NON-UNIVERSAL BEHAVIOUR OF SCALING PROPERTIES FOR GENERALIZED SEMISTANDARD AND STANDARD MAPS

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Abstract. We consider two-dimensional maps generalizing the semistandard map by allowing more general analytic nonlinear terms having only Fourier components f_ν with positive label ν, and study the solutions corresponding to homotopically nontrivial invariant curves with complex rotation number. Then we show that, if the perturbation parameter is suitably rescaled, when the rotation number tends to a rational value non-tangentially to the real axis, the limit of the conjugating function is a well defined analytic function. The rescaling depends not only on the limit value of the rotation number, but also on the map, and it is obtainable by the solution of a Diophantine problem: so no universality property is exhibited. We show also that the rescaling can be different from that of the corresponding generalized standard maps, *i.e.* of the maps having also the Fourier components $f_{-\nu} = f_\nu$. The results allow us to give quantitative bounds, from above and from below, on the radius of convergence of the limit function for generalized standard maps in the case of nonlinear terms which are trigonometric polynomials, solving a problem left open in a previous work of ours.

1. Introduction

In this paper we consider the same problem as in [3] for the generalized semistandard maps (*i.e.* maps generalizing the semistandard map in the sense specified below). The aim is twofold: (1) to show explicitly that generalized standard maps and generalized semistandard maps behave in a different way, so that the fact that the radii of convergence for the Lindstedt series of the standard map and the semistandard map share the same dependence on the rotation number (as proved in [2]) is likely to be considered as an accident; (2) to use the results to prove that a bound that in [3] was proved to hold from below holds also from above.

In particular this will imply a precise scaling law for the infimum of the radii of convergence of the Lindstedt series, where the infimum is taken over all analytic functions with some prefixed suitable norm; see the theorem in sect. 2 for a precise formulation of the results.

So we shall be able to prove a result left as an open problem in [3] (see Remark 3 and sect. 7 in [3]), and provide a quantitative bound from above on the infimum of the radii of convergence of the limit functions of the rescaled conjugating function.

We refer to [1] and [3] for an introduction to the topics discussed in the present paper: in general the full knowledge of [3] is considered as necessary in order to follow the present letter.

2. The main results

We consider maps of the kind

$$T_{\varepsilon,f}: \begin{cases} x' = x + y + \varepsilon f(x), \\ y' = y + \varepsilon f(x), \end{cases} \tag{2.1}$$

where $f(x)$ is a 2π-periodic function of x, analytic in a strip $\mathcal{S} = \{|\operatorname{Im} x| < \xi\}$ of width 2ξ around the real x axis, of the form

$$f(x) = \sum_{\nu \geq 1} f_\nu \, e^{i\nu x}. \tag{2.2}$$

The Fourier coefficients f_ν are assumed to be real. By the analyticity assumptions they decay exponentially:

$$\forall \xi' < \xi \; \exists C_1 : \quad |f_\nu| < C_1 e^{-\xi'|\nu|}. \tag{2.3}$$

We call $\mathcal{F}$ the set of functions of such a form; of course if $f_\nu = 0$ for all $\nu \geq 2$, (2.1) reduces to the semistandard map.

If we denote by $\mathcal{B}_\rho$ the Banach space $\mathcal{B}_\rho = \{g \in \mathcal{F} | \, \|g\|_\rho < \infty\}$, where

$$\|g\|_\rho = \sum_{\nu \in \mathbb{Z}} |g_\nu| \, e^{\rho|\nu|}, \tag{2.4}$$

then $f \in \mathcal{B}_{\xi'} \; \forall \xi' < \xi$. Denote by $\mathcal{S}_1$ the set

$$\mathcal{S}_1 = \{f \in \mathcal{B}_{\xi'} \; \forall \xi' < \xi | \|f\|_0 = 1\}. \tag{2.5}$$

The variable y can be easily eliminated from (2.1) to obtain the second order recurrence

$$x_{n+1} - 2x_n + x_{n-1} = \varepsilon f(x_n). \tag{2.6}$$

On the homotopically non-trivial invariant curves the dynamics is conjugate to rotations by $2\pi\omega$. So we can write

$$x = \alpha + u_0(\alpha, \varepsilon, \omega), \tag{2.7}$$

such that in the coordinate α the dynamics is a rotation by $2\pi\omega$:

$$\alpha' = \alpha + 2\pi\omega. \tag{2.8}$$

We use a subscript 0 for u in order to distinguish between it and the corresponding function for generalized standard maps, [3]: at the end we shall be interested in comparing the two cases, so that it is better since now to use different notations.

It is easy to see that the *conjugating function* u_0 satisfies

$$D_\omega u_0(\alpha, \varepsilon, \omega) = \varepsilon f(\alpha + u_0(\alpha, \varepsilon, \omega)), \tag{2.9}$$

where the operator D_ω, acting on 2π-periodic functions of α, is given by

$$D_\omega \phi(\alpha) = \phi(\alpha + 2\pi\omega) - 2\phi(\alpha) + \phi(\alpha - 2\pi\omega). \tag{2.10}$$

For the semistandard map the parameter ε in (2.1) is not necessary (and in fact it is usually replaced by i). If one considers the conjugating function as a function of $z = e^{i\alpha}$, *i.e.* $u_0(\alpha, i, \omega) \equiv \Phi(z, \omega)$, then $\Phi(z, \omega)$ is expanded in powers of z, and the analyticity domain in z is studied: this leads to a condition $|z| < K$, for some constant K, which is equivalent to require that the modulus of the imaginary part of α can not be too large. Instead, we can fix α to be real and impose such a condition on a auxiliary parameter ε (which plays the rôle of $e^{i\,\mathrm{Im}\,\alpha}$). We prefer to adopt this second convention also in the case of the semistandard map in order to make more transparent the connection with the standard map.

The *Lindstedt series* for the problem is the formal expansion of u_0 as a Taylor series in ε and a Fourier series in α:

$$u_0(\alpha, \varepsilon, \omega) = \sum_{k \ge 1} \varepsilon^k u_0^{(k)}(\alpha, \omega) = \sum_{k \ge 1} \varepsilon^k \sum_{\nu \ge k} e^{i\nu\alpha}\, u_{0,\nu}^{(k)}(\omega), \tag{2.11}$$

where we used that the Fourier label ν has to be positive and not smaller than the Taylor label k (by construction). In the following we shall write $u_{0,\nu}^{(k)}(\omega) = u_{0,\nu}^{(k)}$.

The *radius of convergence* of the Lindstedt series for generalized semistandard maps (2.1) is defined as

$$\bar{\bar{\rho}}_{0,f}(\omega) \equiv \rho_0(\omega) = \inf_{\alpha \in \mathbb{T}} \left(\limsup_{k \to \infty} \left| u_0^{(k)}(\alpha, \omega) \right|^{1/k} \right)^{-1}. \tag{2.12}$$

We are interested in the behaviour of the radius of convergence $\rho_0(\omega)$ of the Lindstedt series (2.11) as the complexified rotation number ω tends to a *resonant* value p/q, with $\gcd(p,q) = 1$. We then consider the q sequences $\tilde{I}_c(f) = \{f_{lq+c}\}_{l\in\mathbb{Z}_+}$, $c = 1,\dots,q$ (recall that f is assumed to have not the Fourier component with $\nu = 0$; see (2.1)). For each sequence $\tilde{I}_c(f)$, $c = 1,\dots,q$, let

$$I_c(f) = \{f_\nu \in \tilde{I}_c(f) | f_\nu \neq 0\}$$

be the set of nonzero values of the sequence $\tilde{I}_c(f)$.

We define the following set of integers:

$$A_{p/q}(f) = \{c \in \{1,\dots,q\} | I_c(f) \neq \emptyset\}. \tag{2.13}$$

Of course $A_{p/q}(f) = \{c_1,\dots,c_M\}$, where $1 \le c_1 < \cdots < c_M \le q$, and $M \le q$.

We define M integers $r_1,\dots,r_M$, with $r_i \ge 0$, and an integer $R > 0$ as those integers which satisfy the following conditions:

$$r_1 c_1 + \cdots + r_M c_M = Rq, \tag{2.14a}$$

$$r_1 + \cdots + r_M = r_0 \ge 1, \tag{2.14b}$$

$$r_0 \text{ is minimal}, \tag{2.14c}$$

where $\{c_1,\dots,c_M\} = A_{p/q}(f)$; note that, when considering generalized semi-standard maps, it is not necessary to introduce the set $\mathcal{A}(f)$ (see the lemma 1 in [3]), as it is straightforward to convince himself by using that all Fourier labels are positive.

The Diophantine problem (2.14) has always a finite, nonzero number of solutions and $r_0 \le q$, as it can be proved by reasoning analogous to the corresponding Lemma 2 in [3]. In particular if $q \in A_{p/q}(f)$ then (2.14) admits the solution $r_M = 1$, $r_i = 0\ \forall i < M$, $c_M = 1$, $R = 1$ and $r_0 = 1$.

Define $r^*(f) \equiv r^*$ as $r^* = r_0$. We introduce the notation r^* simply to make easier the comparison with [3] (where distinction had to be made between r_0 and r^*; see (1.18) in [3]).

We prove the following result.

Theorem. *Let f be any function in $\mathcal{B}_\rho$, with $\rho > 0$. Consider the cone $\mathcal{C}_{p/q,\beta} = \{z \in \mathbb{C} :\ |\operatorname{Im} z| > 0,\ |\operatorname{Re} z - p/q| \le \beta |\operatorname{Im} z|,\ \beta \ge 0\}$; let $\omega \in \mathcal{C}_{p/q,\beta}$. Then the* rescaled conjugating function

$$\bar{u}_{0,p/q}(\alpha,\varepsilon,\omega) = u\left(\alpha, \varepsilon\left(\omega - \frac{p}{q}\right)^{2/r^*(f)}, \omega\right) \tag{2.15}$$

extends to a function continuous in ω in the closure of the cone $\mathcal{C}_{p/q,\beta}$ and analytic in ω in the interior of $\mathcal{C}_{p/q,\beta}$, for any $\beta \ge 0$, analytic in ε for $|\varepsilon| < a$

and analytic in α for $|\operatorname{Im}\alpha| < b$, with a, b two positive constants. In particular, the following limit exists:

$$\bar{\bar{u}}_{0,p/q}(\alpha,\varepsilon) = \lim_{\omega\to p/q} \bar{u}_{0,p/q}(\alpha,\varepsilon,\omega), \tag{2.16}$$

it is independent from the non-tangential path chosen either in the complex upper half plane $\operatorname{Im}\omega > 0$ or in the complex lower half plane $\operatorname{Im}\omega < 0$, and it is analytic for $|\varepsilon| < \bar{\bar{\rho}}_{0,f}(p/q)$, for some $\bar{\bar{\rho}}_{0,f}(p/q) > 0$. Defining

$$\bar{\bar{R}}_0(p/q) = \inf_{f\in\mathcal{S}_1} \bar{\bar{\rho}}_{0,f}(p/q), \qquad \bar{\bar{R}}_{0,r}(p/q) = \inf_{\substack{f\in\mathcal{S}_1,\\ r^*(f)=r}} \bar{\bar{\rho}}_{0,f}(p/q), \tag{2.17}$$

there exist finite strictly positive constants $\bar{\bar{C}}_r, \bar{\bar{C}}_0, \bar{\bar{D}}_r, \bar{\bar{D}}_0$, depending only on p/q, such that one has

$$\bar{\bar{C}}_0 \le \bar{\bar{R}}_0(p/q) \le \bar{\bar{D}}_0, \tag{2.18a}$$

$$\bar{\bar{C}}_r \le \bar{\bar{R}}_{0,r}(p/q) \le \bar{\bar{D}}_r, \tag{2.18b}$$

for all integer $r \le q$.

The proof of the Theorem above essentially follows from the analysis in [3], up to the last assertion on the finiteness of the quantities $\bar{\bar{R}}_0(p/q)$ and $\bar{\bar{R}}_{0,r}(p/q)$: in fact a constructive bound from above requires some extra work, which can be borrowed from the analysis permormed in [4]. Furthermore, as in [4], the result which can be proved for generalized semistandard maps gives information also on the radius of convergence of the limit functions of the corresponding generalized standard maps; see sect. 5.

In the following we shall consider explicitly the case $\omega - p/q = i\eta$, with $\eta \in \mathbb{R}$: the analysis can be easily extended to the case $\zeta \in \mathcal{C}_{p/q,\beta}$ as in [1] and [3].

3. The lower bound

We can consider a tree expansion for the conjugating function as in [3], to which we refer (see also [1]): the only difference is that now $\nu_u \ge 1$ for all $u \in \vartheta$ and (as a consequence) $\nu \ge k$. So we have

$$u^{(k)}_{0,\nu} = \sum_{\vartheta\in\mathcal{T}_{k,\nu}} \operatorname{Val}(\vartheta), \tag{3.1}$$

where

$$\operatorname{Val}(\vartheta) = i^{k-1}\Big[\prod_{u\in\vartheta} \frac{f_{\nu_u}\nu_u^{m_u}}{m_u!}\Big]\Big[\prod_{\ell\in\vartheta} g(\nu_\ell)\Big] \tag{3.2}$$

where the same notations as in [3] are used. In particular

$$g(\nu_\ell) = \frac{1}{\gamma(\nu_\ell)} = \frac{1}{2(\cos 2\pi\omega\nu_\ell - 1)} \tag{3.3}$$

is the *propagator* of the line ℓ. The line ℓ is said to be on scale $n = 0$ if $\nu_\ell \in q\mathbb{N}$ and on scale $n = 1$ otherwise.

The main difference (and simplification) with respect to [3] is that now no resonances are possible as $\nu_u \geq 1$ for all $u \in \vartheta$. So the analogue of Proposition 1 of [3] holds, and the proof is the same (we refer to [3] for the definition of clusters).

Proposition 1. *A cluster has at least r^* nodes.*

Then a result analogous to Lemma 4 in [3] is easily obtained (of course for the number of all lines on scale $n = 0$ as no resonant line is possible), that is, by denoting with $N_0(\vartheta)$ the number of lines in ϑ on scale $n = 0$, one has

$$N_0(\vartheta) \leq \left\lfloor \frac{k}{r^*} \right\rfloor . \tag{3.4}$$

if k is the number of nodes of ϑ.

By defining

$$R_0(\omega) = \inf_{f \in \mathcal{S}_1} \rho_{0,f}(\omega), \qquad R_{0,r}(\omega) = \inf_{\substack{f \in \mathcal{S}_1, \\ r^*(f) = r}} \rho_{0,f}(\omega). \tag{3.5}$$

then one immediately obtains that there exist constants C_0, C_r such that, for η small,

$$R_{0,r}(\omega) \geq C_r \eta^{2/r}, \qquad R_0(\omega) \geq C_0 \eta^2. \tag{3.6}$$

With respect to [3] there is an oversimplification due to the absence of resonances: so no cancellation mechanism has to be discussed in order to obtain the right bound from below of the radius of convergence. Note also that the formal solubility requires no discussion as $\nu \geq k$ yields $\nu_\ell \neq 0$ for any line ℓ in any tree ϑ.

If $\bar{\bar{\rho}}_{0,f}(p/q)$ denotes the radius of convergence for the rescaled function $\bar{\bar{u}}_0$, *i.e.*

$$\bar{\bar{\rho}}_{0,f}(p/q) = \inf_{\alpha \in \mathbb{T}} \left(\limsup_{k \to \infty} \left| \bar{\bar{u}}_0^{(k)}(\alpha) \right|^{1/k} \right)^{-1}, \tag{3.7}$$

we can define, analogously to (3.5),

$$\bar{\bar{R}}_0(p/q) = \inf_{f \in \mathcal{S}_1} \bar{\bar{\rho}}_{0,f}(p/q), \qquad \bar{\bar{R}}_{0,r}(p/q) = \inf_{\substack{f \in \mathcal{S}_1, \\ r^*(f) = r}} \bar{\bar{\rho}}_{0,f}(p/q). \tag{3.8}$$

so that the bounds (3.6) imply, for suitable constants $\bar{\bar{C}}_0, \bar{\bar{C}}_r$ (depending on p/q),

$$\bar{\bar{R}}_{0,r}(p/q) \geq \bar{\bar{C}}_r, \qquad \bar{\bar{R}}_0(p/q) \geq \bar{\bar{C}}_0, \tag{3.9}$$

which give a bound from below on the quantities $\bar{\bar{R}}_{0,r}(p/q)$ and $\bar{\bar{R}}_0(p/q)$.

4. The upper bound

For the rescaled function $\bar{\bar{u}}_0 = \bar{\bar{u}}_{0,p/q}$, one has

$$\bar{\bar{u}}_{0,\nu}^{(k)} = \begin{cases} \sum'_{\vartheta \in \mathcal{T}_{k,\nu}} \mathrm{Val}'(\vartheta), & \text{if } k \in r^* \mathbb{N}, \quad \nu \in q\mathbb{N}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.1}$$

where $\sum'$ means that only trees ϑ with $N_0(\vartheta) = k/r^*$ have to be summed over and $\mathrm{Val}'(\vartheta)$ differs from $\mathrm{Val}(\vartheta)$ as, for ν multiple of q, $\gamma(\nu)$ has to be replaced with $(2\pi i \nu)^2$ in the definition of $g(\nu)$. The proof of (4.1) can be carried out as in [3]. In particular the analysis of [3], sect. 6, shows that $k = nr^*$, with $n \in \mathbb{N}$, implies

$$\nu = nRq + \sum_{u \in \vartheta} l_u q, \tag{4.2}$$

where R is given by (2.14a) and the integers l_u are such that one has $f_{l_u q + c_u} \in I_{c_u}(f)$ (we use once more the notations of [3]).

Of course we can bound from above the quantity $\bar{\bar{R}}_{0,r}(\omega)$ restricting our analysis to the functions (2.2) such that $\nu \leq q$. So in the following we shall consider only such functions. Note that in such a case in (4.2) one has $l_u = 0$ for all $u \in \vartheta$, hence $\nu = nRq$.

By reasoning as in the second part of sect. 2 in [4], we shall prove that there exists a class $\mathcal{S}_1^{(N)}$ of functions $f \in \mathcal{S}_1$ with $\nu \leq N$ (for some integer N to be specified below) such that the following bound holds:

$$\bar{\bar{\rho}}_{0,f}(\omega) \leq C(f), \tag{4.3}$$

for some constant $C(f)$ (depending on f of course). We shall prove also that there exists a constant $\bar{\bar{D}}_r$ such that $C(f) \leq \bar{\bar{D}}_{r(f)}$ uniformly in f in the considered class of functions $\mathcal{S}_1^{(N)}$. So the bounds from above

$$\bar{\bar{R}}_{0,r}(\omega) \leq \bar{\bar{D}}_r, \qquad \bar{\bar{R}}_0(\omega) \leq \bar{\bar{D}}_0, \tag{4.4}$$

immediately follow, for suitable constants $\bar{\bar{D}}_r$ and $\bar{\bar{D}}_0$, and the proof of the theorem is complete.

Now we prove (4.3). Take any function in $\mathcal{S}_1$ of the form

$$f(x) = \sum_{\nu=1}^{N} f_\nu \, e^{i\nu x}, \qquad f_\nu > F > 0, \tag{4.5}$$

for some fixed constant F: this defines the class $\mathcal{S}_1^{(N)}$.

The propagators $g(\nu_\ell)$ in (3.2) are all negative, so that the choice (4.5) and the definition of $\mathrm{Val}'(\vartheta)$ yield that, for any $\vartheta \in \mathcal{T}_{k,\nu}$, $\mathrm{Val}'(\vartheta)$ is given by $(-1)^k$ times i^{k-1} times a positive quantity $P(\vartheta)$. Therefore all trees in $\mathcal{T}_{k,\nu}$ have, apart from a commun factor $[-(-i)^{k-1}]$, a value $P(\vartheta)$ strictly positive: such a property can be fruitfully used in order to prove the following result.

Lemma. *Let $\bar{\bar{u}}_0$ be the limit function* (2.16) *for $f \in \mathcal{S}_1^{(N)}$, i.e. of the form* (4.5)*, and let $r^* = r_0$ and R be the integers appearing in the solution of the Diophantine problem* (2.14)*. Then there exists a universal positive constant D such that*

$$\left| \bar{\bar{u}}_{0,\nu}^{(k)} \right| \geq D^k F^k, \tag{4.6}$$

for any $k = nr^*$, with $n \in \mathbb{N}$, and for any $\nu = nRq$, while $\bar{\bar{u}}_{0,\nu}^{(k)} = 0$ otherwise.

Proof. Set $r^* \equiv r$ and $Rq \equiv Q$. The modulus of each propagator with $\nu_\ell \notin q\mathbb{N}$ can be bounded from above with 4, while the propagators with $\nu_\ell = n_\ell q$, $n_\ell \in \mathbb{N}$, are given by $g(\nu_\ell) = (2\pi i q)^{-2} n_\ell^{-2}$.

Note that any ϑ contributing to $\bar{\bar{u}}_{0,\nu}^{(nr)}$ must have nr nodes, n lines on scale $n = 0$, hence n clusters with r nodes: each cluster contributes Q to ν, so that one must have for the root line a momentum $\nu = rQ$ in order that $\bar{\bar{u}}_{0,\nu}^{(nr)} \neq 0$. This suggests us to write

$$\bar{\bar{u}}_{0,\nu}^{(k)} = |g(\nu)| \left[(-1)^k i^{k-1} \right] \psi_\nu^{(k)}, \tag{4.7}$$

so defining the quantity $\psi_\nu^{(k)} > 0$. We can prove that $\psi_\nu^{(k)}$ satisfies the recursive bound

$$\psi_{nQ}^{(nr)} \geq n^2 8^{1-n} \, |g(Q)|^{n-1} \, \psi_Q^{(r)n}, \tag{4.8}$$

for any $n \in \mathbb{N}$. The proof of such an assertion is by induction on n. The case $n = 1$ is trivial. So assume that (4.8) holds for $n' < n$: we then prove that it holds also for n.

Consider all trees in $\mathcal{T}_{nr,nQ}$ formed in the following way. Consider a tree $\vartheta_1 \in \mathcal{T}_{tr,tQ}$ and attach to its last node u_0 a tree $\vartheta_2 \in \mathcal{T}_{sr,sQ}$. Here $s = \lfloor n/2 \rfloor$, $t = n - s$ (see fig. 1).

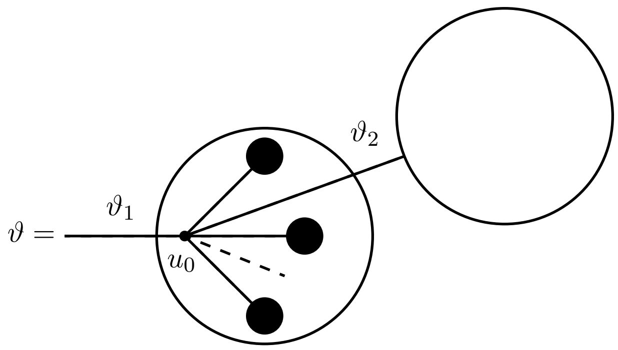

FIGURE 1. A tree $\vartheta \in \mathcal{T}_{nr,nQ}$ obtained by attaching the tree $\vartheta_2 \in \mathcal{T}_{sr,sQ}$ to the last node of a tree $\vartheta_1 \in \mathcal{T}_{tr,tQ}$

The combinatorial factor associated to the node u_0 is $1/(m_{u_0}+1)!$, if m_{u_0} is the number of lines of ϑ_1 entering u_0 (the unit takes into account the subtree ϑ_2 itself). If n is even then $s = t$ and the subtree ϑ_2 has a different number of nodes with respect to any of the m_{u_0} subtrees of ϑ_1 entering u_0, then it is different from all of them: we can insert ϑ_2 in $m_{u_0}+1$ different ways amomg the m_{u_0} subtrees of ϑ_1 entering u_0, so obtaining $m_{u_0}+1$ different trees. If n is odd the same happens except possibly for $m_{u_0} = 1$: in such a case the subtree of ϑ_1 entering u_0 and ϑ_2 can be equal to each other. In conclusion, by counting all the possible trees obtained by changing the order of the root line of ϑ_2 with respect to the lines of ϑ_1 entering u_0, in both cases (n even and n odd) we can bound the combinatorial factor times the number of choices with $1/2$.

By summing over all possible trees ϑ_1 we get a quantity which can be bounded with $\psi^{(tr)}_{tQ}$, while by summing over all the possible trees ϑ_2 which can be attached to the last node u_0 of ϑ_1 we get a quantity which can be bounded with $|g(sQ)|\,\psi^{(sr)}_{sQ}$. Therefore

$$
\begin{aligned}
\psi^{(nr)}_{nQ} &\geq \frac{1}{2}\psi^{(tr)}_{tQ}\,|g(sQ)|\;\psi^{(sr)}_{sQ} \\
&\geq \frac{1}{2s^2}\,|g(Q)|\left(s^2 8^{1-s}\,|g(Q)|^{s-1}\,\psi^{(r)s}_{Q}\right)\left(t^2 8^{1-t}\,|g(Q)|^{t-1}\,\psi^{(r)t}_{Q}\right) \qquad (4.9)\\
&\geq \frac{8t^2}{2}\left(8^{1-n}\,|g(Q)|^{n-1}\,\psi^{(r)n}_{Q}\right),
\end{aligned}
$$

so that (4.8) is proven. Furthermore one immediately checks that there is a constant D_1 such that

$$
\psi^{(r)}_{Q} \geq D_1^r F^r, \qquad (4.10)
$$

so that, by using together the two bounds (4.8) and (4.10), we obtain

$$\left|\bar{\bar{u}}_{0,nQ}^{(nr)}\right| \ge \left(\frac{1}{2\pi qR}\right)^{2n} 8^{-n} D_1^{nr} F^{nr}, \tag{4.11}$$

hence the lemma is proved (and an explicit bound on D is given). □

So we can conclude that there exists a constant $B(f)$ such that

$$\limsup_{k\to\infty}\left|\bar{\bar{u}}_0^{(k)}(\alpha)\right|^{1/k} \ge \limsup_{n\to\infty}\left|\bar{\bar{u}}_{0,nQ}^{(nr)}\right|^{1/nr} \ge B(f), \tag{4.12}$$

concluding the proof of (4.3), with $C(f) = 1/B(f)$.

5. Comparison with the generalized standard maps

Consider functions of the form (4.5). We can also consider a generalized standard map which is of the form (2.1) with

$$f(x) = \sum_{\nu=-N}^{N} f_\nu\, e^{i\nu x}, \tag{5.1}$$

with the same Fourier components f_ν as in (4.5) for $\nu > 0$, and with $f_{-\nu} = f_\nu$ and $f_0 = 0$.

Proposition 2. *Fixed $q, r \in \mathbb{N}$, with $r \le q$, choose N as a solution of*

$$\begin{cases} rN = q, & \text{if } q/r \in \mathbb{N}, \\ q+1 \le rN \le q-1+N, & \text{otherwise.} \end{cases} \tag{5.2}$$

Then the Diophantine problem (2.14) *for the generalized semistandard map with $f(x)$ given by* (4.5) *and the corresponding one (see* (1.17) *in* [3]*) for the generalized standard map with $f(x)$ given by* (5.1) *admit the same solution $(r^*, R, \{r_i\})$, with $r^* = r$.*

Proof. Note first that one has $r^* = 1$ if $r = 1$ and $r^* = r_0$ if $r > 1$ also for the generalized standard maps in the case we are considering: recall (1.18) in [3] and, noting that N given by the second line of (5.2) satisfies $N < q$, use the second assertion in Proposition 1 of [3]. In the latter case $N < q$ implies also that one has $\mathcal{A}(f) = A_{p/q}(f)$ for the generalized standard maps.

In order to have the same solution for both the Diophantine problems, it is enough to show that there exists a solution of (1.17) in [3] such that $r_i' = 0$ $\forall i = 1, \dots, M$. This is easily obtained for functions of the form (5.1), simply showing that, in such a case, a solution (with $R = 1$) of the Diophantine problem (2.14) can be explicitly constructed by the procedure described below, hence realizing that it solves also the Diophantine problem (1.17) in [3].

For fixed q and r, let N be such that (5.2) is satisfied. Note that then both for the function (4.5) in the case of generalized semistandard maps and for the function (5.1) in the case of generalized standard maps, one has $A_{p/q}(f) = \{1,2,3,\dots,N\}$ and $M = N$. Set

$$r_M \equiv r_N = \lfloor q/N \rfloor. \tag{5.3}$$

If $q/r \in \mathbb{N}$ then for $N = q/r$ one has $q/N = r$, so that in (5.3) one has $r_N = r$. Then if one chooses $r_N = r$ and $r_i = 0$ $\forall i < N$, one obtains a solution of the Diophantine problem (2.14) with $r_0 = r_N = r$, so that $r^* = r$.

If $q/r \notin \mathbb{N}$, then N is chosen as in the second line of (5.2). Note that in such a case $r_N < q/N$ and there exists $c \in A_{p/q}(f) \setminus \{N\}$ such that $c = q - r_N N$. Hence $r_N N = q - c$ satisfies $q - N + 1 \le r_N N \le q - 1$, which can be written as $q + 1 \le (r_N + 1)\, N \le q - 1 + N$, implying $r_N + 1 = r$. Then, by choosing r_N as given by (5.3), $r_c = 1$ and $r_i = 0$ for all other i, one obtains a solution of the Diophantine problem (2.14) with $r_0 = r_N + 1 = r$, so that $r^* = r$.

So we are left with checking that any solution of the form we have found in the case of the generalized semistandard map is also a solution of the corresponding Diophantine problem (1.17) of [3] for the generalized standard map. This is quite obvious in the case $q/r \in \mathbb{N}$.

In the other cases one can reason as follows. Suppose that a solution of (1.17) of [3] contains a negative harmonic, say c_i: then $Rq = r_1 + 2r_2 + \ldots + N r_N - c_i$ and $r = r_1 + r_2 + \ldots + r_N + r_i'$. As (1.17c) implies $r_N \ge 1$, we can write Rq as $Rq = r_1 + 2r_2 + \ldots + N(r_N - 1) + (N - c_i)$, with $1 \le N - c_i \equiv c_j \le N - 1$, for some j, so that we can lower the value of r (without changing q), by choosing r_N diminished by one unit and $r_i = 0$, provided that r_j is increased by one unit. One can exclude *a fortiori* the presence of more than one negative harmonic in the solution of the Diophantine problem (1.17) of [3] (for the class of functions we are considering). The condition that r_0 to be minimal requires $R = 1$ as in the case of the generalized semistandard map, so that the proof is complete. □

By denoting with $\bar{\bar{u}}$ the quantity analogous to $\bar{\bar{u}}_0$ for the generalized standard map (see [3]), we have

$$\bar{\bar{u}}_0(\alpha,\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^{nr^*} e^{inRq\alpha} \bar{\bar{u}}_{0,nRq}^{(nr^*)}, \tag{5.4a}$$

$$\bar{\bar{u}}(\alpha,\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^{nr^*} \sum_{\substack{\nu\in\mathbb{Z},\\ |\nu|\le nRq}} e^{i\nu\alpha} \bar{\bar{u}}_{\nu}^{(nr^*)}. \tag{5.4b}$$

Note that $\bar{\bar{u}}^{(nr^*)}_{0,nRq} = \bar{\bar{u}}^{(nr^*)}_{nRq}$, by construction.

It follows from Lemma 3.1, Proposition 3.3 (and the remark following it) in [4] that the radius of convergence $\bar{\bar{\rho}}_f(p/q)$ of the series (5.4b) can be defined as

$$\bar{\bar{\rho}}_f(p/q) = \left(\limsup_{n\to\infty} \max_{|\nu|\le nRq} \left| \bar{\bar{u}}^{(nr)}_{\nu} \right|^{1/nr} \right)^{-1}. \tag{5.5}$$

so that the radius of convergence of the series (5.4b) cannot be larger than the radius of convergence of the series (5.4a). On the other hand the latter is bounded from above as in (4.3): in this way a complete bound, from above and from below, is obtained for the radius of convergence of the rescaled linearizations for generalized standard maps with $f(x)$ given by (5.1), therefore completing the work of [3] to the same level as in [1] for the standard map.

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