Ward identities and Chiral anomaly in the Luttinger liquid

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ABSTRACT. *Systems of interacting non relativistic fermions in $d = 1$, as well as spin chains or interacting bidimensional Ising models, verify an hidden approximate Gauge invariance which can be used to derive suitable Ward identities. Despite the presence of corrections and anomalies, such Ward identities can be implemented in a Renormalization Group approach and used to exploit non-trivial cancellations which allow to control the flow of the running coupling constants; in particular this is achieved combining Ward identities, Dyson equations and suitable correction identities for the extra terms appearing in the Ward identities, due to the presence of cutoffs breaking the local gauge symmetry. The correlations can be computed and show a Luttinger liquid behavior characterized by non universal critical indices, so that the general Luttinger liquid construction for one dimensional systems is completed without any use of exact solutions. The ultraviolet cutoff can be removed and a Quantum Field Theory corresponding to the Thirring model is also constructed.*

1. Introduction

1.1 *Luttinger liquids.*

A key notion in solid state physics is the one of *Fermi liquids*, used to describe systems of *interacting* electrons which, in spite of the interaction, have a physical behavior qualitatively similar to the one of the *free Fermi gas*. In analogy with Fermi liquids, the notion of *Luttinger liquids* has been more recently introduced, to describe systems behaving qualitatively as the Luttinger model, see for instance [A] or [Af]; their correlations have an anomalous behavior described in terms of non universal (*i.e.* nontrivial functions of the coupling) critical indices.

A large number of models, for which an exact solution is lacking (at least for the correlation functions) are indeed believed to be in the same class of universality of the Luttinger model or of its massive version, and indeed in the last decade, starting from [BG], it has been possible to substantiate this assumption on a large class of models, by a quantitative analysis based on Renormalization Group techniques, which at the end allow us to write the correlations and the critical indices as convergent series in the coupling. We mention the Schwinger functions of interacting non relativistic fermions in $d = 1$ (modelling the electronic properties of metals so anisotropic to be considered as one dimensional), in the spinless [BGPS], in the spinning case with repulsive interaction [BoM], or with external periodic or quasi-periodic potentials [M]; the spin-spin correlations of the *Heisenberg XYZ spin chain* [BM1], [BM2]; the thermodynamic functions of classical Ising systems on a bidimensional lattice with quartic interactions like the *Eight-vertex* or the *Ashkin-Teller* models [M1]; and many others, see for instance the review [GM].

In all such models the observables are written as Grassmannian integrals, and a naive evaluation of them in terms of a series expansion in the perturbative parameter does not work; it is however possible, by a multiscale analysis based on Renormalization Group, to write the Grassmannian integrals as series of suitable finitely many parameters, called *running coupling constants*, and this expansion is convergent if the running coupling constants are small. The running coupling constants obey a complicated set of recursive equations, whose right hand side is called, as usual, the *Beta function.* The Beta function can be written as sum of two terms; one, which we call *principal part* of the Beta function, is common to all such models while the other one is model dependent. It turns out that, *if* the principal part of the beta function is asymptotically vanishing, than the flow of the running coupling constants in all such models can be controlled just by dimensional bounds, and the expansion is really convergent; the observables are then expressed by explicit convergent series from which the physical information can be extracted. On the other hand the principal part of the Beta function coincides with the Beta function of a model, which we call *reference model*, describing two kind of fermions with a linear "relativistic" dispersion relation and with momenta restricted by infrared and ultraviolet cutoffs, interacting by a local quartic potential.

In order to prove the vanishing of the Beta function of the reference model (in the form of the bound (3.3) below), one can use an indirect argument, based on the fact that the reference model is close to the Luttinger model, and then use the Luttinger model exact solution of [ML], see [BGPS], [BM3].

In this paper we show that indeed the vanishing of the principal part of the Beta function can be proved without any use of the exact solution, by using only Ward identities based on the approximate chiral gauge invariance of the reference model.

As exact solutions are quite rare and generally peculiar to $d = 1$, while RG analysis and Ward identities are general methods working in any dimension, our results might become relevant for the theory of non-Fermi liquids in $d > 1$. We remember that there is experimental evidence for non Fermi liquid (probably Luttinger) behavior in high T_c superconductors [A], which are essentially planar systems.

Ward identities play a crucial role in Quantum Field Theory and Statistical Mechanics, as they allow to prove cancellations in a non perturbative way. The advantage of reducing the analysis of quantum spin chains or interacting Ising models to the reference model is that such model, formally neglecting the cutoffs, verifies many symmetries which were not verified by the spin chain or spin models; in particular it verifies a *total gauge invariance* symmetry $\psi^{\pm}_{\mathbf{x},\omega} \to e^{\pm i\alpha_{\mathbf{x}}}\psi^{\pm}_{\mathbf{x},\omega}$ and *chiral gauge invariance* $\psi^{\pm}_{\mathbf{x},\omega} \to e^{\pm i\alpha_{\mathbf{x},\omega}}\psi^{\pm}_{\mathbf{x},\omega}$. Such symmetries are *hidden* in Ising or spin chain models, as they do not verify chiral gauge invariance, even if they are "close", in an RG sense, to a model formally verifying them.

1.2 *The reference model.*

The reference model is not Hamiltonian and is defined in terms of *Grassmannian variables*. It describes a system of two kinds of fermions with linear dispersion relation interacting with a local potential; the presence of an ultraviolet and an infrared cutoff makes the model *not* solvable. Given the interval $[0, L]$, the inverse temperature β and the (large) integer N, we introduce in $\Lambda = [0, L] \times [0, \beta]$ a lattice Λ_N, whose sites are given by the *space-time points* $\mathbf{x} = (x, x_0) = (na, n_0 a_0)$, $a = L/N$, $a_0 = \beta/N$, $n, n_0 = 0, 1, \ldots, N-1$. We also consider the set $\mathcal{D}$ of *space-time momenta* $\mathbf{k} = (k, k_0)$, with $k = \frac{2\pi}{L}(n + \frac{1}{2})$ and $k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2})$, $n, n_0 = 0, 1, \ldots, N-1$. With each $\mathbf{k} \in \mathcal{D}$ we associate four Grassmannian variables $\hat\psi^{[h,0]\sigma}_{\mathbf{k},\omega}$, $\sigma, \omega \in \{+, -\}$. Then we define the functional integration $\int \mathcal{D}\psi^{[h,0]}$ as the linear functional on the Grassmann algebra generated by the variables $\hat\psi^{[h,0]\sigma}_{\mathbf{k},\omega}$, such that, given a monomial $Q(\hat\psi)$ in the variables $\hat\psi^{[h,0]\sigma}_{\mathbf{k},\omega}$, its value is 0, except in the case $Q(\hat\psi) = \prod_{\mathbf{k}\in\mathcal{D},\omega=\pm} \hat\psi^{[h,0]-}_{\mathbf{k},\omega} \hat\psi^{[h,0]+}_{\mathbf{k},\omega}$, up to a permutation of the variables. In this case the value of the functional is determined, by using the anticommuting properties of the variables, by $\int \mathcal{D}\psi^{[h,0]} Q(\hat\psi) = 1$.

The lattice Λ_N is introduced only to allow us to perform a non formal treatment of the Grassmannian integrals, as the number of Grassmannian variables is finite, and eventually the limit $N \to \infty$ is taken, see [BM1].

We also define the *Grassmannian field* on the lattice Λ_N as

$$\psi^{[h,0]\sigma}_{\mathbf{x},\omega} = \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}} e^{i\sigma\mathbf{k}\mathbf{x}} \hat\psi^{[h,0]\sigma}_{\mathbf{k},\omega} \ , \quad \mathbf{x} \in \Lambda_N \ . \tag{1.1}$$

Note that $\psi^{[h,0]\sigma}_{\mathbf{x},\omega}$ is antiperiodic both in time and space variables.

We define

$$V(\psi^{[h,0]}) = \lambda \int d\mathbf{x}\, \psi^{[h,0]+}_{\mathbf{x},+} \psi^{[h,0]-}_{\mathbf{x},+} \psi^{[h,0]+}_{\mathbf{x},-} \psi^{[h,0]-}_{\mathbf{x},-} \tag{1.2}$$

and

$$P(d\psi^{[h,0]}) = \mathcal{N}^{-1} \mathcal{D}\psi^{[h,0]} \cdot \exp\left\{ -\frac{1}{L\beta} \sum_{\omega=\pm 1} \sum_{\mathbf{k}\in\mathcal{D}} C_{h,0}(\mathbf{k})(-ik_0 + \omega k) \hat{\psi}^{[h,0]+}_{\mathbf{k},\omega} \hat{\psi}^{[h,0]-}_{\mathbf{k},\omega} \right\} , \tag{1.3}$$

with $\mathcal{N} = \prod_{\mathbf{k}\in\mathcal{D}} [(L\beta)^{-2}(-k_0^2 - k^2) C_{h,0}(\mathbf{k})^2]$ and $\int d\mathbf{x}$ is a shorthand for "$a\ a_0\ \sum_{\mathbf{x}\in\Lambda_N}$". The function $C_{h,0}(\mathbf{k})$ acts as an ultraviolet and infrared cutoff and it is defined in the following way. We introduce a positive number $\gamma > 1$ and a positive function $\chi_0(t) \in C^\infty(\mathbb{R}_+)$ such that

$$\chi_0(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 , \\ 0 & \text{if } t \ge \gamma_0 , \quad 1 < \gamma_0 \le \gamma , \end{cases} \tag{1.4}$$

and we define, for any integer $j \le 0$, $f_j(\mathbf{k}) = \chi_0(\gamma^{-j}|\mathbf{k}|) - \chi_0(\gamma^{-j+1}|\mathbf{k}|)$. Finally we define $\chi_{h,0}(\mathbf{k}) = [C_{h,0}(\mathbf{k})]^{-1} = \sum_{j=h}^{0} f_j(\mathbf{k})$ so that $[C_{h,0}(\mathbf{k})]^{-1}$ is a smooth function with support in the interval $\{\gamma^{h-1} \le |\mathbf{k}| \le \gamma\}$, equal to 1 in the interval $\{\gamma^h \le |\mathbf{k}| \le 1\}$.

We call $\psi^{[h,0]}$ simply ψ and we introduce the *generating functional*

$$\mathcal{W}(\phi, J) = \log \int P(d\psi) e^{-V(\psi) + \sum_\omega \int d\mathbf{x} \left[J_{\mathbf{x},\omega} \psi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega} + \phi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega} + \psi^+_{\mathbf{x},\omega} \phi^-_{\mathbf{x},\omega} \right]} . \tag{1.5}$$

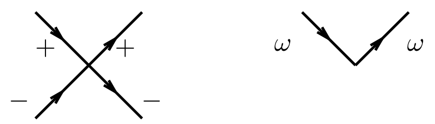


Fig. 1: Graphical representation of the interaction $V(\psi)$ and the density $\psi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega}$

The Grassmannian variables $\phi^\sigma_{\mathbf{x},\omega}$ are antiperiodic in x_0 and x and anticommuting with themselves and $\psi^\sigma_{\mathbf{x},\omega}$, while the variables $J_{\mathbf{x},\omega}$ are periodic and commuting with themselves and all the other variables. The Schwinger functions can be obtained by functional derivatives of (1.5); for instance

$$G^{2,1}_\omega(\mathbf{x}; \mathbf{y}, \mathbf{z}) = \frac{\partial}{\partial J_{\mathbf{x},\omega}} \frac{\partial^2}{\partial \phi^+_{\mathbf{y},+} \partial \phi^-_{\mathbf{z},+}} \mathcal{W}(\phi, J)|_{\phi=J=0} , \tag{1.6}$$

$$G^{4,1}_\omega(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \frac{\partial}{\partial J_{\mathbf{x},\omega}} \frac{\partial^2}{\partial \phi^+_{\mathbf{x}_1,\omega} \partial \phi^-_{\mathbf{x}_2,\omega}} \frac{\partial^2}{\partial \phi^+_{\mathbf{x}_3,-\omega} \partial \phi^-_{\mathbf{x}_4,-\omega}} \mathcal{W}(\phi, J)|_{\phi=J=0} , \tag{1.7}$$

$$G^4_\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \frac{\partial^2}{\partial \phi^+_{\mathbf{x}_1,\omega} \partial \phi^-_{\mathbf{x}_2,\omega}} \frac{\partial^2}{\partial \phi^+_{\mathbf{x}_3,-\omega} \partial \phi^-_{\mathbf{x}_4,-\omega}} \mathcal{W}(\phi, J)|_{\phi=J=0} , \tag{1.8}$$

$$G^2_\omega(\mathbf{y}, \mathbf{z}) = \frac{\partial^2}{\partial \phi^+_{\mathbf{y},\omega} \partial \phi^-_{\mathbf{z},\omega}} \mathcal{W}(\phi, J)|_{\phi=J=0} . \tag{1.9}$$

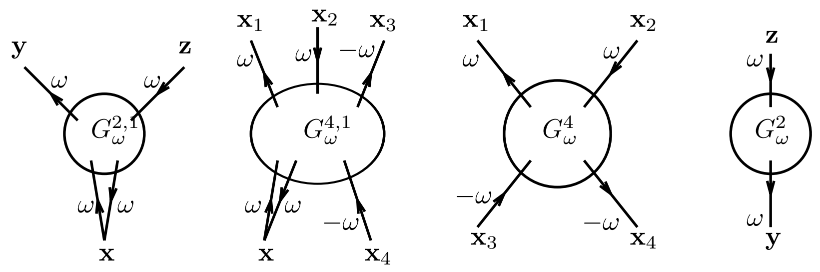


Fig. 2: Graphical representation of the Schwinger functions $G^{2,1}_\omega, G^{4,1}_\omega, G^4_\omega, G^2_\omega$

If $Q(\psi)$ is a monomial in the Grassmannian variables, it is easy to see that $\int P(d\psi)$ is given by the anticommutative *Wick rule*; the corresponding propagator

$$\hat{g}_\omega(\mathbf{k}) = \frac{\chi_{h,0}(\mathbf{k})}{-ik_0 + \omega k} \tag{1.10}$$

is singular as $L, \beta \to \infty$ at $\mathbf{k} = 0$. The simplest way of computing the Grassmannian integral (1.5) is to expand in power series of λ the exponential, obtaining many Grassmannian integrals of monomials, which can be computed by the anticommutative Wick rule. This procedure makes it possible to write series expansions for the Schwinger functions, for example those defined in (1.6)-(1.9), and to prove that they are absolutely convergent, in the limit $L, \beta \to \infty$, for $|\lambda| \le \varepsilon_h$, with $\varepsilon_h \to 0$ as $h \to -\infty$; *in other words, the estimated radius of convergence vanishes as the infrared cutoff is removed*. In this paper we will show that it is possible to modify the expansions, through a *resummation* of the power series in λ, so that it is possible to prove that they are well defined and convergent even when the infrared cutoff is removed ($h \to -\infty$); as a corollary we prove vanishing of the Beta function in the form of the bound (3.3) below.

It is also possible to remove the ultraviolet cutoff, so constructing a relativistic quantum field theory, the massless *Thirring model* [T]; this is shown in the Appendix. Note finally that the reference model, if both the infrared and ultraviolet cutoffs are removed and the local potential is replaced by a short-ranged one, coincides with the *Luttinger model*, which was solved in [ML]; the presence of cutoffs makes however the model (1.5) *not solvable*.

1.3 *Sketch of the proof: the Dyson equation.*

This paper is the conclusion of our construction of $d = 1$ Luttinger liquids *with no use of exact solutions*, started in [BM1], [BM2], [BM3], whose results will be used here.

The analysis starts by expressing the Grassmann integration in (1.5) as the product of many independent integrations, each of them "describing the theory at a certain momentum scale γ^j", with j an integer such that $h \le j \le 0$ and $\gamma > 1$. This allows us to perform the overall functional integration by iteratively integrating the Grassmanian variables of decreasing momentum scale. After $|j|$ integration steps one gets a Grassmannian integral still similar to (1.5), the main differences being that the Grassmannian fields acquire a *wave function renormalization* Z_j, the local terms quartic in the ψ fields in the interaction have coupling λ_j (the effective interaction strength at momentum scale γ^j) and the local terms $J\psi^+_\omega\psi^-_\omega$ have coupling $Z^{(2)}_j$ (the density renormalization). This iterative procedure allows us to get an expansion, resumed in §2, for the Schwinger functions; they are written as series in the set of parameters λ_j, $j = 0, -1, ...h$, called *running coupling constants*. It was proved in [BM1] that, if the running coupling constants are small enough, such expansions are convergent, as a consequence of suitable cancellations due to the anticommutativity of fermions.

However, the property that the running coupling constants are small is not trivial at all; it is related to very complex and intricate cancellations at all orders in the perturbative series, eventually implying that the effective interaction strength λ_j stays close to its initial value λ for any j; while one can easily check that cancellations are present at lowest orders by an explicit computation, to prove directly that the cancellations are present at every order seems essentially impossible. In order to prove that λ_j remains close to λ for any j, we use the fact that the Schwinger functions, even if they are expressed by apparently very different series expansion, are indeed related by remarkable identities; on the other hand the Fourier transform of Schwinger functions computed at the cut-off scale are related to the running coupling constants or to the renormalization constants at the cutoff scale (see Theorem 2.1), so that identities between Schwinger functions imply identities between coupling or renormalization constants.

As $\hat G^4$ computed at the cutoff scale is proportional to λ_h, see (2.37) below, it is natural to write (see §3) a *Dyson* equation for $\hat G^4$:

$$
\begin{aligned}
-\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = \lambda \hat g_-(\mathbf{k}_4)\Big[&\hat G^2_-(\mathbf{k}_3)\hat G^{2,1}_+(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)+\\
&+\frac{1}{L\beta}\sum_{\mathbf{p}} G^{4,1}_+(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})\Big]\;,
\end{aligned}
\tag{1.11}
$$

relating the correlations in (1.6),(1.7),(1.8),(1.9); see Fig. 3.

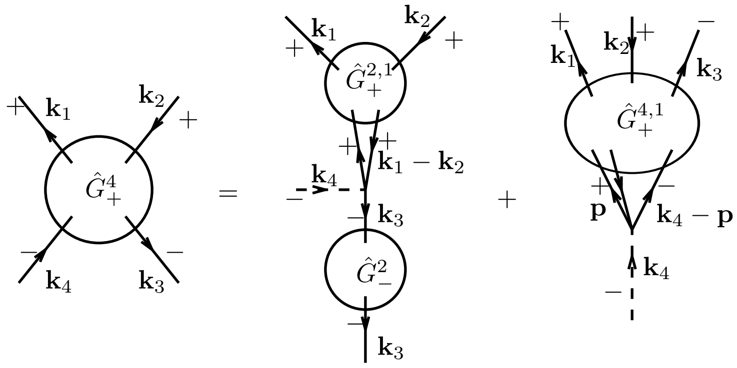


Fig. 3: Graphical representation of the Dyson equation (1.11); the dotted line represents the "bare" propagator $g(\mathbf{k}_4)$

The l.h.s. of the Dyson equation computed at the cutoff scale is indeed proportional to the effective interaction λ_h (see (2.37) below), while the r.h.s. is proportional to λ. *If one does not take into account cancellations in (1.11)*, this equation only allows us to prove that $|\lambda_h| \le C_h|\lambda|$, with C_h diverging as $h \to -\infty$. However, inspired by the analysis in the physical literature, see [DL], [S], [MD], we can try to express $\hat G^{2,1}_\omega$ and $\hat G^{4,1}_\omega$, in the r.h.s. of (1.11), in terms of $\hat G^2_\omega$ and $\hat G^4_\omega$ by suitable *Ward identities* and *correction identities.*

1.4 *Ward identities and the first addend of (1.11).*

To begin with, we consider the first addend in the r.h.s. of the Dyson equation (1.11). A remarkable identity relating $\hat G^{2,1}_+$ to $\hat G^2_+$ can be obtained by the chiral Gauge transformation $\psi^\pm_{\mathbf{x},+} \to e^{\pm i\alpha_{\mathbf{x}}}\psi^\pm_{\mathbf{x},+}$, $\psi^\pm_{\mathbf{x},-} \to \psi^\pm_{\mathbf{x},-}$ in the generating functional (1.5); one obtains the identity (3.14) below, represented pictorially in Fig. 4, with $D_\omega(\mathbf{p}) = -ip_0 + \omega p$.

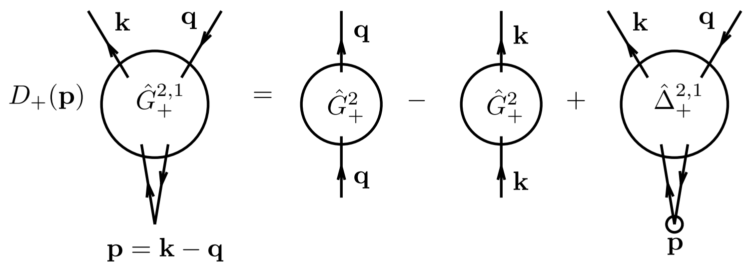


Fig. 4: Graphical representation of the Ward identity (3.14);
the small circle in $\hat\Delta_+^{2,1}$ represents the function C_+ of (3.16)

The above Ward identity provides a relation between $\hat G_+^{2,1}$, $\hat G_+^2$ and a *correction term* $\hat\Delta_+^{2,1}$, which can be obtained, through the analogue of (1.6), from a functional integral very similar to (1.5), with the difference that $\sum_\omega\int d\mathbf{p}d\mathbf{k}\hat J_{\mathbf{p},\omega}\psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k}-\mathbf{p},\omega}$ is replaced by $\int d\mathbf{p}d\mathbf{k}C_+(\mathbf{k},\mathbf{k}-\mathbf{p})\hat J_{\mathbf{p}}\psi^+_{\mathbf{k},+}\psi^-_{\mathbf{k}-\mathbf{p},+}$; the function $C_+(\mathbf{k},\mathbf{k}-\mathbf{p})$, defined in (3.16) below and represented by the small circle in Fig. 4, is vanishing (like the term $\hat\Delta_+^{2,1}$ itself) if $C_{h,0}^{-1}=1$, that is the correction term would vanish if no cutoffs (which break Gauge invariance) were present in the model.

Remark. The above Ward identity is usually stated in the physical literature by neglecting the correction term $\hat\Delta_+^{2,1}$; we shall call in general *formal* Ward identities the Ward identities one obtains by putting equal to 0 the correction terms. The formal Ward identities are generally derived, see [DL], [S], [MD], by neglecting the cutoffs, so that the propagator becomes simply $D_\omega(\mathbf{k})^{-1}$ and one can use the following relation

$$D_\omega(\mathbf{p})^{-1}(D_\omega(\mathbf{k})^{-1}-D_\omega(\mathbf{k}+\mathbf{p})^{-1})=D_\omega(\mathbf{k})^{-1}D_\omega(\mathbf{k}+\mathbf{p})^{-1}\;. \tag{1.12}$$

After the derivation of the formal Ward identities, the cutoffs are introduced in order to have non diverging quantities; this approximation leads however to some well known inconsistencies, see [G].

The use of Ward identities is to provide relations between Schwinger functions, but the correction terms (due to the cutoffs) substantially affect the Ward identities and apparently spoil them of their utility. However there are other remarkable relations connecting the correction terms to the Schwinger functions; such *correction identities* can be proved by performing a careful analysis of the renormalized expansion for the correction terms, and come out of the peculiar properties of the function $C_+(\mathbf{k},\mathbf{k}-\mathbf{p})$, see [BM2] and section 4. For example, the analysis of [BM2] implies that $\hat\Delta_+^{2,1}$ verifies the following *correction identity*, see Fig.5

$$\hat\Delta_+^{2,1}(\mathbf{p},\mathbf{k},\mathbf{q})=\nu_+D_+(\mathbf{p})\hat G_+^{2,1}(\mathbf{p},\mathbf{k},\mathbf{q})+\nu_-D_-(\mathbf{p})\hat G_-^{2,1}(\mathbf{p},\mathbf{k},\mathbf{q})+\hat H_+^{2,1}(\mathbf{p},\mathbf{k},\mathbf{q}) \tag{1.13}$$

where ν_+,ν_- are $O(\lambda)$ and weakly dependent on h, once we prove that λ_j is small enough for $j\ge h$, and $\hat H_+^{2,1}(\mathbf{p},\mathbf{k},\mathbf{q})$ can be obtained through the analogue of (1.6), from a functional integral very similar to (1.5), with the difference that $\sum_\omega\int d\mathbf{p}d\mathbf{k}\hat J_{\mathbf{p},\omega}\psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k}-\mathbf{p},\omega}$ is replaced by

$$\int d\mathbf{p}d\mathbf{k}C_+(\mathbf{k},\mathbf{k}-\mathbf{p})\hat J_{\mathbf{p}}\psi^+_{\mathbf{k},+}\psi^-_{\mathbf{k}-\mathbf{p},+}-\sum_\omega\nu_\omega\int d\mathbf{p}d\mathbf{k}\hat J_{\mathbf{p}}D_\omega(\mathbf{p})\psi^+_{\mathbf{k},\omega}\psi^+_{\mathbf{k}-\mathbf{p},\omega}\;. \tag{1.14}$$

The crucial point is that $\hat H_+^{2,1}$, when computed for momenta at the cut-off scale, *is $O(\gamma^{\theta h})$ smaller, with $0<\theta<1$ a positive constant, with respect to the first two addends of the r.h.s. of (1.13).* In other words the correction identity (1.13) says that the correction term $\hat\Delta_+^{2,1}$, which is usually neglected, can be written in terms of the Schwinger functions $\hat G^{2,1}_+$ and $\hat G^{2,1}_-$ up to the exponentially smaller term $\hat H^{2,1}_+$.

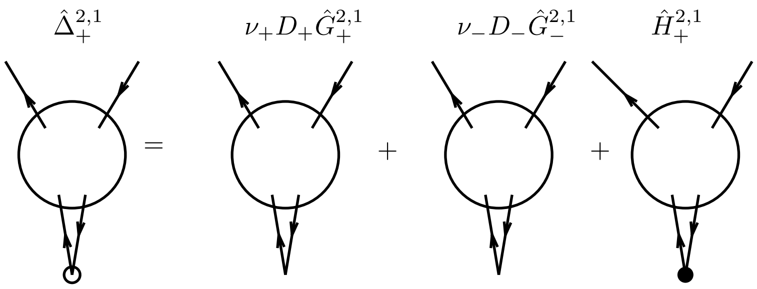


Fig. 5: Graphical representation of the correction identity (1.13); the filled point in the last term represents (1.14)

Note that (1.13) was not explicitly stated in [BM2], but its proof, which we omit here too, is implicitly contained in the proof of Theorem 4 of that paper. One has to use a strategy similar (but much simpler) to that used in §4.3 below.

Inserting the correction identity (1.13) in the Ward identity (3.14), we obtain the new identity

$$(1-\nu_+)D_+(\mathbf{p})\hat G^{2,1}_+(\mathbf{p},\mathbf{k},\mathbf{q}) - \nu_- D_-(\mathbf{p})\hat G^{2,1}_-(\mathbf{p},\mathbf{k},\mathbf{q}) = \hat G^2_+(\mathbf{q}) - \hat G^2_+(\mathbf{k}) + \hat H^{2,1}_+(\mathbf{p},\mathbf{k},\mathbf{q})\;. \quad (1.15)$$

In the same way one can show that the formal Ward identity $D_-(\mathbf{p})\hat G^{2,1}_-(\mathbf{p},\mathbf{k},\mathbf{q}) = 0$ becomes, if the cutoffs are taken into account

$$(1-\nu'_-)D_-(\mathbf{p})\hat G^{2,1}_-(\mathbf{p},\mathbf{k},\mathbf{q}) - \nu'_+ D_+(\mathbf{p})\hat G^{2,1}_+(\mathbf{p},\mathbf{k},\mathbf{q}) = \hat H^{2,1}_-(\mathbf{p},\mathbf{k},\mathbf{q})\;, \quad (1.16)$$

where, by symmetry reasons, $\nu'_\pm = \nu_\mp$ and $H^{2,1}_-$ satisfying a bound similar to that of $H^{2,1}_+$, when computed for momenta at the cutoff scale.

Remark. By removing both the ultraviolet and the infrared cutoff, see the Appendix, *the functions* $\hat H^{2,1}_\pm$ *vanishes*, if we fix the momenta to some cutoff independent values. Hence the formal Ward identities are not true, even after the removal of the infrared and ultraviolet cutoff, but must be replaced by (1.15) and (1.16), with $\hat H^{2,1}_\pm = 0$. In other words, the presence of cutoffs produces modifications to the formal Ward identities, which persist when the cutoffs are removed, a phenomenon known as *anomaly*; in the case of Ward identities based on chiral gauge transformations, one speaks of *chiral anomaly*, see [Z].

The identities (1.15) and (1.16) allow us to write $\hat G^{2,1}_+$ in terms of $\hat G_2$ and $\hat H^{2,1}_\pm$. If we put the expression so obtained in the first addend of (1.11), we can prove that it is indeed proportional to λ with the "right" proportionality constant, see (3.30), (3.31) below. Note that in (3.31) we make reference to a bound explicitly proved in [BM2], which is expressed directly in terms of the function $\Delta^{2,1}_+$. In the following section we shall explain how a similar strategy can be applied to the second addend of (1.11).

1.5 *Ward identities and the second addend of (1.11).*

The analysis of the second addend of (1.11) is more complex, the reason being that $\mathbf{p}$ is integrated instead of being fixed at the infrared cutoff scale, as it was the case for the first addend. If we simply compute $\hat G^{4,1}_\omega$ by our series expansion and we insert it in the second addend of (1.11), we get a "bad" bound, just by dimensional reasons. We can however derive a Ward identity for $\hat G^{4,1}$, in the form of (3.17) below, see Fig. 6.

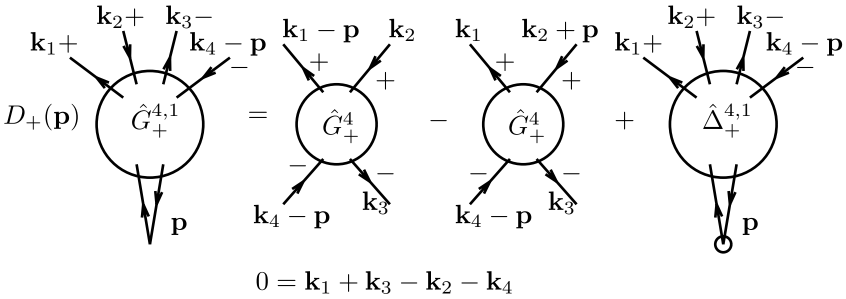


Fig. 6: Graphical representation of the Ward identity (3.17)

If we insert the above identity in the second addend of the Dyson equation, we get three terms (all multiplied by λ); two of them, the ones involving the $\hat G^4_+$ functions (which are of the form $g(\mathbf{k}_4)(L\beta)^{-1}\sum_{\mathbf{p}}\hat G^4_+ D_+(\mathbf{p})^{-1}$), admit good bounds, see (3.33) below. On the contrary, the third term, which is of the form $g(\mathbf{k}_4)(L\beta)^{-1}\sum_{\mathbf{p}}\hat\Delta^{4,1}_+ D_+(\mathbf{p})^{-1}$, has a "bad" bound, see [BM3]. This is not surprising, as also $\hat\Delta^{4,1}_+$ verifies a *correction identity*, which represents it in terms of $\hat G^{4,1}_+$ and $\hat G^{4,1}_-$; see (3.23) below and Fig. 7.

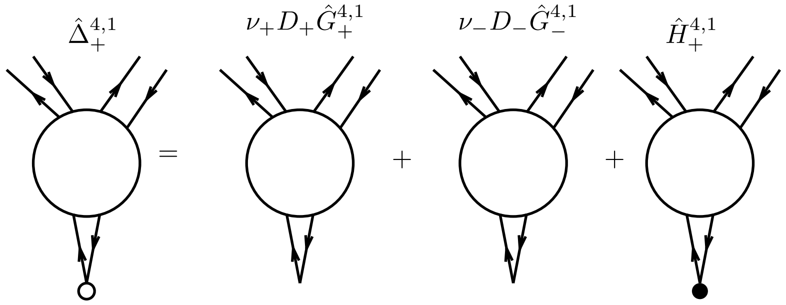


Fig. 7: The correction identity for $\hat\Delta^{4,1}_\omega$; the filled point in the last term represents (1.14)

By combining the above equation and the Ward identities for $\hat G^{4,1}_+$ and $\hat G^{4,1}_-$ we obtain, after some algebra, an equation relating $\hat G^{4,1}_+$ to $\hat G^4_+$ and functions $\hat H^{4,1}_+$ and $\hat H^{4,1}_-$; see (3.24) below. Inserting this expression in the second addend of the r.h.s. of the Dyson equation, we get our final expression, see (3.26) below, for the Dyson equation, containing several terms; among them the ones still requiring a further analysis are the ones involving the functions $\hat H^{4,1}_\pm$, namely $g(\mathbf{k}_4)(L\beta)^{-1}\sum_{\mathbf{p}}\hat H^{4,1}_\pm D_+(\mathbf{p})^{-1}$, represented as in Fig. 8.

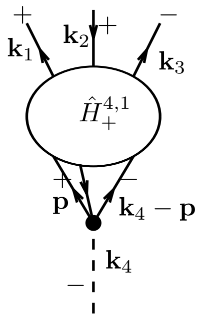


Fig. 8: Graphical representation of the term containing $\hat{H}^{4,1}_{\pm}$

The analysis of such terms is done in §4; we again start by writing for them a Dyson equation similar to (1.11), in which the analogous of the first addend in the l.h.s. vanishes; this Dyson equation allows to write this term in terms of a function $\tilde{G}_4$ similar to G_4.

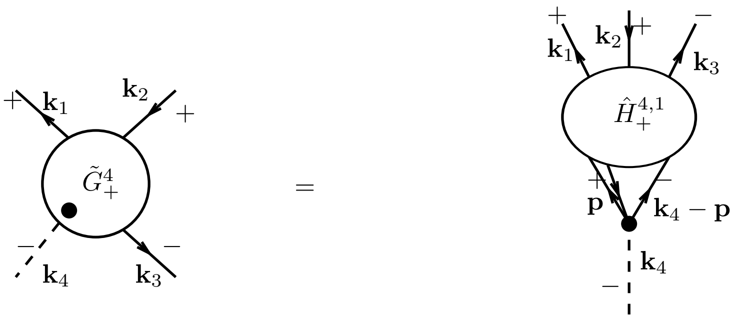


Fig. 9: Graphical representation of the Dyson equation for the correction.

We can study $\tilde{G}_4$ by a multiscale analysis very similar to the one for $\hat{G}^4$; the presence of a "special" vertex (the one associated to the filled point in Fig 9) has however the effect that a new running coupling appears, associated with the local part of the terms with four external lines among which one is the dotted line in Fig. 9, to which the bare propagator $\hat{g}(\mathbf{k}_4)$ is associated; we will call this new running coupling constant $\tilde{\lambda}_j$. It would seem that we have a problem more difficult than our initial one, since we have now to control the flow of two running coupling constants, λ_j and $\tilde{\lambda}_j$, instead of one. However, it turns out, see Lemma 4.2, that λ_j and $\tilde{\lambda}_j$ are not independent but are essentially *proportional*; this follows from a careful analysis of the expansion for $\tilde{\lambda}_j$, based on the properties of the function $C_\omega(\mathbf{k}, \mathbf{k} - \mathbf{p})$. One then gets for $\tilde{G}_4$ a bound very similar to the one for $\hat{G}_4$, except that λ_h is replaced by $\tilde{\lambda}_h$ (but $\tilde{\lambda}_h$ and λ_h are proportional) and there is no wave function renormalization associated to the external line with momentum $\mathbf{k}_4$ (to such line is associated a "bare" instead of a "dressed" propagator, like in $\hat{G}_4$). We can however identify two class of terms in the expansion of $\tilde{G}_4$, see Fig. 10, and summing them has the effect that also the external line with momentum $\mathbf{k}_4$ is dressed by the interaction, and this allows us to complete the proof that $\lambda_h = \lambda + O(\lambda^2)$ for any h.

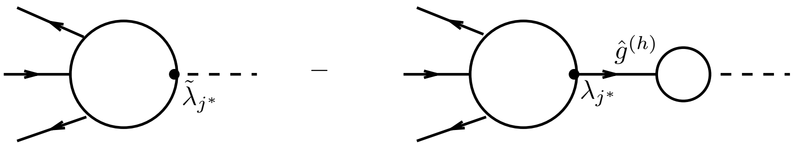


Fig. 10: The last resummation

Finally we show in the Appendix that a simple extension of our analysis implies that also the ultraviolet cutoff can be removed, that is we can construct a QFT corresponding to the Thirring model.

2. Renormalization Group analysis

2.1 *Multiscale integration.*
We resume the Renormalization Group analysis in [BM1] for the generating function (1.5). The functional integration of (1.5) can be performed iteratively in the following way. We prove by induction that, for any negative j, there are a constant E_j, a positive function $\tilde Z_j(\mathbf{k})$ and functionals $\mathcal{V}^{(j)}$ and $\mathcal{B}^{(j)}$ such that

$$e^{\mathcal{W}(\phi,J)} = e^{-L\beta E_j}\int P_{\tilde Z_j,C_{h,j}}(d\psi^{[h,j]})e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})+\mathcal{B}^{(j)}(\sqrt{Z_j}\psi^{[h,j]},\phi,J)}\;, \tag{2.1}$$

where:
1) $P_{\tilde Z_j,C_{h,j}}(d\psi^{[h,j]})$ is the *effective Grassmannian measure at scale* j, equal to, if $Z_j=\max_{\mathbf{k}}\tilde Z_j(\mathbf{k})$,

$$\begin{aligned} P_{\tilde Z_j,C_{h,j}}(d\psi^{[h,j]}) = & \prod_{\mathbf{k}:C_{h,j}(\mathbf{k})>0}\prod_{\omega=\pm1}\frac{d\hat\psi^{[h,j])+}_{\mathbf{k},\omega}d\hat\psi^{[h,j]-}_{\mathbf{k},\omega}}{\mathcal{N}_j(\mathbf{k})}\cdot \\ & \cdot\;\exp\left\{-\frac{1}{L\beta}\sum_{\mathbf{k}}C_{h,j}(\mathbf{k})\tilde Z_j(\mathbf{k})\sum_{\omega\pm1}\hat\psi^{[h,j]+}_{\omega}D_\omega(\mathbf{k})\hat\psi^{[h,j]-}_{\mathbf{k},\omega}\right\}\;, \end{aligned}\tag{2.2}$$

$$\mathcal{N}_j(\mathbf{k}) = (L\beta)^{-1}C_{h,j}(\mathbf{k})\tilde Z_j(\mathbf{k})[-k_0^2-k^2]^{1/2}\;, \tag{2.3}$$

$$C_{h,j}(\mathbf{k})^{-1} = \sum_{r=h}^{j} f_r(\mathbf{k}) \equiv \chi_{h,j}(\mathbf{k}) \quad , \quad D_\omega(\mathbf{k}) = -ik_0+\omega k\;; \tag{2.4}$$

2) the *effective potential on scale* j, $\mathcal{V}^{(j)}(\psi)$, is a sum of monomial of Grassmannian variables multiplied by suitable kernels. *i.e.* it is of the form

$$\mathcal{V}^{(j)}(\psi) = \sum_{n=1}^{\infty}\frac{1}{(L\beta)^{2n}}\sum_{\substack{\mathbf{k}_1,\ldots,\mathbf{k}_{2n}\\ \omega_1,\ldots,\omega_{2n}}}\prod_{i=1}^{2n}\hat\psi^{\sigma_i}_{\mathbf{k}_i,\omega_i}\hat W^{(j)}_{2n,\underline{\omega}}(\mathbf{k}_1,\ldots,\mathbf{k}_{2n-1})\delta\left(\sum_{i=1}^{2n}\sigma_i\mathbf{k}_i\right)\;, \tag{2.5}$$

where $\sigma_i=+$ for $i=1,\ldots,n$, $\sigma_i=-$ for $i=n+1,\ldots,2n$ and $\underline{\omega}=(\omega_1,\ldots,\omega_{2n})$;
3) the *effective source term at scale* j, $\mathcal{B}^{(j)}(\sqrt{Z_j}\psi,\phi,J)$, is a sum of monomials of Grassmannian variables and $\phi^\pm, J$ field, with at least one $\phi^\pm$ or one J field; we shall write it in the form

$$\mathcal{B}^{(j)}(\sqrt{Z_j}\psi,\phi,J) = \mathcal{B}^{(j)}_\phi(\sqrt{Z_j}\psi)+\mathcal{B}^{(j)}_J(\sqrt{Z_j}\psi)+W^{(j)}_R(\sqrt{Z_j}\psi,\phi,J)\;, \tag{2.6}$$

where $\mathcal{B}_\phi^{(j)}(\psi)$ and $\mathcal{B}_J^{(j)}(\psi)$ denote the sums over the terms containing only one ϕ or J field, respectively.

Of course (2.1) is true for $j = 0$, with

$$\begin{aligned}
&\tilde{Z}_0(\mathbf{k}) = 1, \qquad E_0 = 0, \qquad \mathcal{V}^{(0)}(\psi) = V(\psi), \qquad W_R^{(0)} = 0,\\
&\mathcal{B}_\phi^{(0)}(\psi) = \sum_\omega \int d\mathbf{x}[\phi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \psi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}], \qquad \mathcal{B}_J^{(0)}(\psi) = \sum_\omega \int d\mathbf{x} J_{\mathbf{x},\omega}\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} \;.
\end{aligned} \tag{2.7}$$

Let us now assume that (2.1) is satisfied for a certain $j \le 0$ and let us show that it holds also with $j - 1$ in place of j.

In order to perform the integration corresponding to $\psi^{(j)}$, we write the effective potential and the effective source as sum of two terms, according to the following rules.

We split the effective potential $\mathcal{V}^{(j)}$ as $\mathcal{L}\mathcal{V}^{(j)} + \mathcal{R}\mathcal{V}^{(j)}$, where $\mathcal{R} = 1 - \mathcal{L}$ and $\mathcal{L}$, the *localization operator*, is a linear operator on functions of the form (2.5), defined in the following way by its action on the kernels $\hat{W}^{(j)}_{2n,\underline{\omega}}$.

1) If $2n = 4$, then

$$\mathcal{L}\hat{W}^{(j)}_{4,\underline{\omega}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \hat{W}^{(j)}_{4,\underline{\omega}}(\bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}) \;, \tag{2.8}$$

where $\bar{\mathbf{k}}_{\eta\eta'} = (\eta\pi L^{-1}, \eta'\pi\beta^{-1})$. Note that $\mathcal{L}\hat{W}^{(j)}_{4,\underline{\omega}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = 0$, if $\sum_{i=1}^4 \omega_i \ne 0$, by simple symmetry considerations.

2) If $2n = 2$ (in this case there is a non zero contribution only if $\omega_1 = \omega_2$)

$$\mathcal{L}\hat{W}^{(j)}_{2,\underline{\omega}}(\mathbf{k}) = \frac{1}{4}\sum_{\eta,\eta'=\pm 1} \hat{W}^{(j)}_{2,\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'}) \left\{1 + \eta\frac{L}{\pi} + \eta'\frac{\beta}{\pi}k_0\right\} \;, \tag{2.9}$$

3) In all the other cases

$$\mathcal{L}\hat{W}^{(j)}_{2n,\underline{\omega}}(\mathbf{k}_1, \ldots, \mathbf{k}_{2n-1}) = 0 \;. \tag{2.10}$$

These definitions are such that $\mathcal{L}^2 = \mathcal{L}$, a property which plays an important role in the analysis of [BM1]. Moreover, by using the symmetries of the model, it is easy to see that

$$\mathcal{L}\mathcal{V}^{(j)}(\psi^{[h,j]}) = z_j F_\zeta^{[h,j]} + a_j F_\alpha^{[h,j]} + l_j F_\lambda^{[h,j]} \;, \tag{2.11}$$

where z_j, a_j and l_j are real numbers and

$$F_\alpha^{[h,j]} = \sum_\omega \frac{\omega}{(L\beta)} \sum_{\mathbf{k}:C^{-1}_{h,j}(\mathbf{k})>0} k\hat{\psi}^{[h,j]+}_{\mathbf{k},\omega}\hat{\psi}^{[h,j]-}_{\mathbf{k},\omega} = \sum_\omega i\omega \int_\Lambda d\mathbf{x}\psi^{[h,j]+}_{\mathbf{x},\omega}\partial_x\psi^{[h,j]-}_{\mathbf{x},\omega} \;, \tag{2.12}$$

$$F_\zeta^{[h,j]} = \sum_\omega \frac{1}{(L\beta)} \sum_{\mathbf{k}:C^{-1}_{h,j}(\mathbf{k})>0} (-ik_0)\hat{\psi}^{[h,j]+}_{\mathbf{k},\omega}\hat{\psi}^{[h,j]-}_{\mathbf{k}',\omega} = -\sum_\omega \int_\Lambda d\mathbf{x}\psi^{[h,j]+}_{\mathbf{x},\omega}\partial_0\psi^{[h,j]-}_{\mathbf{x},\omega} \;, \tag{2.13}$$

$$F_\lambda^{[h,j]} = \frac{1}{(L\beta)^4} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4:C^{-1}_{h,j}(\mathbf{k}_i)>0} \hat{\psi}^{[h,j]+}_{\mathbf{k}_1,+}\hat{\psi}^{[h,j]-}_{\mathbf{k}_2,+}\hat{\psi}^{[h,j]+}_{\mathbf{k}_3,-}\hat{\psi}^{[h,j]-}_{\mathbf{k}_4,-}\delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4). \tag{2.14}$$

∂_x and ∂_0 are defined in an obvious way, so that the second equality in (2.12) and (2.13) is satisfied; if $N = \infty$ they are simply the partial derivative with respect to x and x_0. Note that $\mathcal{L}\mathcal{V}^{(0)} = \mathcal{V}^{(0)}$, hence $l_0 = \lambda$, $a_0 = z_0 = 0$.

In the limit $L = \beta = \infty$ one has $a_j = z_j$ as a trivial consequence of the symmetries of the propagator. If L, β are finite this is not true and by dimensional arguments it follows that $z_j - a_j$ is of order $\gamma^{-j}\max\{L^{-1}, \beta^{-1}\}$.

Analogously we write $\mathcal{B}^{(j)} = \mathcal{L}\mathcal{B}^{(j)} + \mathcal{R}\mathcal{B}^{(j)}$, $\mathcal{R} = 1 - \mathcal{L}$, according to the following definition. First of all, we put $\mathcal{L}W_R^{(j)} = W_R^{(j)}$. Let us consider now $\mathcal{B}_J^{(j)}(\sqrt{Z_j}\psi)$. It is easy to see that the field J is equivalent, from the point of view of dimensional considerations, to two ψ fields. Hence, the only terms which need to be renormalized are those of second order in ψ, which are indeed marginal. We shall use for them the definition

$$\begin{aligned}\mathcal{B}_J^{(j,2)}(\sqrt{Z_j}\psi) &= \sum_{\omega,\tilde\omega}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}B_{\omega,\tilde\omega}(\mathbf{x},\mathbf{y},\mathbf{z})J_{\mathbf{x},\omega}(\sqrt{Z_j}\psi^+_{\mathbf{y},\tilde\omega})(\sqrt{Z_j}\psi^-_{\mathbf{z},\tilde\omega}) = \\ &= \sum_{\omega,\tilde\omega}\int \frac{d\mathbf{p}}{(2\pi)^2}\frac{d\mathbf{k}}{(2\pi)^2}\hat B_{\omega,\tilde\omega}(\mathbf{p},\mathbf{k})\hat J(\mathbf{p})(\sqrt{Z_j}\hat\psi^+_{\mathbf{p}+\mathbf{k},\tilde\omega})(\sqrt{Z_j}\hat\psi^-_{\mathbf{k},\tilde\omega})\ .\end{aligned} \tag{2.15}$$

We regularize $\mathcal{B}_J^{(j,2)}(\sqrt{Z_j}\psi)$, in analogy to what we did for the effective potential, by decomposing it as the sum of $\mathcal{L}\mathcal{B}_J^{(j,2)}(\sqrt{Z_j}\psi)$ and $\mathcal{R}\mathcal{B}_J^{(j,2)}(\sqrt{Z_j}\psi)$, where $\mathcal{L}$ is defined through its action on $\hat B_\omega(\mathbf{p},\mathbf{k})$ in the following way:

$$\mathcal{L}\hat B_{\omega,\tilde\omega}(\mathbf{p},\mathbf{k}) = \frac{1}{4}\sum_{\eta,\eta'=\pm1}\hat B_{\omega,\tilde\omega}(\bar p_\eta, \bar k_{\eta,\eta'})\ , \tag{2.16}$$

where $\bar k_{\eta,\eta'}$ was defined above and $\bar p_\eta = (0, 2\pi\eta'/\beta)$. In the limit $L = \beta = \infty$ it reduces simply to $\mathcal{L}\hat B_{\omega,\tilde\omega}(\mathbf{p},\mathbf{k}) = \hat B_{\omega,\tilde\omega}(0,0)$.

This definition apparently implies that we have to introduce two new renormalization constants. However, one can easily show that, in the limit $L,\beta\to\infty$, $\hat B_{\omega,-\omega}(0,0) = 0$, while, at finite L and β, $\mathcal{L}B_{\omega,-\omega}$ behaves as an irrelevant term, see [BM1].

The previous considerations imply that we can write

$$\mathcal{L}\mathcal{B}_J^{(j,2)}(\sqrt{Z_j}\psi) = \sum_\omega \frac{Z_j^{(2)}}{Z_j}\int d\mathbf{x}J_{\mathbf{x},\omega}(\sqrt{Z_j}\psi^+_{\mathbf{x},\omega})(\sqrt{Z_j}\psi^-_{\mathbf{x},\omega})\ , \tag{2.17}$$

which defines the renormalization constant $Z_j^{(2)}$.

Finally we have to define $\mathcal{L}$ for $\mathcal{B}_\phi^{(j)}(\sqrt{Z_j}\psi)$; we want to show that, by a suitable choice of the localization procedure, if $j\le -1$, it can be written in the form

$$\begin{aligned}\mathcal{B}_\phi^{(j)}(\sqrt{Z_j}\psi) &= \sum_\omega\sum_{i=j+1}^{0}\int d\mathbf{x}d\mathbf{y}\ \cdot \\ &\cdot\left[\phi^+_{\mathbf{x},\omega}g_\omega^{Q,(i)}(\mathbf{x}-\mathbf{y})\frac{\partial}{\partial\psi^+_{\mathbf{y}\omega}}\mathcal{V}^{(j)}(\sqrt{Z_j}\psi) + \frac{\partial}{\partial\psi^-_{\mathbf{y},\omega}}\mathcal{V}^{(j)}(\sqrt{Z_j}\psi)g_\omega^{Q,(i)}(\mathbf{y}-\mathbf{x})\phi^-_{\mathbf{x},\omega}\right] + \\ &+\sum_\omega\int\frac{d\mathbf{k}}{(2\pi)^2}\left[\hat\psi^{[h,j]+}_{\mathbf{k},\omega}\hat Q_\omega^{(j+1)}(\mathbf{k})\hat\phi^-_{\mathbf{k},\omega} + \hat\phi^+_{\mathbf{k},\omega}\hat Q_\omega^{(j+1)}(\mathbf{k})\hat\psi^{[h,j]-}_{\mathbf{k},\omega}\right]\ ,\end{aligned} \tag{2.18}$$

where $\hat g_\omega^{Q,(i)}(\mathbf{k}) = \hat g_\omega^{(i)}(\mathbf{k})\hat Q_\omega^{(i)}(\mathbf{k})$, with

$$\hat g_\omega^{(j)}(\mathbf{k}) = \frac{1}{Z_{j-1}}\frac{\tilde f_j(\mathbf{k})}{D_\omega(\mathbf{k})}\ , \tag{2.19}$$

$\tilde f_j(\mathbf{k}) = f_j(\mathbf{k})Z_{j-1}[\tilde Z_{j-1}(\mathbf{k})]^{-1}$ and $Q_\omega^{(j)}(\mathbf{k})$ defined inductively by the relations

$$\hat Q_\omega^{(j)}(\mathbf{k}) = \hat Q_\omega^{(j+1)}(\mathbf{k}) - z_jZ_jD_\omega(\mathbf{k})\sum_{i=j+1}^{0}\hat g_\omega^{Q,(i)}(\mathbf{k})\ ,\quad \hat Q_\omega^{(0)}(\mathbf{k}) = 1\ . \tag{2.20}$$

Note that $\hat g^{(j)}_\omega(\mathbf{k})$ does not depend on the infrared cutoff for $j>h$ and that (even for $j=h$) $\hat g^{(j)}(\mathbf{k})$ is of size $Z_{j-1}^{-1}\gamma^{-j}$, see discussion in §3 of [BM2], after eq. (60). Moreover the propagator $\hat g^{Q,(i)}_\omega(\mathbf{k})$ is equivalent to $\hat g^{(i)}_\omega(\mathbf{k})$, as concerns the dimensional bounds.

The $\mathcal{L}$ operation for $\mathcal{B}^{(j)}_\phi$ is defined by decomposing $\mathcal{V}^{(j)}$ in the r.h.s. of (2.18) as $\mathcal{L}\mathcal{V}^{(j)}+\mathcal{R}\mathcal{V}^{(j)}$, $\mathcal{L}\mathcal{V}^{(j)}$ being defined by (2.11).

After writing $\mathcal{V}^{(j)}=\mathcal{L}\mathcal{V}^{(j)}+\mathcal{R}\mathcal{V}^{(j)}$ and $\mathcal{B}^{(j)}=\mathcal{L}\mathcal{B}^{(j)}+\mathcal{R}\mathcal{B}^{(j)}$, the next step is to *renormalize* the free measure $P_{\tilde Z_j,C_{h,j}}(d\psi^{[h,j]})$, by adding to it part of the r.h.s. of (2.11). We get

$$\begin{aligned}&\int P_{\tilde Z_j,C_{h,j}}(d\psi^{[h,j]})\,e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})+\mathcal{B}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})}=\\&=e^{-L\beta t_j}\int P_{\tilde Z_{j-1},C_{h,j}}(d\psi^{[h,j]})\,e^{-\tilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})+\tilde{\mathcal{B}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})}\;,\end{aligned}\tag{2.21}$$

where

$$\tilde Z_{j-1}(\mathbf{k})=Z_j(\mathbf{k})[1+\chi_{h,j}(\mathbf{k})z_j]\;,\tag{2.22}$$

$$\tilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})=\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})-z_jZ_j[F_\zeta^{[h,j]}+F_\alpha^{[h,j]}]\,,\tag{2.23}$$

and the factor $\exp(-L\beta t_j)$ in (2.21) takes into account the different normalization of the two measures. Moreover

$$\tilde{\mathcal{B}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})=\tilde{\mathcal{B}}^{(j)}_\phi(\sqrt{Z_j}\psi^{[h,j]})+\mathcal{B}^{(j)}_J(\sqrt{Z_j}\psi^{[h,j]})+W_R^{(j)}\;,\tag{2.24}$$

where $\tilde{\mathcal{B}}^{(j)}_\phi$ is obtained from $\mathcal{B}^{(j)}_\phi$ by inserting (2.23) in the second line of (2.18) and by absorbing the terms proportional to z_j in the terms in the third line of (2.18).

If $j>h$, the r.h.s of (2.21) can be written as

$$\begin{aligned}&e^{-L\beta t_j}\int P_{\tilde Z_{j-1},C_{h,j-1}}(d\psi^{[h,j-1]})\int P_{Z_{j-1},\tilde f_j^{-1}}(d\psi^{(j)})\\&e^{-\tilde{\mathcal{V}}^{(j)}\left(\sqrt{Z_j}[\psi^{[h,j-1]}+\psi^{(j)}]\right)+\tilde{\mathcal{B}}^{(j)}\left(\sqrt{Z_j}[\psi^{[h,j-1]}+\psi^{(j)}]\right)}\;,\end{aligned}\tag{2.25}$$

where $P_{Z_{j-1},\tilde f_j^{-1}}(d\psi^{(j)})$ is the integration with propagator $\hat g^{(j)}_\omega(\mathbf{k})$.

We now *rescale* the field so that

$$\tilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})=\hat{\mathcal{V}}^{(j)}(\sqrt{Z_{j-1}}\psi^{[h,j]})\quad,\quad\tilde{\mathcal{B}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})=\hat{\mathcal{B}}^{(j)}(\sqrt{Z_{j-1}}\psi^{[h,j]})\;;\tag{2.26}$$

it follows that (in the limit $L,\beta=\infty$, so that $a_j=z_j$, see above)

$$\mathcal{L}\hat{\mathcal{V}}^{(j)}(\psi^{[h,j]})=\lambda_jF_\lambda^{[h,j]}\;,\tag{2.27}$$

where $\lambda_j=(Z_jZ_{j-1}^{-1})^2l_j$. If we now define

$$\begin{aligned}&e^{-\mathcal{V}^{(j-1)}\left(\sqrt{Z_{j-1}}\psi^{[h,j-1]}\right)+\mathcal{B}^{(j-1)}\left(\sqrt{Z_{j-1}}\psi^{[h,j-1]}\right)-L\beta E_j}=\\&=\int P_{Z_{j-1},\tilde f_j^{-1}}(d\psi^{(j)})\,e^{-\hat{\mathcal{V}}^{(j)}\left(\sqrt{Z_{j-1}}[\psi^{[h,j-1]}+\psi^{(j)}]\right)+\hat{\mathcal{B}}^{(j)}\left(\sqrt{Z_{j-1}}[\psi^{[h,j-1]}+\psi^{(j)}]\right)}\;,\end{aligned}\tag{2.28}$$

it is easy to see that $\mathcal{V}^{(j-1)}$ and $\mathcal{B}^{(j-1)}$ are of the same form of $\mathcal{V}^{(j)}$ and $\mathcal{B}^{(j)}$ and that the procedure can be iterated. Note that the above procedure allows, in particular, to write the running coupling constant λ_j, $0<j\le h$, in terms of $\lambda_{j'}$, $0\ge j'\ge j+1$:

$$\lambda_j=\beta_j^{(h)}(\lambda_{j+1},\ldots,\lambda_0)\quad,\quad\lambda_0=\lambda,\tag{2.29}$$

The function $\beta_j^{(h)}(\lambda_{j+1},...,\lambda_0)$ is called the *Beta function*. By the remark above on the independence of scale j propagators of h for $j > h$, it is independent of h, for $j > h$.

2.2 *Tree expansion.*

At the end of the iterative integration procedure, we get

$$\mathcal{W}(\varphi, J) = -L\beta E_{L,\beta} + \sum_{m^\phi+n^J\geq 1} S^{(h)}_{2m^\phi,n^J}(\phi, J) \; , \tag{2.30}$$

where $E_{L,\beta}$ is the *free energy* and $S^{(h)}_{2m^\phi,n^J}(\phi, J)$ are suitable functional, which can be expanded, as well as $E_{L,\beta}$, the effective potentials and the various terms in the r.h.s. of (2.6) and (2.5), in terms of trees (for an updated introduction to trees formalism see also [GM]). This expansion, which is indeed a finite sum for finite values of N, L, β, is explained in detail in [BGPS] and [BM1], which we shall refer to often in the following.

Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the *unlabelled tree*, so that r is not a branching point. Two unlabelled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide.

n will be called the *order* of the unlabelled tree and the branching points will be called the *non trivial vertices*. The unlabelled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order.

We shall consider also the *labelled trees* (to be called simply trees in the following), see Fig. 11; they are defined by associating some labels with the unlabelled trees, as explained in the following items.

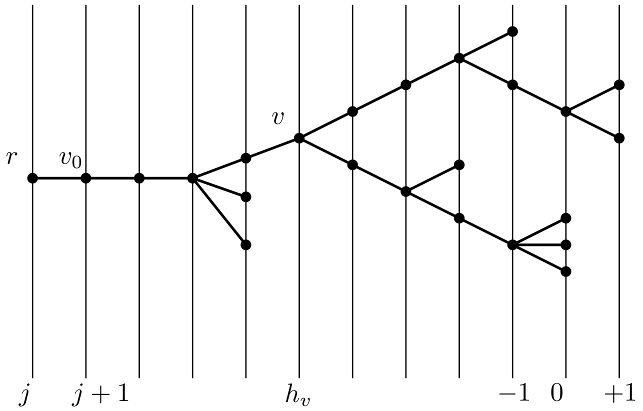


Fig. 11: A tree τ and its labels.

1) We associate a label $j \leq 0$ with the root and we denote $\mathcal{T}_{j,n}$ the corresponding set of labelled trees with n endpoints. Moreover, we introduce a family of vertical lines, labelled by an integer taking values in $[j, 1]$, and we represent any tree $\tau \in \mathcal{T}_{j,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > j$, to be called the *scale* of v, while the root is on the line with index j. There is the constraint that, if v is an endpoint, $h_v > j + 1$.

The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. The definition of h_v is extended in an obvious way to the trivial vertices and the endpoints.

Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$. Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $j + 1$.

2) There are two kind of endpoints, *normal* and *special*.

With each normal endpoint v of scale h_v we associate the local term $\mathcal{L}\hat{\mathcal{V}}^{(h_v)}(\psi^{[h,h_v-1]})$ of (2.27) and one space-time point $\mathbf{x}_v$. We shall say that the endpoint is of type λ.

There are two types of special endpoints, to be called of type ϕ and J; the first one is associated with the terms in the third line of (2.18), the second one with the terms in the r.h.s. of (2.17). Given $v \in \tau$, we shall call n_v^ϕ and n_v^J the number of endpoints of type ϕ and J following v in the tree, while n_v will denote the number of normal endpoints following v. Analogously, given τ, we shall call n_τ^ϕ and n_τ^J the number of endpoint of type ϕ and J, while n_τ will denote the number of normal endpoints. Finally, $\mathcal{T}_{j,n,n^\phi,n^J}$ will denote the set of trees belonging to $\mathcal{T}_{j,n}$ with n normal endpoints, n^ϕ endpoints of type ϕ and n^J endpoints of type J. Given a vertex v, which is not an endpoint, $\mathbf{x}_v$ will denote the family of all space-time points associated with one of the endpoints following v.

3) There is an important constraint on the scale indices of the endpoints. In fact, if v is an endpoint normal or of type J, $h_v = h_{v'} + 1$, if v' is the non trivial vertex immediately preceding v. This constraint takes into account the fact that at least one of the ψ fields associated with an endpoint normal or of type J has to be contracted in a propagator of scale $h_{v'}$, as a consequence of our definitions.

On the contrary, if v is an endpoint of type ϕ, we shall only impose the condition that $h_v \geq h_{v'}+1$. In this case the only ψ field associated with v is contracted in a propagator of scale $h_v - 1$, instead of $h_{v'}$.

4) If v is not an endpoint, the *cluster* L_v with frequency h_v is the set of endpoints following the vertex v; if v is an endpoint, it is itself a (*trivial*) cluster. The tree provides an organization of endpoints into a hierarchy of clusters.

5) We associate with any vertex v of the tree a set P_v, the *external fields* of v. The set P_v includes both the field variables of type ψ which belong to one of the endpoints following v and are not yet contracted at scale h_v (in the iterative integration procedure), to be called *normal external fields*, and those which belong to an endpoint normal or of type J and are contracted with a field variable belonging to an endpoint $\tilde{v}$ of type ϕ through a propagator $g^{Q,(h_{\tilde{v}}-1)}$, to be called *special external fields* of v.

These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \ldots, v_{s_v}$ are the s_v vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i} \backslash Q_{v_i}$, whose union will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$, that is if v is a non trivial vertex.

Moreover, if the set P_{v_0} contains only special external fields, that is if $|P_{v_0}| = n^\phi$, and $\tilde{v}_0$ is the vertex immediately following v_0, then $|P_{v_0}| < |P_{\tilde{v}_0}|$.

2.3 *Dimensional bounds.*

We can write

$$
\begin{aligned}
&S^{(h)}_{2m^\phi,n^J}(\phi,J) = \\
&= \sum_{n=0}^{\infty} \sum_{j_0=h-1}^{-1} \sum_{\substack{\tau \in \mathcal{T}_{j_0,n,2m^\phi,n^J} \\ |P_{v_0}|=2m^\phi}} \sum_{\underline{\omega}} \int d\underline{\mathbf{x}} \prod_{i=1}^{2m^\phi} \phi^{\sigma_i}_{\mathbf{x}_i,\omega_i} \prod_{r=1}^{n^J} J_{\mathbf{x}_{2m^\phi+r},\omega_{2m^\phi+r}} S_{2m^\phi,n^J,\tau,\underline{\omega}}(\underline{\mathbf{x}}) \;,
\end{aligned}
\tag{2.31}
$$

where $\underline{\omega} = \underline{\omega} = \{\omega_1, \ldots, \omega_{2m^\phi+n^J}\}$, $\underline{\mathbf{x}} = \{\mathbf{x}_1, \ldots, \mathbf{x}_{2m^\phi+n^J}\}$ and $\sigma_i = +$ if i is odd, $\sigma_i = -$ if i is even.

Let us define $\bar{\lambda}_j = \max_{k \geq j} |\lambda_k|$; in §3 of [BM1] it is proved that the kernels satisfy the following bound:

$$
\begin{aligned}
\int d\underline{\mathbf{x}}|S_{2m^\phi,n^J,\tau,\underline{\omega}}(\underline{\mathbf{x}})| \le L\beta\ (C\bar\lambda_{j_0})^n\gamma^{-j_0(-2+m^\phi+n^J)}\prod_{i=1}^{2m^\phi}\frac{\gamma^{-h_i}}{(Z_{h_i})^{1/2}}\cdot \\
\cdot\prod_{r=1}^{n^J}\frac{Z^{(2)}_{\bar h_r}}{Z_{\bar h_r}}\prod_{\text{v not e.p}}(\frac{Z_{h_v}}{Z_{h_v-1}})^{|P_v|/2}\gamma^{-d_v}\ ,
\end{aligned}
\tag{2.32}
$$

where h_i is the scale of the propagator linking the i-th endpoint of type ϕ to the tree, $\bar h_r$ is the scale of the r-th endpoint of type J and

$$d_v = -2 + |P_v|/2 + n^J_v + \tilde z(P_v)\ , \tag{2.33}$$

with

$$\tilde z(P_v) = \begin{cases} z(P_v) & \text{if } n^\phi_v \le 1\ , n^J_v = 0\ , \\ 1 & \text{if } n^\phi_v = 0\ , n^J_v = 1\ , |P_v| = 2\ , \\ 0 & \text{otherwise} \end{cases} \tag{2.34}$$

and $z(P_v) = 1$ if $|P_v| = 4$, $z(P_v) = 2$ if $|P_v| = 2$ and zero otherwise.

As explained in §5 of [BM1], one can sum over the trees τ only if $d_v > 0$. While it is not true in general that $d_v > 0$ in (2.32), it is true for the trees contributing to $\hat G^{2,1}_\omega$, $\hat G^2_\omega$, $\hat G^4_+$ with external momenta computed at the cutoff scale; hence by using the bound (2.32), one can prove, see [BM2] §3.5, the following theorem.

Theorem 2.1 *There exists ε_0 such that, if $\bar\lambda_h \le \varepsilon_0$ and $|\bar{\mathbf{k}}| = \gamma^h$, then*

$$\hat G^{2,1}_\omega(2\bar{\mathbf{k}},\bar{\mathbf{k}}) = -\frac{Z^{(2)}_h}{Z^2_h D_\omega(\bar{\mathbf{k}})^2}[1 + O(\bar\lambda^2_h)]\ , \tag{2.35}$$

$$\hat G^2_\omega(\bar{\mathbf{k}}) = \frac{1}{Z_h D_\omega(\bar{\mathbf{k}})}[1 + O(\bar\lambda^2_h)]\ , \tag{2.36}$$

$$\hat G^4_+(\bar{\mathbf{k}},-\bar{\mathbf{k}},-\bar{\mathbf{k}}) = Z^{-2}_h|\bar{\mathbf{k}}|^{-4}[-\lambda_h + O(\bar\lambda^2_h)]\ . \tag{2.37}$$

The expansion for (2.35), (2.36), (2.37) in terms of the running coupling constant λ_j is convergent *if* λ_j is small enough for all $j \ge h$. This property is surely true if $|h|$ is at most of order $|\lambda|^{-1}$, but to prove that it is true for any $|h|$ is quite nontrivial, as it is consequence of intricate cancellations which are present in the Beta function. In the following section we will show, by using Ward identities and a Dyson equation, that indeed λ_j is small enough for all $j \ge h$ *for any* h, that is uniformly in the infrared cutoff, so that the above theorem can be applied.

3. Vanishing of Beta function

3.1 *The main theorem*

The main result of this paper is the following theorem.

Theorem 3.1 *The model (1.5) is well defined in the limit $h \to -\infty$. In fact there are constants ε_1 and c_2 such that $|\lambda| \le \varepsilon_1$ implies $\bar\lambda_j \le c_2\varepsilon_1$, for any $j < 0$.*

Proof - The proof of Theorem 3.1 is done by contradiction. Assume that there exists a $h \le 0$ such that

$$\bar\lambda_{h+1} \le c_2\varepsilon_1 < |\lambda_h| \le 2c_2\varepsilon_1 \le \varepsilon_0\ , \tag{3.1}$$

where ε_0 is the same as in Theorem 2.1. We show that this is not possible, if ε_1, c_2 are suitably chosen.
Let us consider the model with cutoff γ^h. In the following sections we shall prove that, in this model,

$$|\lambda_h - \lambda| \leq c_3 \bar{\lambda}_{h+1}^2 \; . \tag{3.2}$$

However, as a consequence of the remark after (2.19), λ_j is the same, for any $j \geq h$, in the model with or without cutoff; in fact $g^{(j)}(\mathbf{k})$ does not depend on h for $j > h$, and λ_h only depends on the propagators $g^{(j)}(\mathbf{k})$ with $j > h$ by definition. Hence, from (3.2) we get the bound $|\lambda_h| \leq \varepsilon_1 + c_3 c_2^2 \varepsilon_1^2$, which is in contradiction with (3.1) if, for instance, $c_2 = 2$ and $\varepsilon_1 \leq 1/(4c_3)$. ∎

Theorem 3.1 implies, as proved in [BM3], that

$$|\beta_{j,\lambda}(\lambda_h, .., \lambda_h)| \leq C |\lambda_h|^2 \gamma^{\tau j} \; , \tag{3.3}$$

a property called *vanishing of Beta function*, which implies that there exists $\lim_{j\to-\infty} \lambda_j$ and that this limit is an analytic function of λ. In its turn, the existence of this limit implies that there exist the limits

$$\lim_{j\to-\infty} \log \frac{Z_{j-1}^{(2)}}{Z_j^{(2)}} = \eta_2(\lambda) \quad , \quad \lim_{k\to-\infty} \log \frac{Z_{j-1}}{Z_j} = \eta(\lambda) \; , \tag{3.4}$$

with $\eta(\lambda) = a_2\lambda^2 + O(\lambda^3)$ and $\eta_1(\lambda) = a_2\lambda^2 + O(\lambda^3)$, $a_2 > 0$.

3.2 *The Dyson equation.*
Let us now prove the bound (3.2). We define

$$G^{4,1}_\omega(\mathbf{z}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = < \rho_{\mathbf{z},\omega}; \psi^-_{\mathbf{x}_1,+}; \psi^+_{\mathbf{x}_2,+}; \psi^-_{\mathbf{x}_3,-}; \psi^+_{\mathbf{x}_4,-} >^T \; , \tag{3.5}$$

$$G^4_+(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = < \psi^-_{\mathbf{x}_1,+}\psi^+_{\mathbf{x}_2,+}; \psi^-_{\mathbf{x}_3,-}\psi^+_{\mathbf{x}_4,-} >^T \; , \tag{3.6}$$

where

$$\rho_{\mathbf{x},\omega} = \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} \; . \tag{3.7}$$

Moreover, we shall denote by $\hat{G}^{4,1}_{+\omega}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ and $\hat{G}^4_+(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ the corresponding Fourier transforms, deprived of the momentum conservation delta. Note that, as a consequence of (1.1), if the ψ^+ momenta are interpreted as "ingoing momenta" in the usual graph pictures, then the ψ^- momenta are "outgoing momenta"; our definition of Fourier transform is such that even $\mathbf{p}$, the momentum associated with the ρ field, is an ingoing momentum. Hence, the momentum conservation implies that $\mathbf{k}_1 + \mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_4 + \mathbf{p}$, in the case of $\hat{G}^{4,1}_\omega(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ and $\mathbf{k}_1 + \mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_4$ in the case of $\hat{G}^4_+(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$.
It is possible to derive a Dyson equation which, combined with the Ward identity (4.9) of ref. [BM3], gives a relation between G^4, G^2 and $G^{2,1}$.
If $Z = \int P(d\psi)\exp\{-V(\psi)\}$ and $< \cdot >$ denotes the expectation with respect to $Z^{-1}\int P(d\psi)$ $\exp\{-V(\psi)\}$,

$$G^4_+(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = < \psi^-_{\mathbf{x}_1,+}\psi^+_{\mathbf{x}_2,+}\psi^-_{\mathbf{x}_3,-}\psi^+_{\mathbf{x}_4,-} > - G^2_+(\mathbf{x}_1, \mathbf{x}_2) G^2_-(\mathbf{x}_3, \mathbf{x}_4) \; , \tag{3.8}$$

where we used the fact that $< \psi^-_{\mathbf{x},\omega}\psi^+_{\mathbf{y},-\omega} >= 0$.
Let $g_\omega(\mathbf{x})$ be the free propagator, whose Fourier transform is $g_\omega(\mathbf{k}) = \chi_{h,0}(\mathbf{k})/(-ik_0 + \omega k)$. Then, we can write the above equation as

$$\begin{aligned}
&G^4_+(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = -\lambda \int d\mathbf{z}\; g_-(\mathbf{z} - \mathbf{x}_4) < \psi^-_{\mathbf{x}_1,+}\psi^+_{\mathbf{x}_2,+}\psi^-_{\mathbf{x}_3,-}\psi^+_{\mathbf{z},-}\psi^+_{\mathbf{z},+}\psi^-_{\mathbf{z},+} > + \\
&+ \lambda\, G^2_+(\mathbf{x}_1, \mathbf{x}_2) \int d\mathbf{z}\; g_-(\mathbf{z} - \mathbf{x}_4) < \psi^-_{\mathbf{x}_3,-}\psi^+_{\mathbf{z},-}\psi^+_{\mathbf{z},+}\psi^-_{\mathbf{z},+} >= \\
&= -\lambda \int d\mathbf{z} g_{-1}(\mathbf{z} - \mathbf{x}_4) < [\psi^-_{\mathbf{x}_1,+}\psi^+_{\mathbf{x}_2,+}]\, ; [\psi^-_{\mathbf{x}_3,-}\psi^+_{\mathbf{z},-}\psi^+_{\mathbf{z},+}\psi^-_{\mathbf{z},+}] >^T \; .
\end{aligned} \tag{3.9}$$

From (3.9) we get

$$
\begin{aligned}
&- G^4_+(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = \lambda\int d\mathbf{z} g_-(\mathbf{z}-\mathbf{x}_4) < \psi^-_{\mathbf{x}_1,+};\psi^+_{\mathbf{x}_2,+};\rho_{\mathbf{z},+} >^T < \psi^-_{\mathbf{x}_3,-}\psi^+_{\mathbf{z},-} > + \\
&\quad + \lambda\int d\mathbf{z} g_-(\mathbf{z}-\mathbf{x}_4) < \rho_{\mathbf{z},+};\psi^-_{\mathbf{x}_1,+};\psi^+_{\mathbf{x}_2,+};\psi^-_{\mathbf{x}_3,-};\psi^+_{\mathbf{z},-} >^T + \\
&\quad + \lambda\int d\mathbf{z} g_-(\mathbf{z}-\mathbf{x}_4) < \psi^-_{\mathbf{x}_1,+};\psi^+_{\mathbf{x}_2,+};\psi^-_{\mathbf{x}_3,-};\psi^+_{\mathbf{z},-} >^T < \rho_{\mathbf{z},+} > \ .
\end{aligned}
\tag{3.10}
$$

The last addend is vanishing, since $< \rho_{\mathbf{z},\omega} >= 0$ by the propagator parity properties. In terms of Fourier transforms, we get the *Dyson equation*

$$
\begin{aligned}
-\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = \lambda \hat g_-(\mathbf{k}_4)\Big[&\hat G^2_-(\mathbf{k}_3)\hat G^{2,1}_+(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)+ \\
&+\frac{1}{L\beta}\sum_{\mathbf{p}} G^{4,1}_+(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})\Big] \ ,
\end{aligned}
\tag{3.11}
$$

see Fig. 3.

Let us now suppose that $|\mathbf{k}_4| \le \gamma^h$; then the support properties of the propagators imply that $|\mathbf{p}| \le \gamma+\gamma^h \le 2\gamma$, hence we can freely multiply $G^{4,1}_+$ in the r.h.s. of (3.11) by the compact support function $\chi_0(\gamma^{-j_m}|\mathbf{p}|)$, with $j_m = [1+\log_\gamma 2]+1$, χ_0 being defined as in (1.4). It follows that (3.11) can be written as

$$
\begin{aligned}
&- \hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = \lambda \hat g_-(\mathbf{k}_4)\Big[\hat G^2_-(\mathbf{k}_3)\hat G^{2,1}_+(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)+ \\
&+\frac{1}{L\beta}\sum_{\mathbf{p}}\chi_M(\mathbf{p}) G^{4,1}_+(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) + \frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p}) G^{4,1}_+(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})\Big] \ ,
\end{aligned}
\tag{3.12}
$$

where $\chi_M(\mathbf{p})$ is a compact support function vanishing for $|\mathbf{p}| \ge \gamma^{h+j_m-1}$ and

$$
\tilde\chi_M(\mathbf{p}) = \sum_{h_p=h+j_m}^{j_m} f_{h_p}(\mathbf{p}) \ . \tag{3.13}
$$

Note that the decomposition of the $\mathbf{p}$ sum is done so that $\tilde\chi_M(\mathbf{p}) = 0$ if $|\mathbf{p}| \le 2\gamma^h$.

Remark. The l.h.s. of the identity (3.12) is, by (2.37), of order $\lambda_h\gamma^{-4h}Z_h^{-2}$; if we can prove that the l.h.s. is proportional to λ with essentially the same proportionality constants, we get that $\lambda_h \simeq \lambda$. This cannot be achieved if we simply use (2.35), (2.36) and the analogous bound for $\hat G^{4,1}$, given in Lemma A1.2 of [BM3]; for instance, by using (2.35) and (2.36), we see that the first addend in the r.h.s. of (3.12) is of size $\gamma^{-2h}Z_h^{(2)}Z_h^{-2}\lambda[1+O(\bar\lambda_h^2)]$. We have to take into account some crucial cancellations in the perturbative expansion, and this will be done by expressing $\hat G^{2,1}$ and $\hat G^{4,1}$ in terms of other functional integrals by suitable Ward identities which at the end will allow us to prove (3.2).

3.3 *Ward identities.*

By doing in (1.5) the chiral Gauge transformation $\psi^\sigma_{\mathbf{x},+} \to e^{i\sigma\alpha_{\mathbf{x},+}}\psi^\sigma_{\mathbf{x},+}$, $\psi^\sigma_{\mathbf{x},-} \to \psi^\sigma_{\mathbf{x},-}$, one obtains, see [BM2], the Ward identity, see Fig. 4

$$
D_+(\mathbf{p})G^{2,1}_+(\mathbf{p},\mathbf{k},\mathbf{q}) = G^2_+(\mathbf{q}) - G^2_+(\mathbf{k}) + \Delta^{2,1}_+(\mathbf{p},\mathbf{k},\mathbf{q}) \ , \tag{3.14}
$$

with

$$
\Delta^{2,1}_+(\mathbf{p},\mathbf{k},\mathbf{q}) = \frac{1}{\beta L}\sum_{\mathbf{k}} C_+(\mathbf{k},\mathbf{k}-\mathbf{p}) < \hat\psi^+_{\mathbf{k},+}\hat\psi^-_{\mathbf{k}-\mathbf{p},+};\hat\psi^-_{\mathbf{k},+}\hat\psi^+_{\mathbf{q},+} >^T \tag{3.15}
$$

and

$$C_\omega(\mathbf{k}^+,\mathbf{k}^-) = [C_{h,0}(\mathbf{k}^-)-1]D_\omega(\mathbf{k}^-) - [C_{h,0}(\mathbf{k}^+)-1]D_\omega(\mathbf{k}^+) \; . \tag{3.16}$$

In the same way, we get two other Ward identities

$$\begin{aligned} D_+(\mathbf{p})G^{4,1}_+(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) &= G^4_+(\mathbf{k}_1-\mathbf{p},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) - \\ &- G^4_+(\mathbf{k}_1,\mathbf{k}_2+\mathbf{p},\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) + \Delta^{4,1}_+ \; , \end{aligned} \tag{3.17}$$

$$\begin{aligned} D_-(\mathbf{p})G^{4,1}_-(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) &= G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3-\mathbf{p},\mathbf{k}_4-\mathbf{p}) - \\ &- G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) + \Delta^{4,1}_- \; , \end{aligned} \tag{3.18}$$

where $\Delta^{4,1}_\pm$ is the "correction term"

$$\Delta^{4,1}_\pm(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{1}{\beta L}\sum_{\mathbf{k}} C_\pm(\mathbf{k},\mathbf{k}-\mathbf{p}) < \hat\psi^+_{\mathbf{k},\pm}\hat\psi^-_{\mathbf{k}-\mathbf{p},\pm};\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\hat\psi^-_{\mathbf{k}_3,-};\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-} >^T \; . \tag{3.19}$$

3.4 *Counterterms*

Eq. (3.17) can be written, by adding and subtracting suitable counterterms $\nu_\pm$, to be fixed properly later, see Fig. 6

$$\begin{aligned} &(1-\nu_+)D_+(\mathbf{p})\hat G^{4,1}_+(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) - \nu_- D_-(\mathbf{p})\hat G^{4,1}_-(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) \\ &= \hat G^4_+(\mathbf{k}_1-\mathbf{p},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) - \hat G^4_+(\mathbf{k}_1,\mathbf{k}_2+\mathbf{p},\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) + H^{4,1}_+(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) \; , \end{aligned} \tag{3.20}$$

where by definition

$$\begin{aligned} H^{4,1}_+(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) &= \frac{1}{\beta L}\sum_{\mathbf{k}} C_+(\mathbf{k},\mathbf{k}-\mathbf{p}) < \hat\psi^+_{\mathbf{k},+}\hat\psi^-_{\mathbf{k}-\mathbf{p},+};\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\hat\psi^-_{\mathbf{k}_3,-};\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-} >^T - \\ &- \frac{1}{\beta L}\sum_{\mathbf{k}}\sum_{\omega} \nu_\omega D_\omega(\mathbf{p}) < \hat\psi^+_{\mathbf{k},\omega}\hat\psi^-_{\mathbf{k}-\mathbf{p},\omega};\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\hat\psi^-_{\mathbf{k}_3,-};\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-} >^T \; . \end{aligned} \tag{3.21}$$

In the same way, eq. (3.18) can be written as

$$\begin{aligned} &(1-\nu'_-)D_-(\mathbf{p})\hat G^{4,1}_-(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) - \nu'_+ D_+(\mathbf{p})\hat G^{4,1}_+(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) = \\ &= \hat G^4_+(\mathbf{k}_1,\mathbf{k}_2\mathbf{k}_3-\mathbf{p},\mathbf{k}_4-\mathbf{p}) - \hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) + \hat H^{4,1}_-(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) \; , \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} H^{4,1}_-(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) &= \frac{1}{\beta L}\sum_{\mathbf{k}} C_-(\mathbf{k},\mathbf{k}-\mathbf{p}) < \hat\psi^+_{\mathbf{k},-}\hat\psi^-_{\mathbf{k}-\mathbf{p},-};\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\hat\psi^-_{\mathbf{k}_3,-};\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-} >^T - \\ &- \frac{1}{\beta L}\sum_{\mathbf{k}}\sum_{\omega} \nu'_\omega D_\omega(\mathbf{p}) < \hat\psi^+_{\mathbf{k},\omega}\hat\psi^-_{\mathbf{k}-\mathbf{p},\omega};\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\hat\psi^-_{\mathbf{k}_3,-};\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-} >^T \; . \end{aligned} \tag{3.23}$$

If we insert in the r.h.s. of (3.20) the value of $\hat G^{4,1}_-$ taken from (3.22), we get

$$\begin{aligned} &(1+A)D_+(\mathbf{p})\hat G^{4,1}_+(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) = [\hat G^4_+(\mathbf{k}_1-\mathbf{p},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) - \\ &\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2+\mathbf{p},\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})] + B[\hat G^4_+(\mathbf{k}_1,\mathbf{k}_3-\mathbf{p},\mathbf{k}_4-\mathbf{p}) - \hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)] + \hat H^{4,1}_+ + B\hat H^{4,1}_- \; , \end{aligned} \tag{3.24}$$

where

$$A = -\nu_+ - \frac{\nu_-\nu'_+}{1-\nu'_-} \quad , \quad B = \frac{\nu_-}{1-\nu'_-} \; . \tag{3.25}$$

If we insert in the last term of the r.h.s. of (3.12) the value of $\hat G^{4,1}_+$ taken from (3.24), we get

$$\begin{aligned}
&-\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)= \qquad (3.26)\\
&=\lambda\hat g_-(\mathbf{k}_4)\left[\hat G^2_-(\mathbf{k}_3)\hat G^{2,1}_+(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)+\frac{1}{L\beta}\sum_{\mathbf{p}}\chi_M(\mathbf{p})G^{4,1}_+(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})\right]+\\
&+\frac{\lambda\hat g_-(\mathbf{k}_4)}{(1+A)}\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{\hat G^4_+(\mathbf{k}_1-\mathbf{p},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})-\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2+\mathbf{p},\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})}{D_+(\mathbf{p})}+\\
&+\frac{\lambda\hat g_-(\mathbf{k}_4)}{(1+A)}\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3-\mathbf{p},\mathbf{k}_4-\mathbf{p})-\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)}{D_+(\mathbf{p})}+\\
&+\frac{\lambda\hat g_-(\mathbf{k}_4)}{(1+A)}\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{\hat H^{4,1}_+(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})+B\hat H^{4,1}_-(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})}{D_+(\mathbf{p})}\;.
\end{aligned}$$

Note that

$$\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)}{D_+(\mathbf{p})}=0\;, \tag{3.27}$$

since $D_+(\mathbf{p})$ is odd. Then, by using also the Ward identity (3.14), we get, if

$$\mathbf{k}_i=\bar{\mathbf{k}}_i\quad,\quad\bar{\mathbf{k}}_1=\bar{\mathbf{k}}_4=-\bar{\mathbf{k}}_2=-\bar{\mathbf{k}}_3=\bar{\mathbf{k}}\quad,\quad|\bar{\mathbf{k}}|=\gamma^h\;, \tag{3.28}$$

the identity

$$\begin{aligned}
&-\hat G^4_+(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4)=\lambda\hat g_-(\bar{\mathbf{k}}_4)\hat G^2_-(\bar{\mathbf{k}}_3)\frac{G^2_+(\bar{\mathbf{k}}_2)-G^2_+(\bar{\mathbf{k}}_1)}{D_+(2\bar{\mathbf{k}})}+ \qquad (3.29)\\
&+\lambda\hat g_-(\bar{\mathbf{k}}_4)\hat G^2_-(\bar{\mathbf{k}}_3)\frac{\Delta^{2,1}_+(2\bar{\mathbf{k}},\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2)}{D_+(2\bar{\mathbf{k}})}+\lambda\hat g_-(\bar{\mathbf{k}}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\chi_M(\mathbf{p})G^{4,1}_+(\mathbf{p};\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4-\mathbf{p})+\\
&+\frac{\lambda\hat g_-(\bar{\mathbf{k}}_4)}{(1+A)}\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{\hat G^4_+(\bar{\mathbf{k}}_1-\mathbf{p},\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4-\mathbf{p})-\hat G^4_+(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2+\mathbf{p},\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4-\mathbf{p})}{D_+(\mathbf{p})}+\\
&+\frac{\lambda\hat g_-(\bar{\mathbf{k}}_4)}{(1+A)}\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{\hat G^4_+(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3-\mathbf{p},\bar{\mathbf{k}}_4-\mathbf{p})}{D_+(\mathbf{p})}+\\
&+\frac{\lambda\hat g_-(\bar{\mathbf{k}}_4)}{(1+A)}\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{\hat H^{4,1}_+(\mathbf{p};\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4-\mathbf{p})+B\hat H^{4,1}_-(\mathbf{p};\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4-\mathbf{p})}{D_+(\mathbf{p})}\;.
\end{aligned}$$

All the terms appearing in the above equation can be expressed in terms of convergent tree expansions. The term in the l.h.s. of (3.29) is given, by (2.37), by

$$\hat G^4_+(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4)=\gamma^{-4h}Z_h^{-2}[-\lambda_h+O(\bar\lambda_h^2)]\;. \tag{3.30}$$

The two terms in the r.h.s., first line, are equal, by (2.36), to $\gamma^{-4h}Z_h^{-2}(\lambda+O(\bar\lambda_h^2))$. The first term in the second line, by (2.36) and eq. (177) of [BM2], can be bounded as

$$\left|\lambda\hat g_-(\bar{\mathbf{k}}_4)\hat G^2_-(\bar{\mathbf{k}}_3)\frac{\Delta^{2,1}_+(2\bar{\mathbf{k}},\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2)}{D_+(2\bar{\mathbf{k}})}\right|\le C\bar\lambda_h^2\frac{\gamma^{-4h}}{Z_h^2}\;, \tag{3.31}$$

while the second term in the second line, by eq. (A1.11) of ref. [BM3], can be bounded as

$$\left|\lambda\hat g_-(\bar{\mathbf{k}}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\chi_M(\mathbf{p})G^{4,1}_+(\mathbf{p};\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4-\mathbf{p})\right|\le C\bar\lambda_h^3\frac{\gamma^{-4h}}{Z_h^2} \tag{3.32}$$

Moreover, by using Lemma A1.1 of [BM3], one sees that

$$\begin{aligned}&\left|\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{\hat G^4_+(\bar{\mathbf{k}}_1-\mathbf{p},\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4-\mathbf{p})-\hat G^4_+(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2+\mathbf{p},\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4-\mathbf{p})}{D_+(\mathbf{p})}\right. +\\ &\left.+\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{\hat G^4_+(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3-\mathbf{p},\bar{\mathbf{k}}_4-\mathbf{p})}{D_+(\mathbf{p})}\right|\le C\bar\lambda_h\frac{\gamma^{-3h}}{Z_h^2}\ .\end{aligned} \tag{3.33}$$

In the following sections we will prove the following Lemma.

LEMMA 3.1 *There exists $\varepsilon_1\le\varepsilon_0$ and four λ-functions $\nu_+,\nu_-,\nu'_+,\nu'_-$ of order λ (uniformly in h), such that, if $\bar\lambda_h\le\varepsilon_1$,*

$$\left|\lambda\hat g_-(\bar{\mathbf{k}}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{\hat H^{4,1}_\pm(\mathbf{p};\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4-\mathbf{p})}{D_+(\mathbf{p})}\right|\le C\bar\lambda_h^2\frac{\gamma^{-4h}}{Z_h^2} \tag{3.34}$$

The above Lemma, together with the identity (3.29), following from the Dyson equation and the Ward identities, and the previous bounds, proved in refs. [BM2] and [BM3] and following from the tree expansion, implies (3.2); this concludes the proof of Theorem 3.1.

4. Proof of Lemma 3.1

4.1 *The corrections.*
We shall prove first the bound (3.34) for $H^{4,1}_+$; the bound for $H^{4,1}_-$ is done essentially in the same way and will be briefly discussed later. By using (3.21), we get

$$\hat g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})D_+^{-1}(\mathbf{p})\hat H^{4,1}_+(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})=$$

$$\begin{aligned}&=\hat g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{C_+(\mathbf{k},\mathbf{k}-\mathbf{p})}{D_+(\mathbf{p})}<\hat\psi^+_{\mathbf{k},+}\hat\psi^-_{\mathbf{k}-\mathbf{p},+};\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\hat\psi^-_{\mathbf{k}_3,-};\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}>^T+\\ &-\nu_-\hat g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})}<\hat\psi^+_{\mathbf{k},-}\hat\psi^-_{\mathbf{k}-\mathbf{p},-};\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\hat\psi^-_{\mathbf{k}_3,+};\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}>^T-\\ &-\nu_+\hat g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}<\hat\psi^+_{\mathbf{k},+}\hat\psi^-_{\mathbf{k}-\mathbf{p},+};\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\hat\psi^-_{\mathbf{k}_3,-};\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}>^T\ .\end{aligned} \tag{4.1}$$

Let us define

$$\tilde G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=\left.\frac{\partial^4}{\partial\phi^+_{\mathbf{k}_1,+}\partial\phi^-_{\mathbf{k}_2,+}\partial\phi^+_{\mathbf{k}_3,-}\partial J_{\mathbf{k}_4}}\tilde W\right|_{\phi=0}\ , \tag{4.2}$$

where

$$\tilde W=\log\int P(d\hat\psi)e^{-T_1(\psi)+\nu_+T_+(\psi)+\nu_-T_-(\psi)}e^{-V(\hat\psi)+\sum_\omega\int d\mathbf{x}[\phi^+_{\mathbf{x},\omega}\hat\psi^-_{\mathbf{x},\omega}+\hat\psi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}]}\ , \tag{4.3}$$

$$T_1(\psi)=\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{C_+(\mathbf{k},\mathbf{k}-\mathbf{p})}{D_+(\mathbf{p})}(\hat\psi^+_{\mathbf{k},+}\hat\psi^-_{\mathbf{k}-\mathbf{p},+})\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}\hat J_{\mathbf{k}_4}\hat g_-(\mathbf{k}_4)\ , \tag{4.4}$$

$$T_+(\psi)=\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}(\hat\psi^+_{\mathbf{k},+}\hat\psi^-_{\mathbf{k}-\mathbf{p},+})\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}\hat J_{\mathbf{k}_4}\hat g_-(\mathbf{k}_4)\ , \tag{4.5}$$

$$T_-(\psi)=\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})}(\hat\psi^+_{\mathbf{k},-}\hat\psi^-_{\mathbf{k}-\mathbf{p},-})\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}\hat J_{\mathbf{k}_4}\hat g_-(\mathbf{k}_4)\;. \tag{4.6}$$

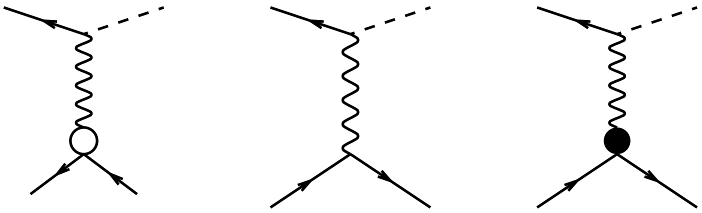


Fig. 12: Graphical representation of T_1, T_+, T_-; the dotted line carries momentum $\bar{\mathbf{k}}_4$, the empty circle represents C_+, the filled one $D_-(\mathbf{p})/D_+(\mathbf{p})$

It is easy to see that $\tilde G^4_+$ is related to (4.1) by an identity similar to (3.9). In fact we can write

$$\begin{aligned}
&-\tilde G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)= \qquad (4.7)\\
&=\hat g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{C_+(\mathbf{k},\mathbf{k}-\mathbf{p})}{D_+(\mathbf{p})}<[\hat\psi^-_{\mathbf{k}_1,+}\hat\psi^+_{\mathbf{k}_2,+}];[\hat\psi^+_{\mathbf{k},+}\hat\psi^-_{\mathbf{k}-\mathbf{p},+}\hat\psi^-_{\mathbf{k}_3,-}\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}]>^T+\\
&-\nu_-\hat g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})}<[\hat\psi^-_{\mathbf{k}_1,+}\hat\psi^+_{\mathbf{k}_2,+}];[\hat\psi^+_{\mathbf{k},-}\hat\psi^-_{\mathbf{k}-\mathbf{p},-}\hat\psi^-_{\mathbf{k}_3,+}\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}]>^T-\\
&-\nu_+\hat g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}<[\hat\psi^-_{\mathbf{k}_1,+}\hat\psi^+_{\mathbf{k}_2,+}];[\hat\psi^+_{\mathbf{k},+}\hat\psi^-_{\mathbf{k}-\mathbf{p},+}\hat\psi^-_{\mathbf{k}_3,-}\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}]>^T\;;
\end{aligned}$$

moreover, if we introduce the definition

$$\delta\rho_{\mathbf{p},+}=\frac{1}{\beta L}\sum_{\mathbf{k}}\frac{C_+(\mathbf{p},\mathbf{k})}{D_+(\mathbf{p})}(\hat\psi^+_{\mathbf{k},+}\hat\psi^-_{\mathbf{k}-\mathbf{p},+})\;, \tag{4.8}$$

the term in the second line of (4.7) can be rewritten as

$$\begin{aligned}
&\hat g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})<[\hat\psi^-_{\mathbf{k}_1,+}\hat\psi^+_{\mathbf{k}_2,+}];[\delta\rho_{\mathbf{p},+}\hat\psi^-_{\mathbf{k}_3,-}\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}]>^T=\\
&=g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\Big[<\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\delta\rho_{\mathbf{p},+}>^T<\hat\psi^-_{\mathbf{k}_3,-}\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}>+ \qquad (4.9)\\
&+<\delta\rho_{\mathbf{p},+};\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\hat\psi^-_{\mathbf{k}_3,-};\hat\psi^+_{\mathbf{k}_4-\mathbf{p},-}>^T\Big]\;,
\end{aligned}$$

where we used the fact that $\mathbf{p}\neq 0$ in the support of $\tilde\chi_M(\mathbf{p})$ and $<\delta\rho_{\mathbf{p},+}>=0$ for $\mathbf{p}\neq 0$. A similar decomposition can be done for the other two terms in the r.h.s. of (4.7); hence, by using (3.21), we get

$$\begin{aligned}
&-\tilde G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{H^{4,1}_+(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})}{D_+(\mathbf{p})}+\\
&+\tilde\chi_M(\mathbf{k}_1-\mathbf{k}_2)g_-(\mathbf{k}_4)G^2_-(\mathbf{k}_3)\Big[<\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\delta\rho_{\mathbf{k}_1-\mathbf{k}_2,+}>^T- \qquad (4.10)\\
&-\nu_+<\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\rho_{\mathbf{k}_1-\mathbf{k}_2,+}>^T-\nu_-\frac{D_-(\mathbf{k}_1-\mathbf{k}_2)}{D_+(\mathbf{k}_1-\mathbf{k}_2)}<\hat\psi^-_{\mathbf{k}_1,+};\hat\psi^+_{\mathbf{k}_2,+};\rho_{\mathbf{k}_1-\mathbf{k}_2,-}>^T\Big]
\end{aligned}$$

We now put $\mathbf{k}_i=\bar{\mathbf{k}}_i$, see (3.28). Since $|\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2|=2\gamma^h$, (3.13) implies that $\tilde\chi_M(\bar{\mathbf{k}}_1-\bar{\mathbf{k}}_2)=0$; hence we get

$$-\tilde G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})\frac{H^{4,1}_+(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})}{D_+(\mathbf{p})}\;. \tag{4.11}$$

Remark. (4.11) says that the last line of the Dyson equation (3.29) can be written as a functional integral very similar to the one for G^4_+ (we are essentially proceding as in the derivation of the Dyson equation, in the opposite direction) except that the interaction V (1.3) is replaced by $V+T_1-\nu_+T_+-\nu_-T_-$; we will evaluate it via a multiscale integration procedure similar to the one for G^4_+, and in the expansion additional running coupling constants will appear; the expansion is convergent again if such new running couplings will remain small uniformly in the infrared cutoff.

4.2 *The properties of $D_\omega(\mathbf{p})^{-1}C_\omega(\mathbf{k},\mathbf{k}-\mathbf{p})$.*

We shall use some properties of the operator $D_\omega(\mathbf{p})^{-1}C_\omega(\mathbf{k},\mathbf{k}-\mathbf{p})$, which were proved in [BM2]. Let us consider first the effect of contracting both $\hat\psi$ fields of $\delta\rho_{\mathbf{p},+}$ on the same or two different scales; in the second case, we also suppose that the regularization procedure (to be defined later, in agreement with this hypothesis) does not act on the propagator of higher scale. Hence, we have to study the quantity

$$\Delta^{(i,j)}_\omega(\mathbf{k}^+,\mathbf{k}^-)=\frac{C_\omega(\mathbf{k}^+,\mathbf{k}^-)}{D_\omega(\mathbf{p})}\tilde g^{(i)}_\omega(\mathbf{k}^+)\tilde g^{(j)}_\omega(\mathbf{k}^-)\;, \tag{4.12}$$

where $\mathbf{p}=\mathbf{k}^+-\mathbf{k}^-$. The crucial observation is that

$$\Delta^{(i,j)}_\omega(\mathbf{k}^+,\mathbf{k}^-)=0\quad,\quad \text{if } h<i,j<0\;, \tag{4.13}$$

since $\chi_{h,0}(\mathbf{k}^\pm)=1$, if $h<i,j<0$. Let us then consider the cases in which $\Delta^{(i,j)}_\omega(\mathbf{k}^+,\mathbf{k}^-)$ is not identically equal to 0. Since $\Delta^{(i,j)}_\omega(\mathbf{k}^+,\mathbf{k}^-)=\Delta^{(j,i)}_\omega(\mathbf{k}^-,\mathbf{k}^+)$, we can restrict the analysis to the case $i\ge j$. We define

$$u_0(\mathbf{k})=\begin{cases}0 & \text{if } |\mathbf{k}|\le 1\\ 1-f_0(\mathbf{k}) & \text{if } 1\le|\mathbf{k}|\end{cases}\quad,\quad u_h(\mathbf{k})=\begin{cases}0 & \text{if } |\mathbf{k}|\ge\gamma^h\\ 1-f_h(\mathbf{k}) & \text{if } |\mathbf{k}|\le\gamma^h\end{cases}\;. \tag{4.14}$$

Then we get, by using (2.19), the fact that $Z_{-1}=Z_0=1$ and $\tilde f_j=f_j$ for $j=0,h$,

$$\Delta^{(0,0)}_\omega(\mathbf{k}^+,\mathbf{k}^-)=\frac{1}{D_\omega(\mathbf{p})}\left[\frac{f_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+)}u_0(\mathbf{k}^-)-\frac{f_0(\mathbf{k}^-)}{D_\omega(\mathbf{k}^-)}u_0(\mathbf{k}^+)\right]\;, \tag{4.15}$$

$$\Delta^{(h,h)}_\omega(\mathbf{k}^+,\mathbf{k}^-)=\frac{1}{D_\omega(\mathbf{p})}\frac{1}{\tilde Z_{h-1}(\mathbf{k}^+)\tilde Z_{h-1}(\mathbf{k}^-)}\left[\frac{f_h(\mathbf{k}^+)u_h(\mathbf{k}^-)}{D_\omega(\mathbf{k}^+)}-\frac{u_h(\mathbf{k}^+)f_h(\mathbf{k}^-)}{D_\omega(\mathbf{k}^-)}\right]\;, \tag{4.16}$$

$$\Delta^{(0,h)}_\omega(\mathbf{k}^+,\mathbf{k}^-)=\frac{1}{D_\omega(\mathbf{p})}\frac{1}{\tilde Z_{h-1}(\mathbf{k}^-)}\left[\frac{f_0(\mathbf{k}^+)u_h(\mathbf{k}^-)}{D_\omega(\mathbf{k}^+)}-\frac{f_h(\mathbf{k}^-)u_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^-)}\right]\;, \tag{4.17}$$

$$\Delta^{(0,j)}_\omega(\mathbf{k}^+,\mathbf{k}^-)=-\frac{1}{Z_{j-1}}\frac{\tilde f_j(\mathbf{k}^-)u_0(\mathbf{k}^+)}{D_\omega(\mathbf{p})D_\omega(\mathbf{k}^-)}\quad,\quad h<j<0\;, \tag{4.18}$$

$$\Delta^{(i,h)}_\omega(\mathbf{k}^+,\mathbf{k}^-)=\frac{1}{Z_{h-1}(\mathbf{k}^-)Z_{i-1}}\frac{\tilde f_i(\mathbf{k}^+)u_h(\mathbf{k}^-)}{D_\omega(\mathbf{p})D_\omega(\mathbf{k}^+)}\quad,\quad j=h<i\le -1\;. \tag{4.19}$$

As an easy consequence of the above equations, as shown in §4.2 of [BM2], one can write, for $0\ge j>h$,

$$\Delta^{(0,j)}_\omega(\mathbf{k}^+,\mathbf{k}^-)=\frac{\mathbf{p}}{D_\omega(\mathbf{p})}\mathbf{S}^{(j)}_\omega(\mathbf{k}^+,\mathbf{k}^-)\;, \tag{4.20}$$

where $S^{(j)}_{\omega,i}(\mathbf{k}^+,\mathbf{k}^-)$ are smooth functions such that

$$|\partial^{m_+}_{\mathbf{k}^+}\partial^{m_j}_{\mathbf{k}^-}S^{(j)}_{\omega,i}(\mathbf{k}^+,\mathbf{k}^-)|\le C_{m_0+m_j}\frac{\gamma^{-j(1+m_j)}}{Z_{j-1}}\;. \tag{4.21}$$

Finally, it is easy to see that, if $0 > i \geq h$,

$$|\Delta_\omega^{(i,h)}(\mathbf{k}^+,\mathbf{k}^-)| \leq C\gamma^{-(i-h)}\frac{\gamma^{-h-i}}{Z_{i-1}} \ . \tag{4.22}$$

Note that, in the r.h.s. of (4.22), there is apparently a Z_{h-1}^{-1} factor missing, but the bound can not be improved; this is a consequence of the fact that $\tilde{Z}_{h-1}(\mathbf{k}) = 0$ for $|\mathbf{k}| \leq \gamma^{h-1}$, see eq. (63) of [BM2], and the support properties of $u_h(\mathbf{k})$. In any case, this is not a problem, since the dimensional dependence of $\Delta^{(i,h)}$ on the field renormalization constants is exactly Z^{-1}. Note also the presence in the bound of the extra factor $\gamma^{-(i-h)}$, with respect to the dimensional bound; it will allow us to avoid renormalization of the marginal terms containing $\Delta^{(i,h)}$.

4.3 *The multiscale expansion of $\tilde{G}_+^4(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4)$: the first integration step.*

The calculation of $\tilde{G}_+^4$ is done via a multiscale expansion; we shall concentrate on the differences with respect to that described in §2, due to the presence in the potential of the terms $T_1(\psi)$ and $T_\pm(\psi)$. Moreover we shall suppose that the momenta $\mathbf{k}_i$ are put equal to $\bar{\mathbf{k}}_i$, defined as in (3.28).

Let us consider the first step of the iterative integration procedure, the integration of the field $\psi^{(0)}$; we shall describe only the terms linear in the external J field, the only ones contributing to $\tilde{G}_+^4$ which were not already discussed. We call $\bar{\mathcal{V}}^{(-1)}(\psi^{[h,-1]})$ the contribution to the effective potential of such terms and we write

$$\bar{\mathcal{V}}^{(-1)}(\psi^{[h,-1]}) = \bar{\mathcal{V}}_{a,1}^{(-1)}(\psi^{[h,-1]}) + \bar{\mathcal{V}}_{a,2}^{(-1)}(\psi^{[h,-1]}) + \bar{\mathcal{V}}_{b,1}^{(-1)}(\psi^{[h,-1]}) + \bar{\mathcal{V}}_{b,2}^{(-1)}(\psi^{[h,-1]}) \ , \tag{4.23}$$

where $\bar{\mathcal{V}}_{a,1}^{(-1)} + \bar{\mathcal{V}}_{a,2}^{(-1)}$ is the sum of the terms in which the field $\hat{\psi}^+_{\bar{\mathbf{k}}_4-\mathbf{p},-}$ appearing in the definition of $T_1(\psi)$ or $T_\pm(\psi)$ is contracted, $\bar{\mathcal{V}}_{a,1}^{(-1)}$ and $\bar{\mathcal{V}}_{a,2}^{(-1)}$ denoting the sum over the terms of this type containing a T_1 or a $T_\pm$ vertex, respectively; $\bar{\mathcal{V}}_{b,1}^{(-1)} + \bar{\mathcal{V}}_{b,2}^{(-1)}$ is the sum of the other terms, that is those where the field $\hat{\psi}^+_{\bar{\mathbf{k}}_4-\mathbf{p},-}$ is an external field, the index $i = 1, 2$ having the same meaning as before.

Note that the condition (3.28) on the external momenta $\mathbf{k}_i$ forbids the presence of vertices of type φ, if $h < 0$, as we shall suppose. Hence, all graphs contributing to $\bar{\mathcal{V}}^{(-1)}$ have, besides the external field of type J, an odd number of external fields of type ψ.

Let us consider first $\bar{\mathcal{V}}_{a,1}^{(-1)}$; we shall still distinguish different group of terms, those where both fields $\hat{\psi}^+_{\mathbf{k},+}$ and $\hat{\psi}^-_{\mathbf{k}-\mathbf{p},+}$ are contracted, those where only one among them is contracted and those where no one is contracted.

If no one of the fields $\hat{\psi}^+_{\mathbf{k},+}$ and $\hat{\psi}^-_{\mathbf{k}-\mathbf{p},+}$ is contracted, we can only have terms with at least four external lines; for the properties of Δ^{ij}, at least one of the fields $\hat{\psi}^+_{\mathbf{k},+}$ and $\hat{\psi}^-_{\mathbf{k}+\mathbf{p},+}$ must be contracted at scale h. If one of these terms has four external lines, hence it is marginal, it has the following form

$$\int d\mathbf{p}\tilde{\chi}_M(\mathbf{p})\hat{\psi}^+_{-,\bar{\mathbf{k}}_4-\mathbf{p}}G_2^{(0)}(\bar{\mathbf{k}}_4-\mathbf{p})\hat{g}_-^{(0)}(\bar{\mathbf{k}}_4-\mathbf{p})\hat{g}_-(\bar{\mathbf{k}}_4)\hat{J}_{\bar{\mathbf{k}}_4}\int d\mathbf{k}\frac{C(\mathbf{k},\mathbf{p})}{D_+(\mathbf{p})}\hat{\psi}^+_{\mathbf{k},+}\hat{\psi}^-_{\mathbf{k}+\mathbf{p},+} \ , \tag{4.24}$$

where $G_2^{(0)}(\mathbf{k})$ is a suitable function which can be expressed as a sum of graphs with an odd number of propagators, hence it vanishes at $\mathbf{k} = 0$. This implies that $G_2^{(0)}(0) = 0$, so that we can regularize it without introducing any running coupling.

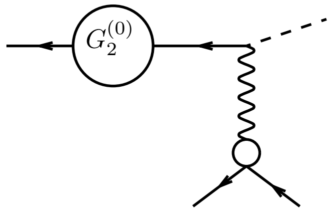


Fig. 13: Graphical representation of (4.24)

If both $\hat{\psi}^+_{\mathbf{k},+}$ and $\hat{\psi}^-_{\mathbf{k}-\mathbf{p},+}$ in $T_1(\psi)$ are contracted, we get terms of the form

$$\tilde{W}^{(-1)}_{n+1}(\bar{\mathbf{k}}_4, \mathbf{k}_1, .., \mathbf{k}_n)\hat{g}_-(\bar{\mathbf{k}}_4)\hat{J}_{\bar{\mathbf{k}}_4}\prod_{i=1}^{n}\hat{\psi}^{\varepsilon_i}_{\mathbf{k}_i} , \tag{4.25}$$

where n is an odd integer. We want to define an $\mathcal{R}$ operation for such terms. There is apparently a problem, as the $\mathcal{R}$ operation involves derivatives and in $\tilde{W}^{(-1)}$ appears the function $\Delta^{(0,0)}$ of the form (4.20) and the cutoff function $\tilde{\chi}_M(\mathbf{p})$, with support on momenta of size γ^h. Hence one can worry about the derivatives of the factor $\tilde{\chi}_M(\mathbf{p})\mathbf{p}D_+(\mathbf{p})^{-1}$. However, as the line of momentum $\bar{\mathbf{k}}_4 - \mathbf{p}$ is necessarily at scale 0 (we are considering terms in which it is contracted), then $|\mathbf{p}| \geq \gamma^{-1} - \gamma^h \geq \gamma^{-1}/2$ (for $|h|$ large enough), so that we can freely multiply by a smooth cutoff function $\bar{\chi}(\mathbf{p})$ restricting $\mathbf{p}$ to the allowed region; this allows us to pass to coordinate space and shows that the $\mathcal{R}$ operation can be defined in the usual way. We define

$$\mathcal{L}\tilde{W}^{(-1)}_4(\bar{\mathbf{k}}_4, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \tilde{W}^{(-1)}_4(0, .., 0) , \tag{4.26}$$

$$\mathcal{L}\tilde{W}^{(-1)}_2(\bar{\mathbf{k}}_4) = \tilde{W}^{(-1)}_2(0) + \bar{\mathbf{k}}_4\partial_{\mathbf{k}}\tilde{W}^{(-1)}_2(0) . \tag{4.27}$$

Note that by parity the first term in (4.27) is vanishing; this means that there are only marginal terms. Note also that the local term proportional to $\hat{J}_{\bar{\mathbf{k}}_4}\hat{\psi}^+_{\bar{\mathbf{k}}_4,-}$ is such that the field $\hat{\psi}^+_{\bar{\mathbf{k}}_4,-}$ can be contracted only at the last scale h; hence it has any influence on the integrations of all the scales $> h$.

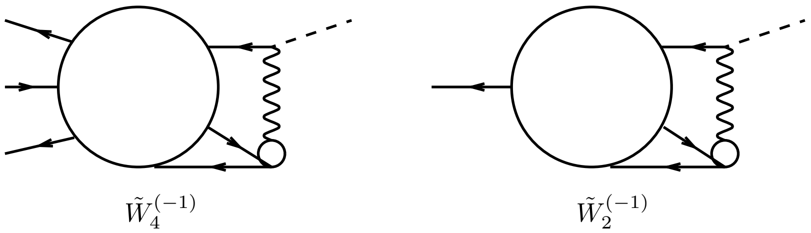


Fig. 14: Graphical representation of $\tilde{W}^{(-1)}_4$ and $\tilde{W}^{(-1)}_2$

If only one among the fields $\hat{\psi}^+_{\mathbf{k},+}$ and $\hat{\psi}^-_{\mathbf{k}-\mathbf{p},+}$ in $T_1(\psi)$ is contracted, we note first that we cannot have terms with two external lines (including $\hat{J}_{\mathbf{k}_4}$); in fact in such a case there is an external line with momentum $\bar{\mathbf{k}}_4$ with $\omega = -$ and the other has $\omega = +$; this is however forbidden by global gauge invariance. Moreover, for the same reasons as before, we do not have to worry about the derivatives of the factor $\tilde{\chi}_M(\mathbf{p})\mathbf{p}D_+(\mathbf{p})^{-1}$, related with the regularization procedure of the terms with four external lines, which have the form

$$\begin{aligned}&\int d\mathbf{k}^+\hat{\psi}^+_{\mathbf{k}_1,+}\hat{\psi}^-_{\mathbf{k}^-,+}\hat{\psi}^+_{\mathbf{k}^-+\bar{\mathbf{k}}_4-\mathbf{k}_1,-}\hat{g}_-(\bar{\mathbf{k}}_4)\hat{J}_{\bar{\mathbf{k}}_4}\tilde{\chi}_M(\mathbf{k}^+-\mathbf{k}^-)\hat{g}^{(0)}_-(\bar{\mathbf{k}}_4-\mathbf{k}^++\mathbf{k}^-)\cdot\\ &\cdot\ G^{(0)}_4(\mathbf{k}^+,\bar{\mathbf{k}}_4,\mathbf{k}_1)\left\{\frac{[C_{h,0}(\mathbf{k}^-)-1]D_+(\mathbf{k}^-)\hat{g}^{(0)}_+(\mathbf{k}^+)}{D_+(\mathbf{k}^+-\mathbf{k}^-)}-\frac{u_0(\mathbf{k}^+)}{D_+(\mathbf{k}^+-\mathbf{k}^-)}\right\} ,\end{aligned} \tag{4.28}$$

or the similar one with the roles of $\mathbf{k}^+$ and $\mathbf{k}^-$ exchanged.

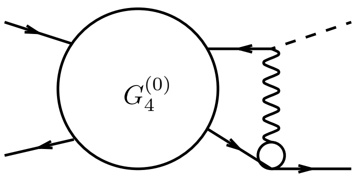


Fig. 15: Graphical representation of a single addend in (4.28)

The two terms in (4.28) must be treated differently, as concerns the regularization procedure. The first term is such that one of the external lines is associated with the operator $[C_{h,0}(\mathbf{k}^-) - 1]D_+(\mathbf{k}^-)D_+(\mathbf{p})^{-1}$. We define $\mathcal{R} = 1$ for such terms; in fact, when such external line is contracted (and this can happen only at scale h), the factor $D_+(\mathbf{k}^-)D_+(\mathbf{p})^{-1}$ produces an extra factor γ^h in the bound, with respect to the dimensional one. This claim simply follows by the observation that $|D_+(\mathbf{p})| \geq 1 - \gamma^{-1}$ as $\mathbf{p} = \mathbf{k}^+ - \mathbf{k}^-$ and $\mathbf{k}^+$ is at scale 0, while $\mathbf{k}^-$, as we said, is at scale h. This factor has the effect that all the marginal terms in the tree path connecting v_0 with the end-point to which is associated the T_1 vertex acquires negative dimension.

The second term in (4.28) can be regularized as above, by subtracting the value of the kernel computed at zero external momenta, *i.e.* for $\mathbf{k}^- = \bar{\mathbf{k}}_4 = \mathbf{k}_1 = 0$. Note that such local part is given by

$$\int d\mathbf{k}^+ \tilde\chi_M(\mathbf{k}^+)\hat g_-^{(0)}(\mathbf{k}^+)G_2^{(0)}(\mathbf{k}^+,0,0)\frac{u_0(\mathbf{k}^+)}{D_+(\mathbf{k}^+)} \;, \tag{4.29}$$

and there is no singularity associated with the factor $D_+(\mathbf{k}^+)^{-1}$, thanks to the support on scale 0 of the propagator $\hat g_-^{(0)}(\mathbf{k}^+)$.

A similar (but simpler) analysis holds for the terms contributing to $\bar{\mathcal{V}}_{a,2}^{(-1)}$, which contain a vertex of type T_+ or T_- and are of order $\lambda\nu_\pm$. Now, the only thing to analyze carefully is the possible singularities associated with the factors $\tilde\chi_M(\mathbf{p})$ and $\mathbf{p}D_+(\mathbf{p})^{-1}$. However, since in these terms the field $\hat\psi^+_{\bar{\mathbf{k}}_4-\mathbf{p},-}$ is contracted, $|\mathbf{p}| \geq \gamma^{-1}/2$, for $|h|$ large enough, a property already used before; hence the regularization procedure can not produce bad dimensional bounds.

We will define $\tilde z_{-1}$ and $\tilde\lambda_{-1}$, so that

$$\mathcal{L}[\bar{\mathcal{V}}_{a,1}^{(-1)} + \bar{\mathcal{V}}_{a,2}^{(-1)}](\psi^{[h,-1]}) = \Big[\tilde\lambda_{-1}\bar F_\lambda^{[h,-1]}(\psi^{[h,-1]}) + \tilde z_{-1}\hat\psi^{[h,-1]+}_{\bar{\mathbf{k}}_4,-}D_-(\bar{\mathbf{k}}_4)\Big]\hat g_-(\bar{\mathbf{k}}_4)\hat J_{\bar{\mathbf{k}}_4} \;, \tag{4.30}$$

where we used the definition

$$\bar F_\lambda^{[h,j]}(\psi^{[h,j]}) = \frac{1}{(L\beta)^4}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3:C_{h,j}^{-1}(\mathbf{k}_i)>0} \hat\psi^{[h,j]+}_{\mathbf{k}_1,+}\hat\psi^{[h,j]-}_{\mathbf{k}_2,+}\hat\psi^{[h,j]+}_{\mathbf{k}_3,-}\delta(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\bar{\mathbf{k}}_4) \;. \tag{4.31}$$

Note that there is no first order contribution to $\tilde\lambda_{-1}$, as follows from a simple calculation, so that $\tilde\lambda_{-1}$ is of order λ^2 or lower. We expect indeed that it satisfies a non zero lower bound of order λ^2, but this will not play any role in the following.

Let us consider now the terms contributing to $\bar{\mathcal{V}}_{b,1}^{(-1)}$, that is those where $\hat\psi^+_{\bar{\mathbf{k}}_4-\mathbf{p}}$ is not contracted and there is a vertex of type T_1.

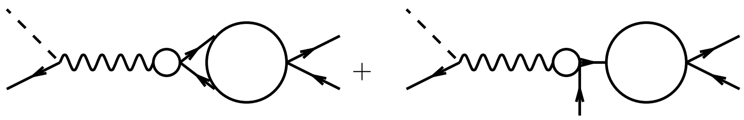
Besides the term of order 0 in λ and $\nu_\pm$, equal to $T_1(\psi^{[h,-1]})$, there are the terms containing at least one vertex λ; among these terms, the only marginal ones (those requiring a regularization) have four external lines (including $\hat J_{\mathbf{k}_4}$), since the oddness of the propagator does not allow tadpoles. These terms are of the form

$$\begin{aligned}\sum_{\tilde\omega}\int d\mathbf{p}\tilde\chi_M(\mathbf{p})\hat\psi^+_{\mathbf{k}^+,\tilde\omega}\int d\mathbf{k}^+\hat\psi^+_{\mathbf{k}^+-\mathbf{p},\tilde\omega}\hat\psi^+_{\bar{\mathbf{k}}_4-\mathbf{p},-}\hat g_-(\bar{\mathbf{k}}_4)\hat J_{\bar{\mathbf{k}}_4}\,\cdot\\ \Big[F^{(-1)}_{2,+,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^+-\mathbf{p}) + F^{(-1)}_{1,+}(\mathbf{k}^+,\mathbf{k}^+-\mathbf{p})\delta_{+,\tilde\omega}\Big] \;,\end{aligned} \tag{4.32}$$

where $F^{(-1)}_{2,+,\tilde\omega}$ and $F^{(-1)}_{1,+}$ are defined as in eq. (132) of [BM2]; they represent the terms in which both or only one of the fields in $\delta\rho_{\mathbf{p},+}$, respectively, are contracted. Both contributions to the r.h.s. of (4.32) are dimensionally marginal; however, the regularization of $F^{(-1)}_{1,+}$ is trivial, as it is of the form

$$F^{(-1)}_{1,+}(\mathbf{k}^+,\mathbf{k}^-) = [\frac{[C_{h,0}(\mathbf{k}^-)-1]D_+(\mathbf{k}^-)\hat g_+^{(0)}(\mathbf{k}^+) - u_0(\mathbf{k}^+)}{D_+(\mathbf{k}^+-\mathbf{k}^-)}G^{(2)}(\mathbf{k}^+) \tag{4.33}$$

or the similar one, obtained exchanging $\mathbf{k}^+$ with $\mathbf{k}^-$.



$F^{(-1)}_{2,+,\tilde\omega}$ $\qquad\qquad$ $F^{(-1)}_{1,+}$

Fig. 16: Graphical representation of (4.33)

By the oddness of the propagator in the momentum, $G^{(2)}(0) = 0$, hence we can regularize such term without introducing any local term, by simply rewriting it as

$$F^{(-1)}_{1,+}(\mathbf{k}^+,\mathbf{k}^-) = [\frac{[C_{h,0}(\mathbf{k}^-)-1]D_+(\mathbf{k}^-)\hat g^{(0)}_+(\mathbf{k}^+) - u_0(\mathbf{k}^+)}{D_+(\mathbf{k}^+-\mathbf{k}^-)}[G^{(2)}(\mathbf{k}^+) - G^{(2)}(0)] \; . \tag{4.34}$$

As shown in [BM2], by using the symmetry property

$$\hat g^{(j)}_\omega(\mathbf{k}) = -i\omega \hat g^{(j)}_\omega(\mathbf{k}^*) \quad , \quad \mathbf{k} = (k,k_0), \quad \mathbf{k}^* = (-k_0, k) \; , \tag{4.35}$$

$F^{(-1)}_{2,+,\tilde\omega}$ can be written as

$$F^{-1}_{2,+,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^-) = \frac{1}{D_+(\mathbf{p})}\left[p_0 A_{0,+,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^-) + p_1 A_{1,+,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^-)\right] \; , \tag{4.36}$$

where $A_{i,+,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^-)$ are functions such that, if we define

$$\mathcal{L}F^{-1}_{2,+,\tilde\omega} = \frac{1}{D_+(\mathbf{p})}\left[p_0 A_{0,+,\tilde\omega}(0,0) + p_1 A_{1,+,\tilde\omega}(0,0)\right] \; , \tag{4.37}$$

then

$$\mathcal{L}F^{-1}_{2,+,+} = Z^{3,+}_{-1} \quad , \quad \mathcal{L}F^{-1}_{2,+,-} = \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} Z^{3,-}_{-1} \; , \tag{4.38}$$

where $Z^{3,+}_{-1}$ and $Z^{3,-}_{-1}$ are suitable real constants. Hence the local part of the marginal term (4.32) is, by definition, equal to

$$Z^{3,+}_{-1} T_+(\psi^{[h,-1]}) + Z^{3,-}_{-1} T_-(\psi^{[h,-1]}) \; . \tag{4.39}$$

Let us finally consider the terms contributing to $\bar{\mathcal{V}}^{(-1)}_{b,2}$, that is those where $\hat\psi^+_{\mathbf{k}_4-\mathbf{p}}$ is not contracted and there is a vertex of type T_+ or T_-. If even this vertex is not contracted, we get a contribution similar to (4.39), with $\nu_\pm$ in place of $Z^{3,\pm}_{-1}$. Among the terms with at least one vertex λ, there is, as before, no term with two external lines; hence the only marginal terms have four external lines and can be written in the form

$$\begin{aligned} &\int d\mathbf{p} \tilde\chi_M(\mathbf{p}) \hat J_{\mathbf{k}_4} \hat g_-(\mathbf{k}_4) \int d\mathbf{k}^+ \sum_{\tilde\omega} \hat\psi^+_{\mathbf{k}^+,\tilde\omega} \hat\psi^-_{\mathbf{k}^+-\mathbf{p},\tilde\omega} \cdot \\ &\cdot \left[\nu_+ G^{(0)}_{+,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^+-\mathbf{p}) + \nu_- \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} G^{(0)}_{-,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^+-\mathbf{p})\right] \; . \end{aligned} \tag{4.40}$$

By using the symmetry property (4.35) of the propagators, it is easy to show that $G^{(0)}_{\omega,-\omega}(0,0) = 0$. Hence, if we regularize (4.40) by subtracting $G^{(0)}_{\omega,\tilde\omega}(0,0)$ to $G^{(0)}_{\omega,\tilde\omega}(\mathbf{k}^+,\mathbf{k}^+-\mathbf{p})$, we still get a local term of the form (4.39).

By collecting all the local term, we can write

$$\mathcal{L}[\bar{\mathcal{V}}^{(-1)}_{b,1}+\bar{\mathcal{V}}^{(-1)}_{b,2}](\psi^{[h,-1]})=\nu_{-1,+}T_+(\psi^{[h,-1]})+\nu_{-1,-}T_-(\psi^{[h,-1]})\ , \tag{4.41}$$

where $\nu_{-1,\omega}=\nu_\omega+Z^{3,\omega}_{-1}+G^{(0)}_{\omega,\omega}(0,0)$. Hence

$$\begin{aligned}&\bar{\mathcal{V}}^{(-1)}(\psi^{[h,-1]})=T_1(\psi^{[h,-1]})+\nu_{-1,+}T_+(\psi^{[h,-1]})+\nu_{-1,-}T_-(\psi^{[h,-1]})+\\&+\left[\tilde\lambda_{-1}\bar F^{[h,-1]}_\lambda(\psi^{[h,-1]})+\tilde z_{-1}\hat\psi^{[h,-1]+}_{\bar{\mathbf{k}}_4,-}D_-(\bar{\mathbf{k}}_4)\right]\hat g_-(\bar{\mathbf{k}}_4)\hat J_{\bar{\mathbf{k}}_4}+\bar{\mathcal{V}}^{(-1)}_R(\psi^{[h,-1]})\ ,\end{aligned} \tag{4.42}$$

where $\bar{\mathcal{V}}^{(-1)}_R(\psi^{[h,-1]})$ is the sum of all irrelevant terms linear in the external field J.

4.4 *The multiscale expansion of $\tilde G^4_+(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4)$: the higher scales integration.*

The integration of the field $\psi^{(-1)}$ is done in a similar way; we shall call $\bar{\mathcal{V}}^{(-2)}(\psi^{[h,-2]})$ the sum over all terms linear in J. As before, the condition (3.28) on the external momenta $\mathbf{k}_i$ forbids the presence of vertices of type φ, if $h<-1$, as we shall suppose.

The main difference is that there is no contribution obtained by contracting both field variables belonging to $\delta\rho$ in $T_1(\psi)$ at scale -1, because of (4.13). It is instead possible to get marginal terms with four external lines (two is impossible), such that one of these fields is contracted at scale -1. However, in this case, the second field variable will be necessarily contracted at scale h, so that we can put $\mathcal{R}=1$ for such terms; in fact, the extra factor $\gamma^{-(-1-h)}$ coming from the bound (4.22) after the integration of the last scale field, has the effect of automatically regularize them, and even the terms containing them as subgraphs.

The terms with a T_1 vertex, such that the field variables belonging to $\delta\rho$ are not contracted, can be treated as in §4.3, hence do not need a regularization.

It follows that, if the irrelevant part $\bar{\mathcal{V}}^{(-1)}_R$ were absent in the r.h.s. of (4.42), then the regularization procedure would not produce any local term proportional to $\bar F^{[h,-1]}_\lambda(\psi^{[h,-2]})$, starting from a graph containing a T_1 vertex.

It is easy to see that all other terms containing a vertex of type T_1 or $T_\pm$ can be treated as in §4.3. Moreover, the support properties of $\hat g_-(\bar{\mathbf{k}}_4)$ immediately implies that it is not possible to produce a graph contributing to $\bar{\mathcal{V}}^{(-2)}$, containing the $\tilde z_{-1}$ vertex. Hence, in order to complete the analysis of $\bar{\mathcal{V}}^{(-2)}$, we still have to consider the marginal terms containing the $\tilde\lambda_{-1}$ vertex, for which we simply apply the localization procedure defined in (4.26), (4.27). We shall define two new constants $\tilde\lambda_{-2}$ and $\tilde z_{-2}$, so that $\tilde\lambda_{-2}(Z_{-2})^2$ is the coefficient of the local term proportional to $\bar F^{[h,-1]}_\lambda(\psi^{[h,-2]})$, while $\tilde z_{-2}Z_{-2}\hat\psi^{[h,-2]+}_{\bar{\mathbf{k}}_4,-}D_-(\bar{\mathbf{k}}_4)\hat g_-(\bar{\mathbf{k}}_4)\hat J_{\bar{\mathbf{k}}_4}$ denotes the sum of all local terms with two external lines produced in the second integration step.

The above procedure can be iterated up to scale $h+1$, without any important difference. In particular, for all marginal terms (necessarily with four external lines) such that one of the field variables belonging to $\delta\rho$ in $T_1(\psi)$ is contracted at scale $i\ge j$, we put $\mathcal{R}=1$. We can do that, because, in this case, the second field variable belonging to $\delta\rho$ has to be contracted at scale h, so that the extra factor $\gamma^{-(i-h)}$ of (4.22) has the effect of automatically regularize their contribution to the tree expansion of $\tilde G^4_+(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4)$, to be described later.

Note that, as in the case $j=-1$, there is no problem connected with the presence of the factors $\tilde\chi(\mathbf{p})$ and $D_-(\mathbf{p})D_+(\mathbf{p})^{-1}$. In fact, if the field $\hat\psi^+_{\bar{\mathbf{k}}_4-\mathbf{p},-}$ appearing in the definition of $T_1(\psi)$ or $T_\pm(\psi)$ is contracted on scale j, each momentum derivative related with the regularization procedure produces the right γ^{-j} dimensional factor, since $\mathbf{p}$ is of order γ^j and the derivatives of $\tilde\chi(\mathbf{p})$ are different from 0 only for momenta of order γ^h. If, on the contrary, the field $\hat\psi^+_{\bar{\mathbf{k}}_4-\mathbf{p},-}$ is not contracted, then the renormalization procedure is tuned so that $\tilde\chi(\mathbf{p})$ and $D_-(\mathbf{p})D_+(\mathbf{p})^{-1}$ are not affected by the regularization procedure.

At step $-j$, we get an expression of the form

$$\bar{\mathcal{V}}^{(j)}(\psi^{[h,j]}) = T_1(\psi^{[h,j]}) + \nu_{j,+}T_+(\psi^{[h,j]}) + \nu_{j,-}T_-(\psi^{[h,j]})+ \\ + \left[\tilde{\lambda}_j Z_j^2 \bar{F}_\lambda^{[h,j]}(\psi^{[h,j]}) + \sum_{i=j}^{-1} \tilde{z}_i Z_i \hat{\psi}^{[h,j]+}_{\bar{\mathbf{k}}_4,-} D_-(\bar{\mathbf{k}}_4)\right] \hat{g}_-(\bar{\mathbf{k}}_4)\hat{J}_{\bar{\mathbf{k}}_4} + \bar{\mathcal{V}}_R^j(\psi^{[h,-1]}) \;, \tag{4.43}$$

where $\bar{\mathcal{V}}_R^j(\psi^{[h,-1]})$ is thought as a convergent tree expansion (under the hypothesis that $\bar{\lambda}_h$ is small enough), to be described in §4.5. Since $Z_{-1} = 1$, this expression is in agreement with (4.42).

The expansion of $\tilde{G}^4_+(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$ is obtained by building all possible graphs with four external lines, which contain one term taken from the expansion of $\bar{\mathcal{V}}^{(h)}(\psi^{(h)})$ and an arbitrary number of terms taken from the effective potential $\mathcal{V}^{(h)}(\psi^{(h)})$. One of the external lines is associated with the free propagator $g_-(\bar{\mathbf{k}}_4)$, the other three are associated with propagators of scale h and momenta $\bar{\mathbf{k}}_i$, $i = 1, 2, 3$.

Remark. With respect to the expansion for G^4_+, there are three additional quartic running coupling constants, $\nu_{j,+}, \nu_{j,-}$ and $\tilde{\lambda}_j$. Note that they are all $O(\lambda)$, despite of the fact that the interaction T_1 has a coupling $O(1)$; this is a crucial property, which follows from the properties in §4.2, implying that either T_1 is contracted at scale 0, or it gives no contribution to the running coupling constants. At a first sight, it seems that now we have a problem more difficult than the initial one; we started from the expansion for G^4_+, which is convergent if the running coupling λ_j is small, and we have reduced the problem to that of controlling the flow of four running coupling constants, $\nu_{+,j}$, $\nu_{j,-}$, λ_j, $\tilde{\lambda}_j$. However, we will see that, under the hypothesis $\bar{\lambda}_h \le \varepsilon_1$, also the flow of $\nu_{j,+}$, $\nu_{j,-}$, $\tilde{\lambda}_j$ is bounded; one uses the counterterms ν_+, ν_- (this is the reason why we introduced them in §3) to impose that $\nu_{+,j}, \nu_{j,-}$ are decreasing and vanishing at $j = h$, and then that the beta functions for $\tilde{\lambda}_j$ and λ_j are identical up to exponentially decaying $O(\gamma^{\tau j})$ terms.

4.5 *The tree structure of the expansion.*

In order to describe the tree expansion of $\bar{\mathcal{V}}^{(j)}(\psi^{[h,j]})$, $j \in [h+1, -1]$, and $\tilde{G}^4_+(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$, we have to modify the tree definition given in §2, after Fig. 11, in the following way.

1) Instead of two, there are six types of special endpoints. There are still the endpoints of type φ, defined exactly as before, but there is no endpoint of type J. In addition, we have special endpoints of type T_1, T_+, T_-, $\tilde{\lambda}$ and $\tilde{z}$, associated in an obvious way with the five local terms of (4.43).

2) There are only trees with one and only one special endpoint of type different from φ.

3) The scale index is $\le +1$ for the endpoints of type T_1, T_+ or T_- , while it is ≤ 0 for the endpoints of type $\tilde{\lambda}$ or $\tilde{z}$. Moreover, the scale index of an endpoint v of type T_1, T_+, T_- or $\tilde{\lambda}$ is equal to $h_{v'} + 1$, if v' is the non trivial vertex immediately preceding v.

4) If the tree has more than one endpoint and one of them is of type $\tilde{z}$, the vertex v_0 of the tree must have scale h and its scale index is equal to any value between $h + 1$ and 0.

5) Given a tree with one endpoint v_1 of type T_1 and scale index $h_{v_1} = +1$, the $\mathcal{R}$ operation in the non trivial vertices of the path $\mathcal{C}$ connecting v_1 to v_0 depends on the set P_1 of external lines in the vertex v_1' (of scale 0) immediately preceding v_1.

If P_1 contains no one of the two external lines of v_1 belonging to the $\delta\rho$ part of the corresponding $T_1(\psi)$ term, then $\mathcal{R}$ is defined in agreement with the localization procedure bringing to (4.38) for all vertices $v \in \mathcal{C}$, such that $v > v_2$, v_2 being the higher vertex, possibly coinciding with v_1', whose set of external lines does not contain the field $\hat{\psi}^+_{\bar{\mathbf{k}}_4-\mathbf{p}}$ of $T_1(\psi)$. For the remaining vertices of $\mathcal{C}$, $\mathcal{R}$ is defined in the usual way.

If P_1 contains both the two external lines of v_1 belonging to $\delta\rho$ (hence the line of momentum $\bar{\mathbf{k}}_4 - \mathbf{p}$ can not belong to P_1), we define $\mathcal{R}$ in agreement with the remark following (4.24), up to the higher vertex $v_2 < v_1$, where at least one of the lines of $\delta\rho$ does not belong to P_1 anymore. For $v \le v_2$, $\mathcal{R}$ is defined in the usual way.

If P_1 contains only one of the two external lines of v_1 belonging to $\delta\rho$ and one defines v_2 as before, $\mathcal{R}$ is defined along $\mathcal{C}$ in agreement with the obvious generalization of (4.34), for $v > v_2$. In v_2 one has to introduce a new label to distinguish two cases, related with the two different terms in the braces of the r.h.s. of (4.28). In the first case, $\mathcal{R} = 1$ for all $v \le v_2$, in the second case $\mathcal{R}$ is defined in the usual way.

If $h_{v_1} \le 0$ and we define P_1 and v_2 as before, the set P_1, as well as the set P_v for all $v < v_1$, must contain at least one of the two external lines of v_1 belonging to $\delta\rho$. Moreover, if P_1 contains one of these lines, then $\mathcal{R} = 1$ for all $v \le v_2$.

6) A similar, but simpler, discussion can be done for the trees containing an endpoint of type $T_\pm$. We do not give the details, but only stress that there is now no vertex v with $|P_v| = 2$ or $|P_v| = 4$, for which $\mathcal{R} = 1$.

4.6 *The flow of $\nu_{j,\pm}$.*

The definitions of the previous sections imply that there is no contribution to $\nu_{j,\pm}$, coming from trees with a special endpoint of type $\tilde\lambda$ or $\tilde z$. Moreover, because of the symmetry (4.35) of the propagators (see remark after (4.40)), $\nu_{j,+}$ gets no contribution from trees with a special endpoint of type $\nu_{j,-}$, and viceversa. Finally, and very important, if a tree contributing to $\nu_{j,\pm}$ has an endpoint of type T_1, this endpoint must have scale index $+1$.

The following Lemma has an important role in the following.

LEMMA 4.1 *If $\bar\lambda_h$ is small enough (uniformly in h), it is possible to choose ν_+ and ν_- so that $\nu_{h,\omega} = 0$ and*

$$|\nu_{j,\omega}| \le c_0 \bar\lambda_h \gamma^{\theta j} \quad , \quad h+1 \le j \le 0\,, \tag{4.44}$$

where $0 < \theta < 1/4$, c_0 is a suitable constant, independent of h, and $\nu_{0,\omega} = \nu_\omega$.

PROOF - The previous remarks imply that there exists $\bar\varepsilon_1 \le \bar\varepsilon_0$, such that, if $\bar\lambda_h \le \bar\varepsilon_1$, we can write

$$\nu_{j-1,\omega} = \nu_{j,\omega} + \beta^{(j)}_{\nu,\omega}(\lambda_j, \nu_{j,\omega}; \ldots; \lambda_0, \nu_{0,\omega}) \quad , \quad h+1 \le j \le 0\,, \tag{4.45}$$

with $\lambda_0 = \lambda$, $\nu_{0,\omega} = \nu_\omega$ and

$$\beta^{(j)}_{\nu,\omega}(\lambda_j, \nu_{j,\omega}; \ldots; \lambda_0, \nu_{0,\omega}) = \beta^{(j,1)}_{\nu,\omega}(\lambda_j; \ldots; \lambda_0) + \sum_{j'=j}^{0} \nu_{j',\omega} \tilde\beta^{(j,j')}_{\nu,\omega}(\lambda_j; \ldots; \lambda_0)\,. \tag{4.46}$$

Moreover given a positive $\theta < 1/4$, there are constants c_1 and c_2 such that

$$|\beta^{(j,1)}_{\nu,\omega}(\lambda_j; \ldots; \lambda_0)| \le c_1 \bar\lambda_h \gamma^{2\theta j} \quad , \quad |\tilde\beta^{(j,j')}_{\nu,\omega}(\lambda_j; \ldots; \lambda_0)| \le c_2 \bar\lambda_h^2 \gamma^{2\theta(j-j')}\,. \tag{4.47}$$

This follows from the fact that $\beta^{(j,1)}_{\nu,\omega}$ and $\tilde\beta^{(j,j')}_{\nu,\omega}$ are given by a sum of trees verifying the bound (2.32) with $d_v > 0$, with at least an end-point respectively at scale 0 and at scale j', hence one can improve the bound respectively by a factor $\gamma^{2\theta j}$ and $\gamma^{2\theta(j-j')}$. In the following we shall call this property the *short memory property*. Note that the bound of $\tilde\beta^{(j,j')}_{\nu,\omega}$ is of order $\bar\lambda_h^2$, instead of $\bar\lambda_h$, because of the symmetry (4.35), but a bound of order $\bar\lambda_h$ would be sufficient.

By a simple iteration, (4.45) can also be written in the form

$$\nu_{j-1,\omega} = \nu_{0,\omega} + \sum_{j'=j}^{0} \beta^{(j')}_{\nu,\omega}(\lambda_{j'}, \nu_{j',\omega}; \ldots; \lambda_0, \nu_{0,\omega})\,. \tag{4.48}$$

We want to show that it is possible to choose $\nu_{0,\omega}$, so that $\nu_{0,\omega}$ is of order $\bar\lambda_h$ and $\nu_{h,\omega} = 0$. Since this last condition, by (4.48), is equivalent to

$$\nu_{0,\omega} = -\sum_{j=h+1}^{0} \beta^{(j)}_{\nu,\omega}(\lambda_j, \nu_{j,\omega}; \ldots; \lambda_0, \nu_{0,\omega})\,, \tag{4.49}$$

we see, by inserting (4.49) in the r.h.s. of (4.48), that we have to show that there is a sequence $\underline{\nu} = \{\nu_j,\, h+1 \le j \le 0\}$, such that $\nu_{0,\omega}$ is of order $\bar\lambda_h$ and

$$\nu_j = -\sum_{j'=h+1}^{j} \beta^{(j')}_{\nu,\omega}(\lambda_{j'}, \nu_{j'}, .., \lambda_0, \nu_0)\ . \tag{4.50}$$

In order to prove that, we introduce the space $\mathfrak{M}_\theta$ of the sequences $\underline{\nu} = \{\nu_j,\, h+1 \le j \le 0\}$ such that $|\nu_j| \le c\bar\lambda_h \gamma^{\theta j}$, for some c; we shall think $\mathfrak{M}_\theta$ as a Banach space with norm $||\underline{\nu}||_\theta = \sup_{h+1\le j\le 0} |\nu_j|\gamma^{-\theta j}\bar\lambda_h^{-1}$. We then look for a fixed point of the operator $\mathbf{T} : \mathfrak{M}_\theta \to \mathfrak{M}_\theta$ defined as:

$$(\mathbf{T}\underline{\nu})_j = -\sum_{j'=h+1}^{j} \beta^{(j')}_{\nu,\omega}(\lambda_{j'}, \nu_{j'}, .., \lambda_0, \nu_0)\ . \tag{4.51}$$

Note that, if $\bar\lambda_h$ is sufficiently small, then $\mathbf{T}$ leaves invariant the ball $\mathfrak{B}_\theta$ of radius $c_0 = 2c_1 \sum_{n=0}^{\infty} \gamma^{-n}$ of $\mathfrak{M}_\theta$, c_1 being the constant in (4.47). In fact, by (4.46) and (4.47), if $||\underline{\nu}||_\theta \le c_0$, then

$$|(\mathbf{T}\underline{\nu})_j| \le \sum_{j'=h+1}^{j} c_1\bar\lambda_h\gamma^{2\theta j'} + \sum_{j'=h+1}^{j}\sum_{i=j'}^{0} c_0\bar\lambda_h\gamma^{\theta i}c_2\bar\lambda_h^2\gamma^{2\theta(j'-i)} \le c_0\bar\lambda_h\gamma^{\theta j}\ , \tag{4.52}$$

if $2c_2\bar\lambda_h^2(\sum_{n=0}^{\infty}\gamma^{-n})^2 \le 1$.

$\mathbf{T}$ is a also a contraction on $\mathfrak{B}_\theta$, if $\bar\lambda_h$ is sufficiently small; in fact, if $\underline{\nu}, \underline{\nu}' \in \mathfrak{M}_\theta$,

$$\begin{aligned} |(\mathbf{T}\nu)_j - (\mathbf{T}\underline{\nu}')_j| &\le \sum_{j'=h+1}^{j} |\beta^{(j')}_{\nu,\omega}(\lambda_{j'}, \nu_{j'}, .., \lambda_0, \nu_0) - \beta^{(j')}_{\nu,\omega}(\lambda_{j'}, \nu'_{j'}, .., \lambda_0, \nu'_0)| \\ &\le \sum_{j'=h+1}^{j}\sum_{i=j'}^{0} ||\underline{\nu}-\underline{\nu}'||_\theta\bar\lambda_h\gamma^{\theta i}c_2\bar\lambda_h^2\gamma^{2\theta(j'-i)} \le \frac{1}{2}||\underline{\nu}-\underline{\nu}'||_\theta\bar\lambda_h\gamma^{\theta j}\ , \end{aligned} \tag{4.53}$$

if $c_2\bar\lambda_h^2(\sum_{n=0}^{\infty}\gamma^{-n})^2 \le 1/2$. Hence, by the contraction principle, there is a unique fixed point $\underline{\nu}^*$ of $\mathbf{T}$ on $\mathfrak{B}_\theta$. ∎

4.7 *The constants $\tilde\lambda_j$ and $\tilde z_j$.*

We shall now analyze the constants $\tilde\lambda_j$ and $\tilde z_j$, $h \le j \le -1$, appearing in the expansion of $\tilde G^4_+(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$. We shall do that by comparing their values with the values of λ_j and z_j. We start noting that the beta function equation for λ_j can be written as

$$\lambda_{j-1} = \left(\frac{Z_j}{Z_{j-1}}\right)^2 \lambda_j + \beta_j + \beta_j^{(1)}\ , \tag{4.54}$$

where β_j is the sum over the local parts of the trees with at least two endpoints and no endpoint of scale index $+1$, while $\beta_j^{(1)}$ is the similar sum over the trees with at least one endpoint of scale index $+1$.

On the other hand we can write

$$\tilde\lambda_{j-1} = \left(\frac{Z_j}{Z_{j-1}}\right)^2 \tilde\lambda_j + \tilde\beta_j + \tilde\beta_j^{(1)} + \tilde\beta_j^{(T)} + \tilde\beta_j^{(\nu)}\ , \tag{4.55}$$

where

1) $\tilde\beta_j$ is the sum over the local parts of the trees with at least two endpoints, no endpoint of scale index $+1$ and one special endpoint of type $\tilde\lambda$.

2) $\tilde\beta_j^{(1)}+\tilde\beta_j^{(T)}$ is the sum over the trees with at least one endpoint of scale index +1; in this case, the special endpoint can be of type $\tilde\lambda$ or T_1 and, if it is of type T_1, its scale index must be equal to +1. $\tilde\beta_j^{(1)}$ and $\tilde\beta_j^{(T)}$ are, respectively, the sum over the trees with the endpoint of type $\tilde\lambda$ or T_1.
3) $\tilde\beta_j^{(\nu)}$ is the sum over the trees with at least two endpoints, whose special endpoint is of type $T_\pm$.

A crucial role in this paper has the following Lemma.

LEMMA 4.2 *Let $\alpha=\tilde\lambda_h/\lambda_h$; then if $\bar\lambda_h$ is small enough, there exists a constant c, independent of λ, such that $|\alpha|\le c$ and*

$$|\tilde\lambda_j-\alpha\lambda_j|\le c\bar\lambda_h\gamma^{\theta j}\quad,\quad h+1\le j\le -1\ . \tag{4.56}$$

PROOF - The main point is the remark that there is a one to one correspondence between the trees contributing to β_j and the trees contributing to $\tilde\beta_j$. In fact the trees contributing to $\tilde\beta_j$ have only endpoints of type λ, besides the special endpoint v^*, and the external field with $\omega=-$ and $\sigma=-$ has to belong to P_{v^*}. It follows that we can associate uniquely with any tree contributing to $\tilde\beta_j$ a tree contributing to β_j, by simply substituting the special endpoint with a normal endpoint, without changing any label. This correspondence is surjective, since we have imposed the condition that the trees contributing to $\tilde\beta_j$ and β_j do not have endpoints of scale index +1. Hence, we can write

$$\left[\left(\frac{Z_j}{Z_{j-1}}\right)^2-1\right](\tilde\lambda_j-\alpha\lambda_j)+\tilde\beta_j-\alpha\beta_j=\sum_{i=j}^{-1}\beta_{j,i}(\tilde\lambda_i-\alpha\lambda_i)\ , \tag{4.57}$$

where, thanks to the “short memory property” and the fact that $Z_j/Z_{j-1}=1+O(\bar\lambda_j^2)$, the constants $\beta_{j,i}$ satisfy the bound $|\beta_{j,i}|\le C\bar\lambda_j\gamma^{2\theta(j-i)}$, with θ defined as in Lemma 4.1.

Among the four last terms in the r.h.s. of (4.55), the only one depending on the $\tilde\lambda_j$ is $\tilde\beta_j^{(1)}$, which can be written in the form

$$\tilde\beta_j^{(1)}=\sum_{i=j}^{-1}\beta'_{j,i}\tilde\lambda_i\ , \tag{4.58}$$

the $\beta'_{j,i}$ being constants which satisfy the bound $|\beta'_{j,i}|\le C\bar\lambda_j\gamma^{2\theta j}$, since they are related to trees with an endpoint of scale index +1. For the same reasons, we have the bounds $|\tilde\beta_j^{(T)}|\le C\bar\lambda_j\gamma^{2\theta j}$, $|\beta_j^{(1)}|\le C\bar\lambda_j^2\gamma^{2\theta j}$. Finally, by using also Lemma 4.1, we see that $|\tilde\beta_j^{(\nu)}|\le C\bar\lambda_j\bar\lambda_h\gamma^{2\theta j}$.

We now choose α so that

$$\tilde\lambda_h-\alpha\lambda_h=0\ , \tag{4.59}$$

and we put

$$x_j=\tilde\lambda_j-\alpha\lambda_j\quad,\quad h+1\le j\le -1\ . \tag{4.60}$$

We can write

$$x_{j-1}=x_{-1}+\sum_{j'=j}^{-1}\left[\sum_{i=j'}^{-1}\beta_{j',i}x_i+\sum_{i=j'}^{-1}\beta'_{j',i}(x_i+\alpha\lambda_i)+\tilde\beta_{j'}^{(T)}+\tilde\beta_{j'}^{(\nu)}-\alpha\beta_j^{(1)}\right]\ . \tag{4.61}$$

On the other hand, the condition (4.59) implies that

$$x_{-1}=-\sum_{j'=h+1}^{-1}\left[\sum_{i=j'}^{-1}\beta_{j',i}x_i+\sum_{i=j'}^{-1}\beta'_{j',i}(x_i+\alpha\lambda_i)+\tilde\beta_{j'}^{(T)}+\tilde\beta_{j'}^{(\nu)}-\alpha\beta_j^{(1)}\right]\ , \tag{4.62}$$

so that, if $h+1 \le j \le -1$, the x_j satisfy the equation

$$x_j = -\sum_{j'=h+1}^{j}\left[\sum_{i=j'}^{-1}\beta_{j',i}x_i + \sum_{i=j'}^{-1}\beta'_{j',i}(x_i+\alpha\lambda_i) + \tilde\beta^{(T)}_{j'} + \tilde\beta^{(\nu)}_{j'} - \alpha\beta^{(1)}_j\right] . \tag{4.63}$$

We want to show that equation (4.63) has a unique solution satisfying the bound

$$|x_j| \le c_0(1+|\alpha|\bar\lambda_h)\bar\lambda_h\gamma^{\theta j} , \tag{4.64}$$

for a suitable constant c_0, independent of h, if $\bar\lambda_h$ is small enough. Hence we introduce the Banach space $\mathfrak{M}_\theta$ of sequences $\underline{x} = \{x_j, h+1 \le j \le -1\}$ with norm $||\underline{x}||_\theta \stackrel{def}{=} \sup_j |x_j|\gamma^{-\theta j}\bar\lambda_h^{-1}$ and look for a fixed point of the operator $\mathbf{T} : \mathfrak{M}_\theta \to \mathfrak{M}_\theta$ defined by the r.h.s. of (4.63). By using the bounds on the various constants appearing in the definition of $\mathbf{T}$, we can easily prove that there are two constants c_1 and c_2, such that

$$|(\mathbf{T}\underline{x})_j| \le c_1\bar\lambda_h(1+|\alpha|\bar\lambda_h)\gamma^{\theta j} + c_2\bar\lambda_h\sum_{j'=h+1}^{j}\sum_{i=j'}^{-1}\gamma^{2\theta(j'-i)}|x_i| . \tag{4.65}$$

Hence, if we take $c_0 = Mc_1$, $M \ge 2$, the ball $\mathfrak{B}_M$ of radius $c_0(1+|\alpha|\bar\lambda_h)$ in $\mathfrak{M}_\theta$ is invariant under the action of $\mathbf{T}$, if $c_2\bar\lambda_h(\sum_{n=0}^\infty \gamma^{-n})^2 \le 1/2$, since $1/2 \le (M-1)/M$. On the other hand, under the same condition, $\mathbf{T}$ is a contraction in all $\mathfrak{M}_\theta$; in fact, if $\underline{x}, \underline{x}' \in \mathfrak{M}_\theta$, then

$$|(\mathbf{T}\underline{x})_j - (\mathbf{T}\underline{x}')_j| \le c_2\bar\lambda_h^2||\underline{x}-\underline{x}'||\sum_{j'=h+1}^{j}\sum_{i=j'}^{-1}\gamma^{2\theta(j'-i)}\gamma^{\theta i} \le \frac{1}{2}||\underline{x}-\underline{x}'||\bar\lambda_h\gamma^{\theta j} , \tag{4.66}$$

if $c_2\bar\lambda_h(\sum_{n=0}^\infty \gamma^{-n})^2 \le 1/2$. It follows, by the contraction principle, that there is a unique fixed point in the ball $\mathfrak{B}_M$, for any $M \ge 2$, hence a unique fixed point in $\mathfrak{M}_\theta$, satisfying the condition (4.64) with $c_0 = 2c_1$.

To complete the proof, we have to show that α can be bounded uniformly in h. In order to do that, we insert in the l.h.s. of (4.62) the definition of x_{-1} and we bound the r.h.s. by using (4.64) and (4.65); we get

$$|\tilde\lambda_{-1} - \alpha\lambda_{-1}| \le c_3\bar\lambda_h + c_4|\alpha|\bar\lambda_h^2 , \tag{4.67}$$

for some constants c_3 and c_4. Since $|\lambda_{-1}| \ge c_5|\lambda|$, $\tilde\lambda_{-1} \le c_6|\lambda|$ and $\bar\lambda_h \le 2|\lambda|$ by the inductive hypothesis, we have

$$|\alpha\lambda_{-1}| \le |\tilde\lambda_{-1}| + c_3\bar\lambda_h + c_4|\alpha|\bar\lambda_h^2 \Rightarrow |\alpha| \le (c_6 + 2c_3 + 2c_4|\alpha|\bar\lambda_h)/c_5 , \tag{4.68}$$

so that, $|\alpha| \le 2(c_6+2c_3)/c_5$, if $4c_4\bar\lambda_h \le c_5$. ∎

Remark. The above Lemma is based on the fact that λ_j and $\tilde\lambda_j$ have the same Beta function, up to $O(\gamma^{\theta j})$ terms (note that this is true thanks to our choice of the counterterms $\nu_\pm$, which implies that $\nu_{j,\pm}$ are $O(\gamma^{\theta j})$). Hence if λ_j is small, the same is true for $\tilde\lambda_j$.

We want now to discuss the properties of the constants $\tilde z_j$, $h \le j \le -1$, by comparing them with the constants z_j, which are involved in the renormalization of the free measure, see (2.23). There is a tree expansion for the z_j, which can be written as

$$z_j = \beta_j + \beta^{(1)}_j , \tag{4.69}$$

where β_j is the sum over the trees without endpoints of scale index $+1$, while $\beta_j^{(1)}$ is the sum of the others, satisfying the bound $|\beta_j^{(1)}| \le C\bar\lambda_h^2\gamma^{\theta j}$. The tree expansion of the $\tilde z_j$ can be written as

$$\tilde z_j = \tilde\beta_j + \tilde\beta_j^{(\nu)} + \tilde\beta_j^{(1)} \ , \tag{4.70}$$

where $\tilde\beta_j$ is the sum over the trees without endpoints of scale index $+1$, such that the special endpoint is of type $\tilde\lambda$, $\tilde\beta_j^{(\nu)}$ is the sum over the trees whose special endpoint is of type $T_\pm$, and $\tilde\beta_j^{(1)}$ is the sum over the trees with at least an endpoint of scale index $+1$ (in this case, if the special endpoint is of type T_1, its scale index must be $+1$, see discussion in §4.4).

Since there is no tree contributing to $\tilde\beta_j^{(1)}$ without at least one λ or $\tilde\lambda$ endpoint and since all trees contributing to it satisfy the "short memory property", by using Lemma 4.2 (which implies that $|\tilde\lambda_j| \le C\bar\lambda_h$), we get the bound $|\tilde\beta_j^{(1)}| \le C\bar\lambda_h\gamma^{\theta j}$. In a similar manner, by using Lemma 4.1, we see that $|\tilde\beta_j^{(\nu)}| \le C\bar\lambda_h^2\gamma^{\theta j}$.

Let us now consider β_j and $\tilde\beta_j$. By an argument similar to that used in the proof of Lemma 4.2, we can write

$$\tilde\beta_j - \alpha\beta_j = \sum_{i=j+1}^{-1} \beta_{j,i}(\tilde\lambda_i - \alpha\lambda_i) \ , \tag{4.71}$$

where α is defined as in Lemma 4.2 and $|\beta_{j,i}| \le C\bar\lambda_h\gamma^{2\theta j}$. Hence, Lemma 4.2 implies that

$$|\tilde z_j - \alpha z_j| \le C\bar\lambda_h\gamma^{\theta j} \ . \tag{4.72}$$

4.8 *The bound of* $\tilde G_+^4(\bar{\mathbf k}_1, \bar{\mathbf k}_2, \bar{\mathbf k}_3, \bar{\mathbf k}_4)$.

There are various classes of trees contributing to the tree expansion of $\tilde G_+^4(\bar{\mathbf k}_1, \bar{\mathbf k}_2, \bar{\mathbf k}_3, \bar{\mathbf k}_4)$, depending on the type of the special endpoint. Let us consider first the family $\mathcal T_{\tilde\lambda}$ of the trees with an endpoint of type $\tilde\lambda$. These trees have the same structure of those appearing in the expansion of $G_+^4(\bar{\mathbf k}_1, \bar{\mathbf k}_2, \bar{\mathbf k}_3, \bar{\mathbf k}_4)$, except for the fact that the external (renormalized) propagator of scale h and momentum $\bar{\mathbf k}_4$ is substituted with the free propagator $\hat g_-(\bar{\mathbf k}_4)$. It follows, by using the bound $|\tilde\lambda_j| \le C\bar\lambda_h$, that a tree with n endpoint is bounded by $(C\bar\lambda_h)^n Z_h^{-1}\gamma^{-4h}$, larger for a factor Z_h with respect to what we need.

Let us now consider the family $\mathcal T_{\tilde z}$ of the trees with a special endpoint of type $\tilde z$. Given a tree $\tau \in \mathcal T_{\tilde\lambda}$, we can associate with it the class $\mathcal T_{\tilde z,\tau}$ of all $\tau' \in \mathcal T_{\tilde\lambda}$, obtained by τ in the following way:

1) we substitute the endpoint v^* of type $\tilde\lambda$ of τ with an endpoint of type λ;

2) we link the endpoint v^* to an endpoint of type $\tilde z$ trough a renormalized propagator of scale h.

Note that $\mathcal T_{\tilde z} = \cup_{\tau\in\mathcal T_{\tilde\lambda}}\mathcal T_{\tilde z,\tau}$ and that, if τ has n endpoints, any $\tau' \in \mathcal T_{\tilde z,\tau}$ has $n+1$ endpoints. Moreover, since the value of $\bar{\mathbf k}_4$ has be chosen so that $f_h(\bar{\mathbf k}_4) = 1$, $\hat g_-^{(h)}(\bar{\mathbf k}_4) = Z_{h-1}^{-1}\hat g_-(\bar{\mathbf k}_4)$; hence it is easy to show that the sum of the values of a tree $\tau \in \mathcal T_{\tilde\lambda}$, such the special endpoint has scale index j^*+1, and of all $\tau' \in \mathcal T_{\tilde z,\tau}$ is obtained from the value of τ, by substituting $\tilde\lambda_{j^*}$ with

$$\Lambda_{j^*} = \tilde\lambda_{j^*} - \lambda_{j^*}\frac{\sum_{j=h}^{-1}\tilde z_j Z_j}{Z_{h-1}} \ , \tag{4.73}$$

see Fig. 10.

On the other hand, (4.72) and the bound $Z_j \le \gamma^{-C\bar\lambda_h^2 j}$, see [BM1], imply that, if $\bar\lambda_h$ is small enough

$$\sum_{j=h}^{-1} |\tilde z_j Z_j - \alpha z_j Z_j| \le \sum_{j=h}^{-1} C\bar\lambda_h\gamma^{\theta j}Z_j \le C\bar\lambda_h \ . \tag{4.74}$$

It follows, by using also the bound (4.56), that

$$\Lambda_{j^*}=\alpha\lambda_{j^*}\left[1-\frac{\sum_{j=h}^{-1}z_jZ_j}{Z_{h-1}}\right]+\frac{O(\bar\lambda_h)}{Z_h}\ . \tag{4.75}$$

Moreover, since $Z_{j-1}=Z_j(1+z_j)$, for $j\in[-1,h]$, and $Z_{-1}=1$, it is easy to check that

$$Z_{h-1}-\sum_{j=h}^{-1}z_jZ_j=1\ . \tag{4.76}$$

This identity, Lemma 4.2 and (4.75) imply the bound

$$|\Lambda_{j^*}|\le C\frac{\bar\lambda_h}{Z_h}\ , \tag{4.77}$$

which gives us the "missing" Z_h^{-1} factor for the sum over the trees whose special endpoint is of type $\tilde\lambda$ or $\tilde z$.

Let us now consider the family $\mathcal{T}_\nu$ of the trees with a special endpoint of type $T_\pm$. It is easy to see, by using Lemma 4.1 and the "short memory property", that the sum over the trees of this class with $n\ge 0$ normal endpoints is bounded, for $\bar\lambda_h$ small enough, by $(C\bar\lambda_h)^{n+1}Z_h^{-1}\gamma^{-4h}\sum_{j=h}^{-1}Z_j^{-2}\gamma^{2\theta(h-j)}$ $\gamma^{\theta j}\le(C\bar\lambda_h)^{n+1}Z_h^{-3}\gamma^{-(4-\theta)h}$, which is even better of our needs.

We still have to consider the family $\mathcal{T}_1$ of the trees with a special endpoint of type T_1. There is first of all the trivial tree, obtained by contracting all the ψ lines of T_1 on scale h, but its value is 0, because of the support properties of the function $\tilde\chi(\mathbf{p})$. Let us now consider a tree $\tau\in\mathcal{T}_1$ with $n\ge 1$ endpoints of type λ, whose structure is described in item 5) of §4.5, which we shall refer to for notation. If we put $h_{v_1}=j_1+1$ and $h_{v_2}=j_2$, then the dimensional bound of this tree differs from that of a tree with $n+1$ normal endpoints contributing to $G^4_+(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3,\bar{\mathbf{k}}_4)$ for the following reasons:

1) there is a factor Z_h^{-1} missing, because the external (renormalized) propagator of scale h and momentum $\bar{\mathbf{k}}_4$ is substituted with the free propagator $\hat g_-(\bar{\mathbf{k}}_4)$;

2) there is a factor $|\lambda_{j_1}|Z_{j_1}^2$ missing, because there is no external field renormalization in the $T_1(\psi^{[h,j]})$ contribution to $\bar{\mathcal{V}}^{(j)}(\psi^{[h,j]})$, see (4.43);

3) if P_1 contains only one of the two external lines of v_1 belonging to $\delta\rho$, then there is a factor $\gamma^{-(j_2-h)}$ missing, because the absence of regularization in the vertices $v\le v_2$, but this is compensated by the same factor arising because of the bound (4.22), see discussion after (4.28) and in §4.4, so that the "short memory property" is always satisfied;

4) there is a factor Z_h^{-1} missing, because of the remark following (4.22).

It follows that the sum of the values of all trees $\tau\in\mathcal{T}_1$ with $n\ge 1$ normal endpoints, if $\bar\lambda_h$ is small enough, is bounded by $(C\bar\lambda_h)^n\gamma^{-4h}\sum_{j_1=h}^{0}Z_{j_1}^{-2}\gamma^{2\theta(h-j_1)}\le(C\bar\lambda_h)^n\gamma^{-4h}Z_h^{-2}$.

By collecting all the previous bounds, we prove that the bound (3.34) of Lemma 3.1 is satisfied in the case of $H^{4,1}_+$.

Remark. In T_1 and in the Grassmannian monomials multiplying $\nu_{j,+},\nu_{j,-}$, an external line is always associated to a free propagator $\hat g_-(\bar{\mathbf{k}}_4)$; this is due to the fact that, in deriving the Dyson equation (3.10), one extracts a free propagator. Then in the bounds there is a Z_h missing (such propagator is not "dressed" in the multiscale integration procedure), and at the end the crucial identity (4.76) has to be used to "dress" the extracted propagator carrying momentum $\bar{\mathbf{k}}_4$.

4.9 *The bound in the case of $H^{4,1}_-$.*

If we substitute, in the l.h.s. of (4.1) $H^{4,1}_+$ with $H^{4,1}_-$, we can proceed in a similar way. By using (3.23), we get

$$\hat g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde\chi_M(\mathbf{p})D_+^{-1}(\mathbf{p})\hat H^{4,1}_-(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})=$$

$$= \hat{g}_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde{\chi}_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{C_-(\mathbf{k},\mathbf{k}-\mathbf{p})}{D_+(\mathbf{p})} < \hat{\psi}^+_{\mathbf{k},-}\hat{\psi}^-_{\mathbf{k}-\mathbf{p},-};\hat{\psi}^-_{\mathbf{k}_1,+};\hat{\psi}^+_{\mathbf{k}_2,+};\hat{\psi}^-_{\mathbf{k}_3,-};\hat{\psi}^+_{\mathbf{k}_4-\mathbf{p},-} >^T +$$

$$-\nu'_-\hat{g}_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde{\chi}_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} < \hat{\psi}^+_{\mathbf{k},-}\hat{\psi}^-_{\mathbf{k}-\mathbf{p},-};\hat{\psi}^-_{\mathbf{k}_1,+};\hat{\psi}^+_{\mathbf{k}_2,+};\hat{\psi}^-_{\mathbf{k}_3,+};\hat{\psi}^+_{\mathbf{k}_4-\mathbf{p},-} >^T -$$

$$-\nu'_+\hat{g}_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde{\chi}_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}} < \hat{\psi}^+_{\mathbf{k},+}\hat{\psi}^-_{\mathbf{k}-\mathbf{p},+};\hat{\psi}^-_{\mathbf{k}_1,+};\hat{\psi}^+_{\mathbf{k}_2,+};\hat{\psi}^-_{\mathbf{k}_3,-};\hat{\psi}^+_{\mathbf{k}_4-\mathbf{p},-} >^T \ . \tag{4.78}$$

We define $\tilde{G}^4_-(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$ as in (4.2) with $\tilde{W}$ replaced by $\tilde{W}_-$ given by

$$\tilde{W}_- = \log\int P(d\hat{\psi})e^{-T_2(\psi)+\nu'_+T_+(\psi)+\nu'_-T_-(\psi)}e^{-V(\hat{\psi})+\sum_\omega\int d\mathbf{x}[\phi^+_{\mathbf{x},\omega}\hat{\psi}^-_{\mathbf{x},\omega}+\hat{\psi}^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}]} \ , \tag{4.79}$$

$$T_2(\psi) = \frac{1}{L\beta}\sum_{\mathbf{p}}\tilde{\chi}_M(\mathbf{p})\frac{1}{L\beta}\sum_{\mathbf{k}}\frac{C_-(\mathbf{k},\mathbf{k}-\mathbf{p})}{D_+(\mathbf{p})}(\hat{\psi}^+_{\mathbf{k},-}\hat{\psi}^-_{\mathbf{k}-\mathbf{p},-})\hat{\psi}^+_{\mathbf{k}_4-\mathbf{p},-}\hat{J}_{\mathbf{k}_4}\hat{g}(\mathbf{k}_4) \ , \tag{4.80}$$

T_+, T_- being defined as in (4.5), (4.6). By the analogues of (4.9), (4.10) we obtain

$$-\tilde{G}^4_-(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = g_-(\mathbf{k}_4)\frac{1}{L\beta}\sum_{\mathbf{p}}\tilde{\chi}_M(\mathbf{p})\frac{H^{4,1}_-(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})}{D_+(\mathbf{p})} \ . \tag{4.81}$$

The calculation of $\tilde{G}^4_-(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$ is done via a multiscale expansion essentially identical to the one of $\tilde{G}^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$, by taking into account that $\delta\rho_{\mathbf{p},+}$ has to be substituted with

$$\delta\rho_{\mathbf{p},-} = \frac{1}{\beta L}\sum_{\mathbf{k}}\frac{C_-(\mathbf{p},\mathbf{k})}{D_+(\mathbf{p})}(\hat{\psi}^+_{\mathbf{k},-}\hat{\psi}^-_{\mathbf{k}-\mathbf{p},-}) \ . \tag{4.82}$$

Let us consider the first step of the iterative integration procedure and let us call again $\bar{\mathcal{V}}^{(-1)}(\psi^{[h,-1]})$ the contribution to the effective potential of the terms linear in J. Let us now decompose $\bar{\mathcal{V}}^{(-1)}(\psi^{[h,-1]})$ as in (4.23) and let us consider the terms contributing to $\bar{\mathcal{V}}^{(-1)}_{a,1}(\psi^{[h,-1]})$. The analysis goes exactly as before when no one or both the fields $\hat{\psi}^+_{\mathbf{k},-}$ and $\hat{\psi}^-_{\mathbf{k}-\mathbf{p},-}$ of $\delta\rho_{\mathbf{p},-}$ are contracted. This is not true if only one among the fields $\hat{\psi}^+_{\mathbf{k},-}$ and $\hat{\psi}^-_{\mathbf{k}-\mathbf{p},-}$ in $T_2(\psi)$ is contracted, since in this case there are marginal terms with two external lines, which before were absent. The terms with four external lines can be treated as before; one has just to substitute $D_+(\mathbf{k}^-)\hat{g}^{(0)}_+(\mathbf{k}^+)$ with $D_-(\mathbf{k}^-)g^{(0)}_-(\mathbf{k}^+)$ in the r.h.s. of (4.28), but this has no relevant consequence. The terms with two external lines have the form

$$\begin{aligned}&\int d\mathbf{k}^-\hat{\psi}^+_{\bar{\mathbf{k}}_4,-}\hat{g}_-(\bar{\mathbf{k}}_4)\hat{J}_{\bar{\mathbf{k}}_4}\tilde{\chi}_M(\bar{\mathbf{k}}^4-\mathbf{k}^-)G^{(0)}_1(\mathbf{k}_-)\,\cdot\\ &\cdot\left\{\frac{[C^\varepsilon_{h,0}(\bar{\mathbf{k}}_4)-1]D_-(\bar{\mathbf{k}}_4)\hat{g}^{(0)}_-(\mathbf{k}^-)}{D_+(\bar{\mathbf{k}}_4-\mathbf{k}^-)}-\frac{u_0(\mathbf{k}^-)}{D_+(\bar{\mathbf{k}}^4-\mathbf{k}^-)}\right\} \ ,\end{aligned} \tag{4.83}$$

where $G^{(0)}_1(\mathbf{k}_-)$ is a smooth function of order 1 in λ. However, the first term in the braces is equal to 0, since $|\bar{\mathbf{k}}_4| = \gamma^h$ implies that $C^\varepsilon_{h,0}(\bar{\mathbf{k}}_4) - 1 = 0$. Hence the r.h.s. of (4.83) is indeed of the form

$$\int d\mathbf{k}^-\hat{\psi}^+_{\bar{\mathbf{k}}_4,-}\hat{g}_-(\bar{\mathbf{k}}_4)\hat{J}_{\bar{\mathbf{k}}_4}\tilde{\chi}_M(\bar{\mathbf{k}}_4-\mathbf{k}^-)G^{(0)}_1(\mathbf{k}_-)\frac{u_0(\mathbf{k}^-)}{D_+(\bar{\mathbf{k}}^4-\mathbf{k}^-)} \ , \tag{4.84}$$

so that it can be regularized in the usual way.

The analysis of $\bar{\mathcal{V}}^{(-1)}_{a,2}(\psi^{[h,-1]})$ can be done exactly as before. Hence, we can define again $\tilde{\lambda}_{-1}$ and $\tilde{z}_{-1}$ as in (4.30), with $\tilde{\lambda}_{-1} = O(\lambda)$ and $\tilde{z}_{-1} = O(1)$.

Let us consider now the terms contributing to $\bar{\mathcal{V}}^{(-1)}_{b,1}$, that is those where $\hat{\psi}^+_{\bar{\mathbf{k}}_4-\mathbf{p}}$ is not contracted and there is a vertex of type T_2. Again the only marginal terms have four external lines and have the form

$$\begin{aligned}&\sum_{\tilde{\omega}}\int d\mathbf{p}\tilde{\chi}_M(\mathbf{p})\hat{\psi}^+_{\mathbf{k}^+,\tilde{\omega}}\int d\mathbf{k}^+\hat{\psi}^+_{\mathbf{k}^+-\mathbf{p},\tilde{\omega}}\hat{\psi}^+_{\bar{\mathbf{k}}_4-\mathbf{p},-}\hat{g}_-(\bar{\mathbf{k}}_4)\hat{J}_{\bar{\mathbf{k}}_4}\cdot\\&\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})}\left[F^{(-1)}_{2,-,\tilde{\omega}}(\mathbf{k}^+,\mathbf{k}^+-\mathbf{p})+F^{(-1)}_{1,-}(\mathbf{k}^+,\mathbf{k}^+-\mathbf{p})\delta_{-,\tilde{\omega}}\right]\;,\end{aligned}\tag{4.85}$$

where we are using again the definition (132) of [BM2] (hence we have to introduce in the r.h.s. of (4.85) the factor $D_-(\mathbf{p})/D_+(\mathbf{p})$). The analysis of the terms $F^{(-1)}_{1,-}(\mathbf{k}^+,\mathbf{k}^+-\mathbf{p})$ is identical to the one in §4.3, while, as shown in [BM2], the symmetry property (4.35) implies now that, if we define

$$F^{-1}_{2,-,\tilde{\omega}}(\mathbf{k}^+,\mathbf{k}^-)=\frac{1}{D_-(\mathbf{p})}\left[p_0A_{0,-,\tilde{\omega}}(\mathbf{k}^+,\mathbf{k}^-)+p_1A_{1,-,\tilde{\omega}}(\mathbf{k}^+,\mathbf{k}^-)\right]\;,\tag{4.86}$$

and

$$\mathcal{L}F^{-1}_{2,-,\tilde{\omega}}=\frac{1}{D_-(\mathbf{p})}\left[p_0A_{0,-,\tilde{\omega}}(0,0)+p_1A_{1,-,\tilde{\omega}}(0,0)\right]\;,\tag{4.87}$$

then

$$\mathcal{L}F^{-1}_{2,-,+}=Z^{3,-}_{-1}\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})}\quad,\quad\mathcal{L}F^{-1}_{2,-,-}=Z^{3,+}_{-1}\;,\tag{4.88}$$

where $Z^{3,+}_{-1}$ and $Z^{3,-}_{-1}$ are the same real constants appearing in (4.38). Hence, the local part of the marginal term (4.85) is, by definition, equal to

$$Z^{3,+}_{-1}T_+(\psi^{[h,-1]})+Z^{3,-}_{-1}T_-(\psi^{[h,-1]})\;.\tag{4.89}$$

The analysis of $\bar{\mathcal{V}}^{(-1)}_{b,2}$ can be done exactly as before, so that we can write for $\bar{\mathcal{V}}^{(-1)}$ an expression similar to (4.42), with $T_2(\psi^{[h,-1]})$ in place of $T_1(\psi^{[h,-1]})$ and $\nu'_{-1,\pm}$ in place of $\nu_{-1,\pm}$. One can prove that, for simple symmetry reasons, $\nu'_{-1,\pm}=\nu_{-1,\mp}$, if $\nu'_\pm=\nu_\mp$, but this property will not play any role, hence we will not prove it.

The integration of higher scales proceed as in §4.4. In fact, the only real difference we found in the integration of the first scale was in the calculation of the $O(1)$ terms contributing to $\tilde{z}_{-1}$, but these terms are absent in the case of $\tilde{z}_j$, $j\le -2$, because the second term in the expression analogous to (4.83), obtained by contracting on scale $j<0$ only one of the fields of $\delta\rho_{\mathbf{p},-}$, is exactly zero. It follows that the tree structure of the expansion is the same as that described in §4.5 and the constants ν'_ω can be chosen again so that the bound (4.44) is satisfied even by the constants $\nu'_{j,\omega}$.

In the analysis of the constants $\tilde{\lambda}_j$ and $\tilde{z}_j$ there is only one difference, concerning the bound (4.72), which has to be substituted with $\tilde{z}_{-1}-\alpha z_{-1}\le C$, in the case $j=-1$, but it is easy to see that this has no effect on the bound (4.77). It follows that the final considerations of §4.8 stay unchanged and we get for $\tilde{G}^4_-(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$ a bound similar to that proved for $\tilde{G}^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$, so ending the proof of Lemma 3.1.

Appendix A1. The ultraviolet problem and the Thirring model

Thanks to the linearity of the propagator, the above analysis can be used with no essential modifications to construct the massless *Thirring model* (see for instance [Z]), by removing the ultraviolet cutoff. We shall sketch here the main ideas; the details will be published elsewhere.

The Thirring model describes Dirac fermions in $d=1+1$ interacting with a local current-current interaction; its action is

$$\int d\mathbf{x}[-\bar{\psi}_\mathbf{x}\;\partial\!\!\!/\psi_\mathbf{x}-\frac{\lambda}{4}J_\mu(\mathbf{x})J^\mu(\mathbf{x})]\tag{A1.1}$$

where $\not\partial = \gamma_0\partial_{x_0} + \gamma_1\partial_x$, $\mathbf{x} = (x_0, x)$, $\bar\psi_{\mathbf{x}} = \psi^+_{\mathbf{x}}\gamma_0$, $\psi_{\mathbf{x}}$ is a two component spinor field (to not be confused with a Grassmannian field), $J_\mu(\mathbf{x}) = \bar\psi_{\mathbf{x}}\gamma_\mu\psi_{\mathbf{x}}$ and $\gamma_0 = \sigma_1$, $\gamma_1 = \sigma_2$ are Pauli matrices.

The *generating functional* of the Thirring model is the following Grassmannian integral with infrared cutoff γ^h and ultraviolet cutoff γ^N, with h, N integers and $N > 0$

$$\begin{aligned}\mathcal{W}(\phi, J) = \log \int P_{Z_N}(d\psi^{[h,N]}) \exp \Bigg\{ & - V(\psi^{[h,N]}) + \\ + \sum_\omega \int d\mathbf{x} \Big[Z_N^{(2)} J_{\mathbf{x},\omega}\psi^{[h,N]+}_{\mathbf{x},\omega}\psi^{[h,N]-}_{\mathbf{x},\omega} & + \phi^+_{\mathbf{x},\omega}\psi^{[h,N]-}_{\mathbf{x},\omega} + \psi^{[h,N]+}_{\mathbf{x},\omega}\phi^{[h,N]-}_{\mathbf{x},\omega} \Big] \Bigg\} ,\end{aligned} \tag{A1.2}$$

where $P_{Z_N}(d\psi^{[h,N]})$ is given by (1.3), with $C_{h,0}(\mathbf{k})$ replaced by $C_{h,N}(\mathbf{k}) = \sum_{j=h}^N f_j(\mathbf{k})$ and $\psi^{[h,0]\sigma}_{\mathbf{k},\omega}$ replaced by $\sqrt{Z_N}\psi^{[h,N]\sigma}_{\mathbf{k},\omega}$, and $V(\psi^{[h,N]})$ is given by

$$V(\psi^{[h,N]}) = \tilde\lambda_N \int d\mathbf{x}\, \psi^{[h,N]+}_{\mathbf{x},+}\psi^{[h,N]-}_{\mathbf{x},+}\psi^{[h,N]+}_{\mathbf{x},-}\psi^{[h,N]-}_{\mathbf{x},-} \ ; \tag{A1.3}$$

Z_N is the (bare)wave function renormalization, $Z_N^{(2)}$ is the (bare)density renormalization and $\tilde\lambda_N$ is the (bare) interaction. In order to get a nontrivial limit as $N \to \infty$, it is convenient to write $\tilde\lambda_N$ and $Z_N^{(2)}$ in terms of Z_N and two new bare constants, λ_N and c_N, in the following way:

$$\tilde\lambda_N = (Z_N)^2\lambda_N \quad , \quad Z_N^{(2)} = c_N Z_N \ . \tag{A1.4}$$

One expects that the model is well defined if λ_N and c_N converge to finite non zero limits and $Z_N \to 0$, as $N \to \infty$; moreover, in order to apply our perturbative procedure, λ_N has to be small enough, uniformly in N.

The proof of this claim is essentially a corollary of the above analysis for the infrared problem. The RG analysis in §2 can be repeated by allowing the scale index j to be positive or negative. The Ward identity (3.14) holds with a factor $Z_N/Z_N^{(2)}$ multiplying $G^{2,1}_+$ and $\Delta^{2,1}_+$, and we get the identity

$$\left| \frac{Z_h^{(2)}}{c_N Z_h} - 1 \right| \le C\bar\lambda_h^2 \ . \tag{A1.5}$$

In the same way, from the Dyson equation (identical to (1.11), with $\lambda_N Z_N/Z_N^{(2)}$ in place of λ in the r.h.s), and proceeding as in §3 and §4, we get that for any h one has $|\lambda_h - \lambda_N| \le c_3\bar\lambda^2_{h+1}$ (compare with (3.2)), so that the expansion is convergent, if λ_N is small enough. By (3.4), Z_N must be chosen so that $Z_N\gamma^{N\eta(\lambda_N)}$ is convergent for $N \to \infty$ to some constant, which can be fixed by requiring, for instance, that $Z_0 = 1$. In the same way we can fix $\lim_{N\to\infty}\lambda_N$ so that, for instance, $\lambda_0 = \lambda$, with λ small enough; of course $\lambda_N = \lambda + O(\lambda^2)$. The choice of $\lim_{N\to\infty} c_N$ is a free parameter, whose value has no special role.

Finally we shall discuss the form taken from Ward identities when the ultraviolet and the infrared cutoff are removed. The analogous of (3.20) for the model (A1.2) is

$$\begin{aligned}&\frac{Z_N}{Z_N^{(2)}}(1-\nu_+)D_+(\mathbf{p})\hat G^{4,1}_+(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) - \frac{Z_N}{Z_N^{(2)}}\nu_- D_-(\mathbf{p})\hat G^{4,1}_-(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) \\ &= \hat G^4_+(\mathbf{k}_1-\mathbf{p},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) - \hat G^4_+(\mathbf{k}_1,\mathbf{k}_2+\mathbf{p},\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) + Z_N H^{4,1}_+(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p}) \ ,\end{aligned} \tag{A1.6}$$

with $H^{4,1}_+$ given by (3.21) with the cutoff function $C^{-1}_{h,0}$ replaced by $C^{-1}_{h,N}$. The counterterms $\nu_\pm$ are found by a fixed point method as in Lemma 4.1, with the only difference that the ultraviolet scale 0 is replaced by the scale N; $\nu_\pm, \nu'_\pm$ tend to a non vanishing well defined limit as $N \to \infty, h \to -\infty$.

In the same way the analogous of (3.22) is

$$
\begin{aligned}
&\frac{Z_N}{Z_N^{(2)}}(1-\nu'_-)D_-(\mathbf{p})\hat G^{4,1}_-(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})-\frac{Z_N}{Z_N^{(2)}}\nu'_+D_+(\mathbf{p})\hat G^{4,1}_+(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})\\
&=\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3-\mathbf{p},\mathbf{k}_4-\mathbf{p})-\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)+Z_NH^{4,1}_-(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})\ ,\qquad (A1.7)
\end{aligned}
$$

It is a straightforward consequence of our analysis (in particular of the short memory property we used extensively throughout the paper) that in (3.20) and (3.22) for the model (A1.2), if the external momenta have a fixed (*i.e.* independent from h,N) value, then

$$
\lim_{\substack{N\to\infty\\ h\to-\infty}} Z_N\hat H^{4,1}_\pm(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})=0\ . \qquad (A1.8)
$$

Hence, if we sum the two Ward identities above and remember that $\nu_+=\nu'_-$ and $\nu_-=\nu'_+$, we get, in the limit $N\to\infty, h\to-\infty$

$$
\begin{aligned}
&\sum_{\omega=\pm} D_\omega(\mathbf{p})\frac{1-\nu_+-\nu'_-}{c_N}\hat G^{4,1}_\omega(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})=G^4_+(\mathbf{k}_1-\mathbf{p},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})-\\
&-\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2+\mathbf{p},\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})+\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3-\mathbf{p},\mathbf{k}_4-\mathbf{p})-\hat G^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)
\end{aligned}
\qquad (A1.9)
$$

The above Ward identity is identical to the formal one, obtained by a *total* gauge transformation, except for the factor $(1-\nu_+-\nu'_-)/c_N$ multiplying $G^{4,1}$; in other words, the formal Ward identity holds when the cutoffs are removed, up to a finite interaction-dependent renormalization of the density operator. A similar phenomenon appears also in perturbative QED [H] and is called *soft breaking* of gauge invariance. Of course it is possible to choose $Z_N^{(2)}$ so that the formal Ward identity is verified, *i.e.* we can choose $c_N=(1-\nu_+-\nu'_-)$.

On the contrary, the Ward identities (3.20) and (3.22), obtained by a *chiral* gauge transformation, do not tend in the limit to the formal Ward identities (obtained by (3.20) and (3.22) putting $\nu_\pm=\nu'_\pm=0$); beside the renormalization of the density operator, an extra factor appears in the identity, namely $G^{4,1}_-$ in (3.20) or $G^{4,1}_+$ in (3.22). This phenomenon is called *chiral anomaly*, see [Z], and is is present also in perturbative QED.

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