Quantum Mechanics of Damped Systems II. Damping and Parabolic Potential Barrier.

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Abstract

We investigate the resonant states for the parabolic potential barrier known also as inverted or reversed oscillator. They correspond to the poles of meromorphic continuation of the resolvent operator to the complex energy plane. As a byproduct we establish an interesting relation between parabolic cylinder functions (representing energy eigenfunctions of our system) and a class of Gel'fand distributions used in our recent paper.

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1 Introduction

In a recent paper [1] we have investigated a quantization of the simple damped system[1]

$$\dot{u} = -\gamma u \ . \tag{1.1}$$

To quantize this system we double the number of degrees of freedom, i.e. together with (1.1) we consider $\dot{v} = +\gamma v$. The enlarged system is a Hamiltonian one and its quantization leads to the following quantum Hamiltonian:

$$\widehat{H} = -\frac{\gamma}{2}(\widehat{u}\widehat{v} + \widehat{v}\widehat{u}) \ . \tag{1.2}$$

We showed that the above system displays two families of generalized eigenvectors $f_n^\pm$ corresponding to purely imaginary eigenvalues $\widehat{H} f_n^\pm = \pm E_n f_n^\pm$. These eigenvectors are interpreted as resonant states — they correspond to the poles of energy eigenfunctions when continued to the complex energy plane. It turns out that resonant states are responsible for the irreversible behavior. We showed that there are two dense subspaces $\Phi_\pm \in L^2(\mathbb{R})$ such that restriction of the unitary group $U(t) = e^{-i\widehat{H}t}$ to $\Phi_\pm$ does no longer define a group but gives rise to two semigroups: $U_-(t) = U(t)|_{\Phi_-}$ defined for $t \geq 0$ and $U_+(t) = U(t)|_{\Phi_+}$ defined for $t \leq 0$. In

[1] We slightly change the notation: the coordinates (x, p) used in [1] are replaced by (u, v) in the present paper.

the framework of Gel'fand triplets (see e.g. [2]) it means that the quantum version of the damped system (1.1) corresponds to the Gel'fand triplet:

$$\Phi_- \subset L^2(\mathbb{R}) \subset \Phi'_- \ , \tag{1.3}$$

together with the Hamiltonian $\widehat{H}|_{\Phi_-}$. This system serves as a simple example of Arno Bohm theory of resonances [3] (see also [4, 5]) and illustrates mathematical results of [6].

In the present paper we continue to study this system but in a different representation. Let us observe that performing the linear canonical transformation $(u,v) \longrightarrow (x,p)$:

$$u = \frac{\gamma x - p}{\sqrt{2\gamma}} \ , \qquad v = \frac{\gamma x + p}{\sqrt{2\gamma}} \ , \tag{1.4}$$

one obtains for the Hamiltonian

$$\widehat{H} = \frac{1}{2}(\widehat{p}^2 - \gamma^2 \widehat{x}^2) \ . \tag{1.5}$$

It represents the parabolic potential barrier $V(x) = -\gamma^2 x^2/2$ and it was studied by several authors in various contexts [7, 8, 9, 10, 11, 12, 13]. It is well known that this system gives rise to the generalized complex eigenvalues — the physical reason for that is the potential unbounded from below. We find the corresponding energy eigenstates for (1.5). They are given in terms of parabolic cylinder functions $D_\nu(x)$. Using the Gel'fand-Maurin spectral decomposition we find the resolvent operator $R(z,\widehat{H}) = (\widehat{H} - z)^{-1}$ and relate its poles to the resonant states. As a byproduct we established a deep relation between the Gel'fand distributions $u_\pm^\lambda$ [14, 15] (used in [1]) and parabolic cylinder functions $D_\nu(x)$. The details are included in the Appendix.

2 Inverted oscillator and complex eigenvalues

Let us note that $\widehat{H}$ defined in (1.5) corresponds to the Hamiltonian of the harmonic oscillator with purely imaginary frequency $\omega = \pm i\gamma$ (in the literature it is also called an inverted or reversed oscillator). The connection with a harmonic oscillator may be established by the following scaling operator [16]:

$$\widehat{V}_\lambda := \exp\left(\frac{\lambda}{2}(\widehat{x}\widehat{p} + \widehat{p}\widehat{x})\right) \ , \tag{2.1}$$

with $\lambda \in \mathbb{R}$. Using commutation relation $[\widehat{x},\widehat{p}] = i$, this operator may be rewritten as follows

$$\widehat{V}_\lambda = e^{-i\frac{\lambda}{2}}\, e^{\lambda \widehat{x}\widehat{p}} = e^{-i\frac{\lambda}{2}}\, e^{-i\lambda x \partial_x} \ , \tag{2.2}$$

and therefore it defines a complex dilation, i.e. the action of $\widehat{V}_\lambda$ on a function $\varphi = \varphi(x)$ is given by

$$\widehat{V}_\lambda\, \varphi(x) = e^{-i\frac{\lambda}{2}}\, \varphi(e^{-i\lambda}\, x) \ . \tag{2.3}$$

In particular one easily finds:

$$\widehat{V}_\lambda\, \widehat{x}\, \widehat{V}_\lambda^{-1} = e^{-i\lambda}\widehat{x} \ , \qquad \widehat{V}_\lambda\, \widehat{p}\, \widehat{V}_\lambda^{-1} = e^{i\lambda}\widehat{p} \ , \tag{2.4}$$

and hence

$$\widehat{V}_\lambda \,(\widehat{p}^2 - \gamma^2 \widehat{x}^2)\, \widehat{V}_\lambda^{-1} = e^{2i\lambda}\,(\widehat{p}^2 - e^{-4i\lambda}\gamma^2\widehat{x}^2) \ . \tag{2.5}$$

Therefore, for $e^{4i\lambda} = -1$, i.e. $\lambda = \pm\pi/4$, one has

$$\widehat{V}_{\pm\pi/4}\, \widehat{H}\, \widehat{V}_{\pm\pi/4}^{-1} = \pm i \widehat{H}_{\rm ho} \ , \tag{2.6}$$

where

$$\widehat{H}_{\rm ho} = \frac{1}{2}(\widehat{p}^2 + \gamma^2\widehat{x}^2) \ , \tag{2.7}$$

stands for the oscillator Hamiltonian. In particular if $E_n^{\rm ho} = \gamma(n+\frac{1}{2})$ is an oscillator spectrum

$$\widehat{H}_{\rm ho}\psi_n^{\rm ho} = E_n^{\rm ho}\psi_n^{\rm ho} \ , \tag{2.8}$$

then

$$\widehat{H}\mathfrak{f}_n^{\pm} = \pm E_n \mathfrak{f}_n^{\pm} \ , \tag{2.9}$$

with

$$E_n = iE_n^{\rm ho} = i\gamma\left(n + \frac{1}{2}\right) \ , \tag{2.10}$$

and

$$\mathfrak{f}_n^{\pm}(x) = \widehat{V}_{\mp\pi/4}\,\psi_n^{\rm ho}(x) = e^{\pm i\frac{\pi}{8}}\,\psi_n^{\rm ho}(e^{\pm i\frac{\pi}{4}}x) \ . \tag{2.11}$$

Now, recalling that (see e.g. [17])

$$\psi_n^{\rm ho}(x) = N_n\, e^{-\frac{\gamma}{2}x^2} H_n(\sqrt{\gamma}x) \ , \tag{2.12}$$

where H_n stands for the n-th Hermite polynomial and the normalization constant

$$N_n = \left(\frac{\sqrt{\gamma}}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} \ , \tag{2.13}$$

one obtains the following formulae for the generalized eigenvectors of $\widehat{H}$:

$$\mathfrak{f}_n^{\pm}(x) = N_n^{\pm}\, e^{\mp i\frac{\gamma}{2}x^2} H_n(\sqrt{\pm i\gamma}x) \ , \tag{2.14}$$

with

$$N_n^{\pm} = e^{\pm i\frac{\pi}{8}}\, N_n = \left(\frac{\sqrt{\pm i\gamma}}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} \ . \tag{2.15}$$

Clearly, $\mathfrak{f}_n^{\pm}$ are not elements from $L^2(\mathbb{R})$ but they do belong to the dual of the Schwartz space $\mathcal{S}(\mathbb{R}_x)'$, i.e. they are tempered distributions.

Proposition 1 *Two families of generalized eigenvectors $\mathfrak{f}_n^{\pm}$ satisfy the following properties:*

1. *they are conjugated to each other:*

$$\overline{\mathfrak{f}_n^{+}(x)} = \mathfrak{f}_n^{-}(x) \ , \tag{2.16}$$

2. *they are orthonormal*

$$\langle\, \mathfrak{f}^{\pm}_n | \mathfrak{f}^{\mp}_m \,\rangle = \delta_{nm} \ , \tag{2.17}$$

3. *they are complete*

$$\sum_{n=0}^{\infty} \overline{\mathfrak{f}^{\pm}_n(x)}\, \mathfrak{f}^{\mp}_n(x') = \delta(x-x') \ . \tag{2.18}$$

The proof follows immediately from orthonormality and completness of oscillator eigenfunctions $\psi^{\rm ho}_n$. Formula (2.16) implies that $\mathfrak{f}^+_n$ and $\mathfrak{f}^-_n$ are related by the time reversal operator $\mathbf{T}$: $\mathbf{T}\psi := \overline{\psi}$. Recall [1] that in u-representation $\mathbf{T}$ is unitary (it is defined by the Fourier transformation), whereas in x-representation it is antiunitary.

3 Change of representation

It should be clear that there exists relation between generalized eigenvectors $\mathfrak{f}^{\pm}_n(x)$ and $f^{\pm}_n(u)$ found in [1]:

$$f^+_n(u) \ \sim\ u^n \ , \qquad f^-_n(u) \ \sim\ \delta^{(n)}(u) \ . \tag{3.1}$$

They define the same eigenvectors $|\pm n\,\rangle$ but in different representations:

$$\mathfrak{f}^{\pm}_n(x) = \langle\, x| \pm n\,\rangle \ , \qquad f^{\pm}_n(u) = \langle\, u| \pm n\,\rangle \ .$$

To find this relation let us observe that the canonical transformation (1.4) is generated by the following generating function

$$S(x,u) = \frac{\gamma}{2}x^2 - \sqrt{2\gamma}xu + \frac{1}{2}u^2 \ , \tag{3.2}$$

that is,

$$p = \frac{\partial S}{\partial x} \ , \qquad v = -\frac{\partial S}{\partial u} \ . \tag{3.3}$$

Let us define a unitary operator

$$\mathcal{U} \ :\ L^2(\mathbb{R}_u) \ \longrightarrow\ L^2(\mathbb{R}_x) \ ,$$

by

$$f \quad\longrightarrow\quad (\mathcal{U}f)(x) = C\int_{-\infty}^{\infty} f(u)e^{iS(x,u)}\,du \ , \tag{3.4}$$

where the constant 'C' is determined by

$$|C|^2\int_{-\infty}^{\infty} e^{iS(x,u)}\, e^{-iS(x',u)}\,du = \delta(x-x') \ . \tag{3.5}$$

It implies $C = e^{i\alpha}\,C_0$, where α is an arbitrary phase and

$$C_0 = \left(\frac{\gamma}{2\pi^2}\right)^{\frac{1}{4}} \ , \tag{3.6}$$

In the next section it would be clear that a natural choice for the phase is $\alpha = -\pi/8$. Clearly, $\mathcal{U}$ may be extended to act on $\mathcal{S}(\mathbb{R}_u)'$. It is easy to show that

$$\mathcal{U}(\mathcal{S}(\mathbb{R}_u)') \subset \mathcal{S}(\mathbb{R}_x)' \ . \tag{3.7}$$

Proposition 2 *The generalized eigenvectors $\mathfrak{f}_n^\pm \in \mathcal{S}(\mathbb{R}_x)'$ and $f_n^\pm \in \mathcal{S}(\mathbb{R}_u)'$ are related by:*

$$\mathfrak{f}_n^\pm = \mathcal{U}\, f_n^\pm \ . \tag{3.8}$$

Proof. Let us show that $\mathfrak{f}_n^+ = \mathcal{U} f_n^+$, that is

$$\mathfrak{f}_n^+(x) \ \sim \ \int u^n e^{iS(x,u)} du \ . \tag{3.9}$$

Using the definition of $S(x,u)$ one has

$$\begin{aligned} \int u^n e^{iS(x,u)} du &= \left(-i\sqrt{2\pi\gamma}\right)^{-n} e^{i\gamma\frac{x^2}{2}} \frac{d^n}{dx^n} \int e^{iu^2/2 - i\sqrt{2\gamma}xu} du \\ &= \sqrt{-2\pi i}\left(-i\sqrt{2\pi\gamma}\right)^{-n} e^{-i\gamma\frac{x^2}{2}} \left(e^{i\gamma x^2} \frac{d^n}{dx^n} e^{-i\gamma x^2}\right) \ . \end{aligned} \tag{3.10}$$

Now, due to the well known formula for the Hermite polynomials

$$e^{iz^2} \frac{d^n}{dz^n} e^{-iz^2} = (-1)^n H_n(z) \ , \tag{3.11}$$

one obtains

$$\int u^n e^{iS(x,u)} du = \sqrt{-2\pi i} \left(\frac{i}{2}\right)^{-\frac{n}{2}} e^{-i\gamma\frac{x^2}{2}} H_n(\sqrt{i\gamma}x) \sim \mathfrak{f}_n^+(x) \ . \tag{3.12}$$

To prove that $\mathfrak{f}_n^- = \mathcal{U} f_n^-$, let us note that[2]

$$\mathfrak{f}_n^-(x) = \overline{\mathfrak{f}_n^+(x)} \ \sim \ \int u^n e^{-iS(x,u)} du \ . \tag{3.13}$$

Now, taking into account that f_n^+ and f_n^- are related by the Fourier transformation

$$u^n = \sqrt{2\pi}(-i)^n F^{-1}[\delta^{(n)}(k)](u) \ , \tag{3.14}$$

one obtains

$$\int u^n e^{-iS(x,u)} du \ = \ \sqrt{2\pi}(-i)^n \int \delta^{(n)}(u) F^{-1}\left[e^{-iS}\right](u)\, du \ . \tag{3.15}$$

Finally,

$$F^{-1}\left[e^{-iS}\right](u) = \frac{1}{\sqrt{2\pi}} \int e^{-iku}\, e^{-iS(x,k)} dk = \sqrt{-i}\, e^{iS(x,u)} \ , \tag{3.16}$$

and hence

$$\mathfrak{f}_n^-(x) \ \sim \ \int \delta^{(n)}(u) e^{iS(x,u)} du \ , \tag{3.17}$$

which ends the proof. □

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[2] It turns out that a function

$$\widetilde{S}(x,v) = -S(x,v) = -\frac{\gamma}{2}x^2 + \sqrt{2\gamma}xv - \frac{1}{2}v^2 \ ,$$

serves as a generating function for the canonical transformation (1.4):

$$p = \frac{\partial \widetilde{S}}{\partial x} \ , \qquad u = -\frac{\partial \widetilde{S}}{\partial v} \ .$$

4 Energy eigenstates

The spectrum of the self-adjoint operator (1.2) reads $\sigma(\widehat{H}) = (-\infty, \infty)$ and the corresponding energy eigenstates (in u-representation) are given by (cf. section 6 in [1]):

$$\psi_\pm^E(u) = \frac{1}{\sqrt{2\pi\gamma}}\, u_\pm^{-(iE/\gamma+1/2)} \ , \tag{4.1}$$

with $E \in \mathbb{R}$. For the basic properties of the tempered distributions $u_\pm^\lambda \in \mathcal{S}(\mathbb{R}_u)'$ we refer the reader to [14, 15] (see also the Appendix in [1]). Now, using (x, p) coordinates the corresponding eigenvalue problem $\frac{1}{2}(\widehat{p}^2 - \gamma^2\widehat{x}^2)\chi^E = E\chi^E$ reads

$$\partial_x^2 \chi^E(x) + (\gamma^2 x^2 + 2E)\chi^E(x) = 0 \ . \tag{4.2}$$

Introducing a new variable

$$z = \sqrt{2i\gamma}\, x \ , \tag{4.3}$$

the above equation may be rewritten as follows

$$\partial_z^2 \chi^E + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right)\chi^E = 0 \ , \tag{4.4}$$

with

$$\nu = -\left(i\frac{E}{\gamma} + \frac{1}{2}\right) \ , \tag{4.5}$$

which is the defining equation for the parabolic cylinder functions [18, 19, 20]. Its solution $\chi^E(z)$ is a linear combination of $D_\nu(z)$, $D_\nu(-z)$, $D_{-\nu-1}(iz)$ and $D_{-\nu-1}(-iz)$.[3] On the other hand the energy eigenstates in x-representation $\chi^E(x)$ may be obtained by applying the operator $\mathcal{U}$ defined in (3.4) to the corresponding eigenstates in u-representation $\psi_\pm^E(u)$:

$$\chi_\pm^E(x) = (\mathcal{U}\psi_\pm^E)(x) = C\int_{-\infty}^{\infty} \psi_\pm^E(u)\, e^{iS(x,u)}\, du \ . \tag{4.6}$$

Hence

$$\begin{aligned} \chi_+^E(x) &= \frac{C}{\sqrt{2\pi\gamma}}\, e^{i\frac{\gamma}{2}x^2} \int_0^\infty u^\nu\, e^{-i\sqrt{2\gamma}xu + iu^2/2}\, du \\ &= \frac{C}{\sqrt{2\pi\gamma}}\, \sqrt{i}^{\nu+1}\, e^{-\frac{y^2}{4}} \int_0^\infty \xi^\nu\, e^{y\xi - \xi^2/2}\, d\xi \ , \end{aligned} \tag{4.7}$$

with $y = \sqrt{-2i\gamma}\, x$, and using an integral representation for $D_p(y)$ (formula 9.241(2) in [18]):[4]

$$D_p(y) = \frac{e^{-\frac{y^2}{4}}}{\Gamma(-p)} \int_0^\infty \xi^{-p-1}\, e^{-y\xi - \xi^2/2}\, d\xi \ , \tag{4.8}$$

[3] These four functions are linearly dependent. For the linear relation see e.g. formula 9.248 in [18].

[4] The validity of this formula is restricted in [18] for $\mathrm{Re}\, p < 0$. However, as we shall show (see the proof of Proposition 4), it is valid for all $p \in \mathbb{C}$.

one finds

$$\chi_+^E(x) = \frac{C_0}{\sqrt{2\pi\gamma}}\,\sqrt{i}^{\nu+\frac{1}{2}}\,\Gamma(\nu+1)D_{-\nu-1}(-\sqrt{-2i\gamma}x)\ , \tag{4.9}$$

with ν given in (4.5). Similarly, using an obvious relation $(-u)_+^\lambda = u_-^\lambda$, one obtains:

$$\chi_-^E(x) = \frac{C_0}{\sqrt{2\pi\gamma}}\,\sqrt{i}^{\ \nu+\frac{1}{2}}\,\Gamma(\nu+1)D_{-\nu-1}(\sqrt{-2i\gamma}x)\ , \tag{4.10}$$

that is, $\chi_-^E(x) = \chi_+^E(-x)$. Actually, instead of $\chi_\pm^E$ one may use energy eigenstates with the definite parity:

$$\chi_{\rm even}^E = \frac{1}{\sqrt{2}}\left(\chi_+^E + \chi_-^E\right)\ , \tag{4.11}$$

$$\chi_{\rm odd}^E = \frac{1}{\sqrt{2}}\left(\chi_+^E - \chi_-^E\right)\ , \tag{4.12}$$

that is,

$$\mathbf{P}\,\chi_{\rm even}^E = \chi_{\rm even}^E\ , \qquad \mathbf{P}\,\chi_{\rm odd}^E = -\chi_{\rm odd}^E\ , \tag{4.13}$$

where $\mathbf{P}$ stands for the parity operator.

Proposition 3 *Energy eigenstates $\chi_\pm^E$ satisfy:*

$$\int_{-\infty}^{\infty} \overline{\chi_\pm^E(x)}\chi_\pm^{E'}(x)\,dx = \delta(E-E')\ , \tag{4.14}$$

and

$$\int_{-\infty}^{\infty} \overline{\chi_\pm^E(x)}\chi_\pm^E(x')\,dE = \delta(x-x')\ . \tag{4.15}$$

The proof follows immediately from the analogous properties satisfied by energy eigenstates $\psi_\pm^E$ in u-representation [1].

In [1] we have used also another generalized basis $F[\psi_\pm^{-E}](u)$. Now, we find its $\mathcal{U}$ image in $\mathcal{S}(\mathbb{R}_x)'$. Recalling the Fourier transformation of $x_\pm^\lambda$ (see [14] and Appendix in [1]):

$$F[x_\pm^\lambda](u) = \frac{\pm i}{\sqrt{2\pi}}\,e^{\pm i\lambda\frac{\pi}{2}}\,\Gamma(\lambda+1)(u+i0)^{-\lambda-1}\ . \tag{4.16}$$

one has

$$F[\psi_+^{-E}](u) = \frac{1}{\sqrt{2\pi\gamma}}\,\frac{(-i)^\nu}{\sqrt{2\pi}}\,\Gamma(-\nu)(u+i0)^\nu\ . \tag{4.17}$$

Therefore, the corresponding x-representation

$$\eta_+^E(x) = (\mathcal{U}\,F[\psi_+^{-E}])(x)\ , \tag{4.18}$$

is given by

$$\begin{aligned}\eta_+^E(x) &= \frac{C}{\sqrt{2\pi\gamma}}\,\frac{(-i)^\nu}{\sqrt{2\pi}}\,\Gamma(-\nu)\int_{-\infty}^{\infty}(u+i0)^\nu e^{iS(x,u)}du \\ &= \frac{C}{\sqrt{2\pi\gamma}}\,\frac{(-i)^\nu}{\sqrt{2\pi}}\,(2\sqrt{i})^{\nu+1}\,\Gamma(-\nu)\,e^{\frac{y^2}{4}}\int_{-\infty}^{\infty}(\xi+i0)^\nu\,e^{-2\xi^2-2iy\xi}\,d\xi\ ,\end{aligned} \tag{4.19}$$

with $y=\sqrt{2i\gamma}x$. Now, using the following integral representation (formula 9.241(1) in [18])

$$D_\nu(y) = \frac{1}{\sqrt{\pi}}\, 2^{\nu+\frac12}(-i)^\nu e^{\frac{y^2}{4}} \int_{-\infty}^{\infty} (\xi + i0)^\nu\, e^{-2\xi^2+2iy\xi}\, d\xi\ , \tag{4.20}$$

one obtains

$$\eta_+^E(x) = \frac{C_0}{\sqrt{2\pi\gamma}}\, \sqrt{i}^{\ \nu+\frac12}\, \Gamma(-\nu) D_\nu(-\sqrt{2i\gamma}x)\ . \tag{4.21}$$

Similarly one shows that

$$\eta_-^E(x) = (\mathcal{U}\, F[\psi_-^{-E}])(x)\ , \tag{4.22}$$

is given by

$$\eta_-^E(x) = \frac{C_0}{\sqrt{2\pi\gamma}}\, \sqrt{i}^{\ \nu+\frac12}\, \Gamma(-\nu) D_\nu(\sqrt{2i\gamma}x)\ . \tag{4.23}$$

Let us note, that

$$\overline{\nu+1} = -\nu\ , \tag{4.24}$$

and

$$\overline{\sqrt{i}^{\ \nu+\frac12}} = \sqrt{i}^{\ \nu+\frac12}\ . \tag{4.25}$$

Clearly, the transition $\nu+1 \longrightarrow -\nu$ is equivalent to $E \longrightarrow -E$ and it corresponds to the fact that $\widehat{H}\eta_+^E = -E\eta_+^E$ while $\widehat{H}\chi_+^E = +E\chi_+^E$. The symmetry between $\chi_\pm^E$ and $\eta_\pm^E$ fully justifies the specific choice of the phase factor in the constant C. One has

$$\eta_\pm^E(x) = \overline{\chi_\pm^E(x)}\ , \tag{4.26}$$

that is they are related by the time reversal operator $\mathbf{T}$: $\eta_\pm^E = \mathbf{T}\,\chi_\pm^E$. Thus energy eigenstates $\eta_\pm^E$ correspond to the time reversed system. This way all four solutions of (4.4) were used to construct four families of energy eigenstates: χ_+^E, χ_-^E, η_+^E and η_-^E.

5 Analytic continuation, resolvent and resonances

Now, let us continue the energy eigenfunctions $\chi_\pm^E$ and $\eta_\pm^E$ into the energy complex plane $E \in \mathbb{C}$ and let us study its analyticity as functions of E.

Proposition 4 *The parabolic cylinder function $D_\lambda(z)$ is an analytic function of $\lambda \in \mathbb{C}$.*

For the proof see the Appendix. Due to the above proposition the analytic properties of the energy eigenfunctions are entirely governed by the analytic properties of the Γ function which is present in the definition of $\chi_\pm^E$ and $\eta_\pm^E$. Since $\Gamma(\lambda)$ has simple poles at $\lambda=-n$, with $n=0,1,2,\ldots$, functions $\chi_\pm^E$ have poles at $E=-E_n$, whereas functions $\eta_\pm^E$ have poles at $E=E_n$, where E_n is defined in (2.10). Using a well known formula for a residue of the Γ function

$$\mathrm{Res}\,(\Gamma(\lambda);\lambda=-n) = \frac{(-1)^n}{n!}\ , \tag{5.1}$$

one has

$$\mathrm{Res}\,\big(\chi_\pm^E(x);-E_n\big) = \frac{C_0}{\sqrt{2\pi\gamma}}\,\frac{(-1)^n}{n!}\,\sqrt{i}^{\ -n-\frac12}\, D_n(\mp\sqrt{-2i\gamma}x)\ , \tag{5.2}$$

and

$$\mathrm{Res}\left(\eta^E_\pm(x); +E_n\right) = \frac{C_0}{\sqrt{2\pi\gamma}}\,\frac{(-1)^n}{n!}\,\sqrt{i}^{\;n+\frac{1}{2}}\,D_n(\mp\sqrt{2i\gamma}x)\ . \tag{5.3}$$

Hence, using the relation [18, 19, 20]:[5]

$$D_n(z) = 2^{-\frac{n}{2}}\,e^{-\frac{z^2}{4}}\,H_n\left(\frac{z}{\sqrt{2}}\right)\ , \qquad n = 0,1,2,\dots\ , \tag{5.4}$$

together with

$$H_n(-z) = (-1)^n H_n(z)\ , \tag{5.5}$$

one obtains

$$\mathrm{Res}\left(\chi^E_\pm(x); -E_n\right)\ \sim\ \mathfrak{f}^+_n(x)\ , \tag{5.6}$$

and

$$\mathrm{Res}\left(\eta^E_\pm(x); +E_n\right)\ \sim\ \mathfrak{f}^-_n(x)\ . \tag{5.7}$$

Now, it is natural to introduce two Hardy classes of functions [21]. Recall, that a smooth function $f = f(E)$ is in the Hardy class from above $\mathcal{H}^2_+$ (from below $\mathcal{H}^2_-$) if $f(E)$ is a boundary value of an analytic function in the upper, i.e. $\mathrm{Im}\,E \geq 0$ (lower, i.e. $\mathrm{Im}\,E \leq 0$) half complex E-plane vanishing faster than any power of E at the upper (lower) semi-circle $|E| \to \infty$. Define

$$\Phi_- := \left\{\phi \in \mathcal{S}(\mathbb{R}_x)\ \middle|\ f(E) := \langle\,\chi^E_\pm|\phi\,\rangle \in \mathcal{H}^2_-\ \right\}\ , \tag{5.8}$$

and

$$\Phi_+ := \left\{\phi \in \mathcal{S}(\mathbb{R}_x)\ \middle|\ f(E) := \langle\,\eta^E_\pm|\phi\,\rangle \in \mathcal{H}^2_+\ \right\}\ . \tag{5.9}$$

It is evident from (4.26) that $\Phi_+ = \overline{\Phi_-}$, that is

$$\Phi_+ = \mathbf{T}(\Phi_-)\ . \tag{5.10}$$

Due to the Gel'fand-Maurin spectral theorem [22, 23] any function $\phi^- \in \Phi_-$ may be decomposed with respect to $\chi^E_\pm$ family

$$\phi^-(x) = \sum_\pm \int_{-\infty}^{\infty} dE\,\chi^E_\pm(x)\langle\,\chi^E_\pm|\phi^-\,\rangle\ , \tag{5.11}$$

and any function $\phi^+ \in \Phi_+$ may be decomposed with respect to $\eta^E_\pm$ family

$$\phi^+(x) = \sum_\pm \int_{-\infty}^{\infty} dE\,\eta^E_\pm(x)\langle\,\eta^E_\pm|\phi^+\,\rangle\ . \tag{5.12}$$

Applying the Residue Theorem one easily proves the following

[5] In [18] the corresponding equation 9.253 has a wrong sign.

Theorem 1 *For any function $\phi^\pm \in \Phi_\pm$ one has*

$$\phi^-(x) = \sum_{n=0}^{\infty} \mathfrak{f}_n^-(x)\langle\, \mathfrak{f}_n^+|\phi^-\,\rangle \ , \tag{5.13}$$

and

$$\phi^+(x) = \sum_{n=0}^{\infty} \mathfrak{f}_n^+(x)\langle\, \mathfrak{f}_n^-|\phi^+\,\rangle \ . \tag{5.14}$$

The proof goes along the same lines as the corresponding proof of Theorem 2 in [1]. The above theorem implies the following spectral resolutions of the Hamiltonian:

$$\widehat{H} = \sum_{\pm} \int_{-\infty}^{\infty} dE\, E|\chi_\pm^E\,\rangle\langle\,\chi_\pm^E| = -\sum_{n=0}^{\infty} E_n|\mathfrak{f}_n^-\,\rangle\langle\,\mathfrak{f}_n^+| \ , \tag{5.15}$$

on Φ_-, and

$$\widehat{H} = \sum_{\pm} \int_{-\infty}^{\infty} dE\, E|\eta_\pm^E\,\rangle\langle\,\eta_\pm^E| = \sum_{n=0}^{\infty} E_n|\mathfrak{f}_n^+\,\rangle\langle\,\mathfrak{f}_n^-| \ , \tag{5.16}$$

on Φ_+. The same techniques may be applied for the resolvent operator

$$R(z,\widehat{H}) = \frac{1}{\widehat{H} - z} \ . \tag{5.17}$$

One obtains

$$R(z,\widehat{H}) = \sum_{\pm} \int_{-\infty}^{\infty} \frac{dE}{E-z}\, |\chi_\pm^E\,\rangle\langle\,\chi_\pm^E| \ = \sum_{n=0}^{\infty} \frac{1}{-E_n - z}\, |\mathfrak{f}_n^-\,\rangle\langle\,\mathfrak{f}_n^+| \ , \tag{5.18}$$

on Φ_-, and

$$R(z,\widehat{H}) = \sum_{\pm} \int_{-\infty}^{\infty} \frac{dE}{E-z}\, |\eta_\pm^E\,\rangle\langle\,\eta_\pm^E| \ = \sum_{n=0}^{\infty} \frac{1}{E_n - z}\, |\mathfrak{f}_n^+\,\rangle\langle\,\mathfrak{f}_n^-| \ , \tag{5.19}$$

on Φ_+. Hence, $R(z,\widehat{H})|_{\Phi_-}$ has poles at $z = -E_n$, and $R(z,\widehat{H})|_{\Phi_+}$ has poles at $z = E_n$. As usual eigenvectors $\mathfrak{f}_n^-$ and $\mathfrak{f}_n^+$ corresponding to poles of the resolvent are interpreted as resonant states. Note, that

$$-\frac{1}{2\pi i}\oint_{\gamma_n} R(z,\widehat{H})dz = |\mathfrak{f}_n^+\,\rangle\langle\,\mathfrak{f}_n^-| := \widehat{P}_n \ , \tag{5.20}$$

where γ_n is a closed curve that encircles the singularity $z = E_n$. Clearly,

$$\widehat{P}_n \cdot \widehat{P}_m = \delta_{nm}\widehat{P}_n \ , \tag{5.21}$$

and the spectral decomposition of $\widehat{H}$ may be written as follows:

$$\widehat{H} = \sum_{n=0}^{\infty} E_n\widehat{P}_n = -\sum_{n=0}^{\infty} E_n\widehat{P}_n^\dagger \ . \tag{5.22}$$

Finally, let us note, that restriction of the unitary group $U(t) = e^{-i\widehat{H}t}$ defined on the Hilbert space $L^2(\mathbb{R})$ to $\Phi_\pm$ no longer defines a group. It gives rise to two semigroups:

$$U_-(t) \ : \ \Phi_- \ \longrightarrow \ \Phi_- \ , \qquad \text{for} \quad t \geq 0 \ , \tag{5.23}$$

and

$$U_+(t) \ : \ \Phi_+ \ \longrightarrow \ \Phi_+ \ , \qquad \text{for} \quad t \leq 0 \ . \tag{5.24}$$

Using (5.15), (5.16) and the formula for $E_n = i\gamma(n + \frac{1}{2})$ one finds:

$$\phi^-(t) = U_-(t)\phi^- = \sum_{n=0}^{\infty} e^{-\gamma(n+\frac{1}{2})t}\, \widehat{P}_n^\dagger \phi^- \ , \tag{5.25}$$

for $t \geq 0$, and

$$\phi^+(t) = U_+(t)\phi^+ = \sum_{n=0}^{\infty} e^{\gamma(n+\frac{1}{2})t}\, \widehat{P}_n \phi^+ \ , \tag{5.26}$$

for $t \leq 0$. We stress that ϕ_t^- (ϕ_t^+) does belong to $L^2(\mathbb{R})$ also for $t < 0$ $(t > 0)$. However, $\phi_t^- \in \Phi_-$ $(\phi_t^+ \in \Phi_+)$ only for $t \geq 0$ $(t \leq 0)$. This way the irreversibility enters the dynamics of the reversed oscillator by restricting it to the dense subspace $\Phi_\pm$ of $L^2(\mathbb{R})$.

6 Scattering vs. resonant states

To compare the physical properties of energy eigenstates $\chi_\pm^E$ and $\eta_\pm^E$ and resonant states $\mathfrak{f}_n^\pm$ let us investigate its asymptotic behavior at $x \longrightarrow \pm\infty$. Following [20] (see also [10, 11]) one finds[6]

$$\chi_-^E(x \to +\infty) \ \sim \ \sqrt{\frac{1}{x}}\, \exp\left[i \left(\frac{\gamma}{2}x^2 + \frac{E}{\gamma}\log(\sqrt{2\gamma}x) + \frac{\pi}{4}\frac{E}{\gamma} + \frac{\pi}{8} \right) \right] \ , \tag{6.1}$$

and

$$\begin{aligned} \chi_-^E(x \to -\infty) \ \ &\sim \ \ i\sqrt{\frac{1}{x}} \left\{ \left(1 + e^{-2\pi\frac{E}{\gamma}} \right) \exp\left[-i \left(\frac{\gamma}{2}x^2 + \frac{E}{\gamma}\log(\sqrt{2\gamma}x) - \frac{\pi}{4}\frac{E}{\gamma} + \frac{3\pi}{8} + \phi \right) \right] \right. \\ &- \ \ \left. e^{-\pi\frac{E}{\gamma}} \exp\left[i \left(\frac{\gamma}{2}x^2 - \frac{E}{\gamma}\log(\sqrt{2\gamma}x) - \frac{\pi}{4}\frac{E}{\gamma} + \frac{\pi}{8} \right) \right] \right\} \ , \end{aligned} \tag{6.2}$$

where $\phi = \arg\Gamma(-i\frac{E}{\gamma} + \frac{1}{2}) = \Gamma(\nu + 1)$. Hence energy eigenstates χ_-^E represent scattering states (see [10] for more details). The same is true for χ_+^E and $\eta_\pm^E$. In particular one finds for

[6] Putting $a = -E/\gamma$ in equation 19.17.9 in [20] and using relation 19.3.1

$$U(a, x) = D_{-a-\frac{1}{2}}(x) \ ,$$

one finds:

$$U\left(-i\frac{E}{\gamma}, \sqrt{2\gamma}xe^{-\frac{1}{4}i\pi} \right) = D_{-\nu-1}(\sqrt{-2i\gamma}x) \ \sim \ \chi_-^E(x) \ .$$

the reflection and transmission amplitudes R and T for $\chi_\pm^E$ scattering states [8, 10]:

$$R(\chi_\pm^E) = -\frac{i}{\sqrt{2\pi}}\, e^{-\frac{\pi E}{2\gamma}}\, \Gamma\left(\frac{1}{2} - i\frac{E}{\gamma}\right) , \tag{6.3}$$

$$T(\chi_\pm^E) = \frac{1}{\sqrt{2\pi}}\, e^{\frac{\pi E}{2\gamma}}\, \Gamma\left(\frac{1}{2} - i\frac{E}{\gamma}\right) . \tag{6.4}$$

Clearly, computing R and T for time-reversed $\eta_\pm^E$ scattering states one finds:

$$R(\eta_\pm^E) = \overline{R(\chi_\pm^E)} , \qquad T(\eta_\pm^E) = \overline{T(\chi_\pm^E)} . \tag{6.5}$$

Note, that $R(\chi_\pm^E)$ and $T(\chi_\pm^E)$ have poles at $E = -E_n$, whereas $R(\eta_\pm^E)$ and $T(\eta_\pm^E)$ have poles at $E = +E_n$. Obviously, the corresponding reflection and transition coefficients $|R|^2$ and $|T|^2$ are time-reversal invariant.

On the other hand the eigenstates $\mathfrak{f}_n^\pm$ behave as follows:

$$\mathfrak{f}_n^+(x \to \pm\infty) \sim (\pm\sqrt{i\gamma}x)^n\, e^{-i\frac{\gamma}{2}x^2} , \tag{6.6}$$

and

$$\mathfrak{f}_n^-(x \to \pm\infty) \sim (\pm\sqrt{-i\gamma}x)^n\, e^{i\frac{\gamma}{2}x^2} . \tag{6.7}$$

Note, that $\mathfrak{f}_n^-$ are purely outgoing states, whereas $\mathfrak{f}_n^+$ are purely ingoing states. Moreover, resonant states have Breit-Wigner energy distribution. Indeed,

$$\langle\, \chi_-^E | \mathfrak{f}_n^+ \,\rangle \sim \Gamma(-\nu) \int_{-\infty}^{\infty} D_\nu(\sqrt{2\gamma i}x)\, \mathfrak{f}_n^+(x)\, dx . \tag{6.8}$$

Now, D_ν is an entire function of ν and $\Gamma(-\nu)$ has poles at $\nu = k \in \mathbb{N}$. In the domain where $n + 1 > \mathrm{Re}\,\nu \geq 1$ one has

$$\Gamma(-\nu) = \text{analytical part } + \sum_{k=0}^{n} \frac{(-1)^k}{k!(k-\nu)^k} . \tag{6.9}$$

Hence,

$$\begin{aligned} \langle\, \chi_-^E | \mathfrak{f}_n^+ \,\rangle &\sim \text{analytical function of } E + \sum_{k=0}^{n} \frac{(-1)^k}{k!\left(k + i\frac{E}{\gamma} + \frac{1}{2}\right)} \langle\, \mathfrak{f}_k^- | \mathfrak{f}_n^+ \,\rangle \\ &\sim \text{analytical function of } E + \frac{\gamma}{E - E_n} , \end{aligned} \tag{6.10}$$

which is consistent with the Breit-Wigner formula.

A Appendix

The integral formula 9.241(2) in [18]

$$D_\lambda(y) = \frac{e^{-\frac{y^2}{4}}}{\Gamma(-\lambda)} \int_{-\infty}^{\infty} \xi_+^{-\lambda-1}\, e^{-y\xi - \xi^2/2}\, d\xi , \tag{A.1}$$

contains two objects: $\Gamma(-\lambda)$ and a distribution $\xi^{-\lambda-1}$ which are singular for $\lambda = 0, 1, 2, \ldots$. However, it is easy to see [14] that

$$\left.\frac{\xi_+^{-\lambda-1}}{\Gamma(-\lambda)}\right|_{p=n} = \delta^{(n)}(\xi) \ , \tag{A.2}$$

which shows that (A.1) defines an entire function of $\lambda \in \mathbb{C}$. The same is true for

$$D_\lambda(y) = \frac{e^{-\frac{y^2}{4}}}{\Gamma(-\lambda)} \int_{-\infty}^{\infty} \xi_-^{-\lambda-1}\, e^{y\xi-\xi^2/2}\, d\xi \ , \tag{A.3}$$

due to

$$\left.\frac{\xi_-^{-\lambda-1}}{\Gamma(-\lambda)}\right|_{\lambda=n} = (-1)^n \delta^{(n)}(\xi) \ . \tag{A.4}$$

The second integral representation given by 9.241(1) in [18]

$$D_\lambda(y) = \frac{1}{\sqrt{\pi}}\, 2^{\lambda+\frac{1}{2}}(-i)^\lambda e^{\frac{y^2}{4}} \int_{-\infty}^{\infty} (\xi + i0)^\lambda\, e^{-2\xi^2+2iy\xi}\, d\xi \ , \tag{A.5}$$

where $(\xi + i0)^\lambda = \xi_+^\lambda + e^{i\pi\lambda}\xi_-^\lambda$, seems to have poles at $\lambda = -1, -2, \ldots$. However, the limit $\lim_{\lambda\to -n}(\xi + i0)^\lambda$ is well defined [14]

$$(\xi + i0)^{-n} = \xi^{-n} - \frac{i\pi(-1)^{n-1}}{(n-1)!}\, \delta^{(n-1)}(\xi) \ . \tag{A.6}$$

Thus, formula (A.5) also defines an entire function of λ.

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