\cdot v$, will be proportional to $|v|^2$ for small $|v|$. In other words, $n(\omega) \sim \omega^2$ is the correct density of low frequency oscillators needed, as already remarked in Section 2 in the more particular context considered there.

In conclusion, a reasonable class of models in which a *low speed* particle may be expected to experience dissipation through linear friction due to inelastic scattering in a homogeneous medium is given by Hamiltonians of the type (6.3) with the conditions (6.9)-(6.10) and (6.4)-(6.5) for which moreover (6.16) is satisfied.

Note that the "low speed" restriction above is important, since we now that the friction force vanishes as $|v| \to \infty$. Let us define $V_0 > 0$ to be the first maximum of $|f_r(|v|)|$. We expect linear friction to hold only for speeds much smaller than V_0. In addition, if the model is to predict behaviour compatible with equation (1.1), the friction constant $\tilde{\gamma}$ (which has the dimension of an inverse time) determines a relaxation time for the system. Certainly, we expect the particle to travel a "macroscopic" distance on such a time, so we obtain the inequalities

$$R_1 << \frac{|v|}{\tilde{\gamma}} << \frac{V_0}{\tilde{\gamma}}$$

so that

$$V_0 >> \tilde{\gamma} R_1. \tag{6.17}$$

Here R_1 is the (effective) diameter of the support of σ_1. In other words, the models we constructed all contain two "speed scales" and hence a dimensionless parameter $V_0/(\tilde{\gamma} R_1)$. We expect this parameter has to be large for the model to display behaviour compatible with (1.1). This is the origin of the condition "c large enough" in the theorems of this paper. Indeed, in that case $V_0 = cW_0$ and $\tilde{\gamma} = \tilde{\gamma}_*/c^3$ where W_0 and $\tilde{\gamma}_*$ are independent of c so that the above condition becomes

$$c^4 >> \frac{\tilde{\gamma}_* R_1}{W_0}.$$

It is not clear whether the results of this paper will hold for all the above translationally invariant models satisfying (6.16), in the regime (6.17). Indeed, the model we studied in this paper has one extra feature: thanks to Huyghens principle, the membranes rapidly evacuate all energy that is put into them out of reach of the particle (*i.e.* away from $y = 0$). It is not obvious this feature is needed, but it is certainly used in our proofs.

We end this section with a comment on the condition for linear friction proposed in [CL]. Even though it is not possible to arrange things so that $f(v) = -\tilde{\gamma} v$ for all v, it is nevertheless possible to assure this for all $|v| \leq V_M$, as follows. Suppose that

$$u(\omega) = u_0, \ \forall \omega \leq \Omega, \ \text{and} \ u(\omega) = 0, \ \forall \omega > \Omega. \tag{6.18}$$

Then

$$f_r(|v|) = \frac{\sqrt{2\pi} u_0}{\| \sigma_1 \|^2} \left[\int_0^{\Omega/|v|} d\xi \ \hat{K}(\xi) \right] |v|$$

from which the claim follows if $\hat{K}$ is of compact support. If $\hat{K}$ is only rapidly decreasing (such as when σ_1 is of compact support), then the result is still true to very good approximation. The condition for models of the type (6.3) to produce linear friction proposed in [CL] reduces to (6.18) when applied to our class of translationally invariant models. This condition is of course much more restrictive than (6.16), the latter having the added advantage of having a simple physical interpretation.

In more realistic models, one ought to couple the oscillators at different points in space. This is easily done in the context of our models by changing the potential energy of the field into

$$\int dx \ dy \ c_1^2 |\nabla_x \phi(x, y)|^2 + c_2^2 |\nabla_y \phi(x, y)|^2.$$

It turns out, however, that in that case the force $f(v)$ vanishes identically for all $|v| \leq c_1$. In such models, the friction force is therefore proportional to higher derivatives of q. In particular, this is the case when $c_1 = c_2$, as in the model for radiation damping studied in [KKS1][KKS2][KS]. This leads to some very different behaviour. For example, in that case, there exist constant speed solutions for the particle in absence of an external potential V. In a confining potential, the particle still converges exponentially fast to a minimum of the potential, but this time the exponential rate does also depend on the shape of the potential.

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