# On the spectral analysis and scattering theory of the Laplacian on the halfplane with a periodic perturbation on the boundary

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ABSTRACT. We study the spectrum of the Laplacian $H^{(\sigma)} = -\Delta$ on $L_2(\mathbb{R}^2_+)$ corresponding to the boundary condition $\frac{\partial u}{\partial \nu} + \sigma u = 0$ for a wide class of periodic functions $\sigma$. The Floquet decomposition leads to problems on a non-compact cell, which are analyzed in detail. This allows us to prove under the condition $\sigma \geq 0$ that $H^{(\sigma)}$ is unitarily equivalent to the Neumann Laplacian $H^{(0)}$, the equivalence being provided by the wave operators. In the general case the existence of additional channels of scattering is investigated, which are due to (possibly embedded) eigenvalues of the problems from the Floquet decomposition.

# Introduction

**1.** The present text studies the Laplacian

$$H^{(\sigma)} u = -\Delta u \qquad \text{on } \mathbb{R}^2_+ \tag{0.0.1}$$

on the halfplane together with a boundary condition of the third type

$$\frac{\partial u}{\partial \nu} + \sigma u = 0 \qquad \text{on } \mathbb{R} \times \{0\}, \tag{0.0.2}$$

where the function $\sigma : \mathbb{R} \to \mathbb{R}$ is assumed to be periodic. Moreover, let

$$\sigma \in L_q(\mathbb{T}) \qquad \text{for some } q > 1.$$

Under this condition $H^{(\sigma)}$ can be defined as a self-adjoint operator in $L_2(\mathbb{R}^2_+)$ by means of the closed and lower semibounded quadratic form

$$\int_{\mathbb{R}^2_+} |Du(x)|^2 \, dx + \int_{\mathbb{R}} \sigma(x_1) |u(x_1, 0)|^2 \, dx_1, \qquad u \in H^1(\mathbb{R}^2_+).$$

Our goal is to analyze the spectrum of $H^{(\sigma)}$ and to develop a scattering theory, where we view $H^{(\sigma)}$ as a (rather singular) perturbation of $H^{(0)}$, the Neumann Laplacian on $\mathbb{R}^2_+$.
Our main result is that under the assumption

$$\sigma \geq 0 \qquad \text{a.e.} \tag{0.0.3}$$

the wave operators exist and provide a unitary equivalence between $H^{(\sigma)}$ and $H^{(0)}$. In particular, $H^{(\sigma)}$ is absolutely continuous.

**2.** In the physical interpretation, $H^{(\sigma)}$ is the Hamiltonian of a two-dimensional quantum-mechanical system which consists of an electron in the upper halfplane and a crystal that fills the lower halfplane. The electron can not enter the crystal but interacts non-trivially with the surface of the crystal, described by the function $\sigma$. The condition (0.0.3) means that the interaction is repulsive, so one expects the particle to behave like a free particle in the distant future and the distant past, when it is far away from the crystal surface. This is exactly what the unitarity of the wave operators means.
We emphasize that if (0.0.3) is violated, there may appear surface states which propagate along the boundary and decay exponentially away from the boundary.

**3.** Our problem can be considered as a scattering problem on the "obstacle" $\Omega = \mathbb{R}^2_-$ with non-compact boundary. For general obstacles $\Omega$ there are rather complete results in the case of a Dirichlet boundary condition on the boundary of the obstacle, however, the case of a Neumann, or more generally of a third kind boundary condition seems to be open (cf. Section 17 in [Ya2]). In particular, we are not aware of a prior work considering a periodic third type condition. The presented results seem to be new.

**4.** Let us explain some of the mathematical ideas involved. A precise definition of the operator $H^{(\sigma)}$ in terms of a quadratic form is given in Section 2.1. By means of the Bloch-Floquet theory we write $H^{(\sigma)}$ in Section 1.4 as a direct integral

$$\int_{-1/2}^{1/2} \bigoplus H^{(\sigma)}(k)\, dk$$

with fiber operators $H^{(\sigma)}(k)$ acting on $L_2(\Pi)$ where $\Pi := (-\pi, \pi) \times \mathbb{R}_+$ is the halfstrip. Functions in the domain of $H^{(\sigma)}(k)$ satisfy the third type condition (0.0.2) on $(-\pi, \pi) \times \{0\}$ (at least if $\sigma$ is smooth), so $H^{(\sigma)}(k)$ differs from $H^{(0)}(k)$ by a relatively compact form perturbation. This makes a rather detailed analysis of the operators $H^{(\sigma)}(k)$ possible. It is carried out in Section 3 and constitutes the main part of the text.
Our approach leans on a quadratic form version of the resolvent identity which we present in Subsection 3.1.1 following [Ya1]. A similar approach has been successfully applied to study periodic Schrödinger operators (cf. [BSu] and [BShtSu]). In our case it allows to show that the difference of resolvents of $H^{(\sigma)}(k)$ and $H^{(0)}(k)$ belongs to the trace class, and from the Birman-Kreĭn theorem (which is sometimes called Birman-Kuroda theorem, unaware of [BKr]) we deduce in Section 3.1 information on the wave operators and the absolutely continuous spectrum. Using the same representation we can prove a limiting absorption principle in Section 3.2, which implies the absence of singular continuous spectrum.
The point spectrum is treated in Section 3.3. In the spirit of the Birman-Schwinger principle we transform the eigenvalue problem for $H^{(\sigma)}(k)$ and $\lambda \in \mathbb{R}$

to the problem whether 0 is an eigenvalue of a certain "discrete pseudo-differential operator" of order one on $L_2(\mathbb{T})$. In this way we reduce the problem of (possibly embedded) eigenvalues to the study of operators with compact resolvent. The Analytic Fredholm Alternative and a continuity principle yield information on the location of eigenvalues. Under the condition (0.0.3) we can prove the absence of positive eigenvalues. This completes the study of the spectrum of the operators $H^{(\sigma)}(k)$ in the case $\sigma \geq 0$.
In Section 3.4 we study the scattering matrix in more detail and derive a representation in terms of generalized eigenfunctions.
Finally, in Section 4.2 we deduce our main result (Theorem 4.2.1) about $H^{(\sigma)}$ under the condition (0.0.3) from the results about the fiber operators $H^{(\sigma)}(k)$. We supplement this theorem with information on the case of general $\sigma$, when $H^{(\sigma)}$ may have negative spectrum with band structure.

**5.** In the case when condition (0.0.3) is violated there remain some open questions. The eigenvalues of the fiber operators $H^{(\sigma)}(k)$ may generate "new" zones in the spectrum of the operator $H^{(\sigma)}$, which produce waveguide effects. This phenomenon is worth a more detailed study.


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# Contents

# Chapter 1

# Preliminary material

## 1.1 Notation

In this section we introduce the notation we use for some geometric objects in $\mathbb{R}^2_+$ and for spaces of smooth functions. General conventions are made.

**1. Sets.** We shall work in two-dimensional Euclidean space $\mathbb{R}^2$ with coordinates denoted by $x = (x_1, x_2)$. In particular, we consider the *halfplane*

$$\mathbb{R}^2_+ := \{x \in \mathbb{R}^2_+ :\ x_2 > 0\} = \mathbb{R} \times \mathbb{R}_+,$$

and the *halfstrip*

$$\Pi := \{x \in \mathbb{R}^2_+ :\ -\pi < x_1 < \pi,\ x_2 > 0\} = (-\pi, \pi) \times \mathbb{R}_+,$$

where $\mathbb{R}_+ := (0, +\infty)$. We find it convenient to denote the coordinates on $\mathbb{R}$ (or $(-\pi, \pi)$) and $\mathbb{R}_+$ by $x_1$ and $x_2$, respectively, even if working in a one-dimensional context.
Moreover, we need the lattice $2\pi\mathbb{Z}$. Unless stated otherwise, periodicity conditions are understood with repect to this lattice. We think of the corresponding torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ as the interval $[-\pi, \pi]$ with endpoints identified.

**2. Spaces of differentiable functions.** As usual, partial differentiation (in the classical and in the distributional sense) with respect to the coordinate $x_j$ is denoted by $\frac{\partial}{\partial x_j}$, and $D_j := \frac{1}{i}\frac{\partial}{\partial x_j}$. If $x_j$ is the coordinate in a one-dimensional set, we write $\frac{d}{dx_j}$ instead of $\frac{\partial}{\partial x_j}$. For higher order derivatives we use the usual multi-index notation.
Let $d \in \mathbb{N}$ (we are particularly interested in the cases $d = 1, 2$). For an open set $\Omega \subset \mathbb{R}^d$ we denote by $C^\infty(\Omega)$ the space of complex-valued functions on $\Omega$ which have continuous derivatives in $\Omega$ up to any order. $C_0^\infty(\Omega)$ consists of those functions in $C^\infty(\Omega)$ whose support is a compact subset of $\Omega$.

We need some more notation when periodicity is involved. In the case of the torus we denote by $C^\infty(\mathbb{T})$ the space of functions in $C^\infty(-\pi,\pi)$ which can be extended ($2\pi$-)periodically to a function in $C^\infty(\mathbb{R})$. In the case of the halfstrip we denote by $\tilde{C}^\infty(\Pi)$ the space of functions in $C^\infty(\Pi)$ which can be extended ($2\pi$-)periodically with respect to $x_1$ to a function in $C^\infty(\mathbb{R}^2_+)$.
We also need the notion of the Schwartz class on an arbitrary open set $\Omega \subset \mathbb{R}^d$. Namely, we say that $u \in \mathcal{S}(\Omega)$ iff $u \in C^\infty(\Omega)$ and

$$\sup_{x\in\Omega}(1+|x|^N)|D^\alpha u(x)| < \infty, \qquad N \in \mathbb{N}_0,\, \alpha \in \mathbb{N}_0^d.$$

**3. Conventions.** All vector spaces are considered over the field $\mathbb{C}$. All subspaces of Hilbert spaces are assumed to be closed.
Unless stated otherwise, we choose the branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ that satisfies $\operatorname{Re}\sqrt{\cdot} > 0$.
Statements and formulae which contain the double index "$\pm$" are understood as two independent assertions.

## 1.2 Sobolev spaces

In this section we present some material on Sobolev spaces. We do not aim at completeness, but rather try to standardize notation.
We are particularly interested in Sobolev spaces with periodicity conditions. After having recalled in Subsection 1 the Fourier expansion in $L_2$-spaces, we introduce periodic Sobolev spaces in Subsection 2. Finally, we present some auxiliary statements related to Sobolev embedding theorems which will be used frequently in Chapter 3.

**1. $L_2$-spaces.** For an open set $\Omega \subset \mathbb{R}^d$ we denote by $L_2(\Omega)$ the Hilbert space of (equivalence classes of) square-integrable complex-valued functions on $\Omega$, endowed with the scalar product

$$(u,v)_{L_2(\Omega)} := \int_\Omega u(x)\overline{v(x)}\,dx, \qquad u,\, v \in L_2(\Omega),$$

and the corresponding norm

$$\|u\|^2_{L_2(\Omega)} := \int_\Omega |u(x)|^2\,dx, \qquad u \in L_2(\Omega).$$

We usually omit the index $L_2(\Omega)$.
The space $L_2(\mathbb{T})$ may be formally identified with $L_2(-\pi,\pi)$. It has a complete orthonormal system $\{e_n :\ n \in \mathbb{Z}\}$ consisting of the trigonometric functions

$$e_n(x_1) := \frac{1}{\sqrt{2\pi}} e^{inx_1}, \qquad x_1 \in \mathbb{T}.$$

Every function $f \in L_2(\mathbb{T})$ can be expanded in its ($L_2$-convergent) *Fourier series*

$$f = \sum_{n\in\mathbb{Z}} \hat{f}_n e_n \tag{1.2.1}$$

with *Fourier coefficients*

$$\hat{f}_n := (f, e_n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x_1) e^{-inx_1}\, dx_1, \qquad n \in \mathbb{Z}. \tag{1.2.2}$$

Parseval's equality states that

$$\|f\|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2. \tag{1.2.3}$$

Let us pass to the case of the halfstrip. Considering $L_2(\Pi)$ as $L_2(-\pi,\pi) \otimes L_2(\mathbb{R}_+)$, the expansion (1.2.1) induces an ($L_2(\Pi)$-convergent) expansion

$$u = \sum_{n\in\mathbb{Z}} e_n \otimes \hat{u}_n \tag{1.2.4}$$

for any $u \in L_2(\Pi)$, where

$$\hat{u}_n(x_2) := (u(.,x_2), e_n)_{L_2(\mathbb{T})} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u(x) e^{-inx_1}\, dx_1, \qquad x_2 \in \mathbb{R}_+,$$

for all $n \in \mathbb{Z}$. This integral exists for a.e. $x_2 \in \mathbb{R}_+$, $\hat{u}_n$ belongs to $L_2(\mathbb{R}_+)$ and

$$\|u\|^2_{L_2(\Pi)} = \sum_{n\in\mathbb{Z}} \|\hat{u}_n\|^2_{L_2(\mathbb{R}_+)}. \tag{1.2.5}$$

Hence we have a decomposition of the Hilbert space

$$L_2(\Pi) = \sum_{n\in\mathbb{Z}} \bigoplus \operatorname{span}\{e_n\} \otimes L_2(\mathbb{R}_+). \tag{1.2.6}$$

**2. Periodic Sobolev spaces.** For an open set $\Omega \subset \mathbb{R}^d$ we denote by $H^s(\Omega)$ the Sobolev space of order $s \in \mathbb{R}$ (with integrability index 2). For definitions as well as the Sobolev embedding theorems we refer to [A].

For $s = m \in \mathbb{N}_0$ we use the standard scalar product

$$(u,v)_{H^m(\Omega)} := \sum_{|\alpha|\le m} \int_\Omega D^\alpha u(x) \overline{D^\alpha v(x)}\, dx, \qquad u,\, v \in H^m(\Omega),$$

with corresponding norm

$$\|u\|^2_{H^m(\Omega)} := \sum_{|\alpha|\le m} \int_\Omega |D^\alpha u(x)|^2\, dx, \qquad u \in H^m(\Omega).$$

In the context of the Floquet-Bloch theory in Section 4 we will see that there appears an additional parameter. In the case $m = 1$ we consider

$$\|u\|^2_{H^1(\Omega),k} := \int_\Omega \left( |(D_1 + k)u(x)|^2 + \sum_{j=2}^{d} |D_j u(x)|^2 + |u(x)|^2 \right) dx, \qquad u \in H^1(\Omega), \tag{1.2.7}$$

where $k \in [-\frac{1}{2}, \frac{1}{2}]$. One easily establishes the following

**Lemma 1.2.1.** *Let $k \in [\frac{1}{2}, \frac{1}{2}]$, then*

$$\frac{1}{2}\|u\|^2_{H^1(\Omega)} \le \|u\|^2_{H^1(\Omega),k} \le 2\|u\|^2_{H^1(\Omega)}, \qquad u \in H^1(\Omega).$$

Let us pass to the definition of periodic Sobolev spaces. We begin with the case of the torus. For $s \ge 0$ we denote by $H^s(\mathbb{T})$ the closure of $C^\infty(\mathbb{T})$ in $H^s(-\pi,\pi)$. One has $H^s(\mathbb{T}) = H^s(-\pi,\pi)$ for $0 \le s < \frac{1}{2}$ and $H^s(\mathbb{T}) = \{f \in H^s(-\pi,\pi) : \frac{d^l f}{dx_1^l}(\pi) = \frac{d^l f}{dx_1^l}(-\pi),\ l = 0, ..., [s - \frac{1}{2}]\}$ for $s > \frac{1}{2}$, $s \ne m + \frac{1}{2}$, $m \in \mathbb{N}_0$. Here $[r]$ denotes the integer part of $r$ and the boundary values of $f$ exist by the Sobolev embedding theorem. Finally, for $s = m + \frac{1}{2}$, $m \in \mathbb{N}_0$, one has

$$H^{m+\frac{1}{2}}(\mathbb{T}) = \{f \in H^m(\mathbb{T}) :\ I[\tfrac{d^m f}{dx_1^m}] < \infty\},$$

where $I[g] := \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|g(x_1) - g(y_1)|^2}{|\sin(\frac{x_1 - y_1}{2})|^2}\, dx_1 dy_1$.
Another important characterization of $H^s(\mathbb{T})$ is possible in terms of Fourier coefficients,

$$H^s(\mathbb{T}) = \Big\{ f \in L_2(\mathbb{T}) :\ \textstyle\sum_{n\in\mathbb{Z}} (1 + n^2)^s |\hat{f}_n|^2 < \infty \Big\}.$$

Throughout the following we will use in $H^s(\mathbb{T})$ the scalar product

$$(f, g)_{H^s(\mathbb{T})} = \sum_{n\in\mathbb{Z}} (1 + n^2)^s \hat{f}_n \overline{\hat{g}_n}, \qquad f,\, g \in H^s(\mathbb{T}),$$

and the corresponding norm

$$\|f\|^2_{H^s(\mathbb{T})} = \sum_{n\in\mathbb{Z}} (1 + n^2)^s |\hat{f}_n|^2, \qquad f \in H^s(\mathbb{T}).$$

We proceed to the case of the halfstrip. For $m \in \mathbb{N}$ we denote by $\tilde{H}^m(\Pi)$ the closure of $\tilde{C}^\infty(\Pi) \cap H^m(\Pi)$ in $H^m(\Pi)$. A function $u \in H^m(\Pi)$ belongs to $\tilde{H}^m(\Pi)$ iff it is periodic with respect to $x_1$ in the sense of the Sobolev trace theorem, i.e.

$$\frac{\partial^l u}{\partial x_1^l}(\pi, .) = \frac{\partial^l u}{\partial x_1^l}(-\pi, .)$$

for all $0 \le l < m$. It is convenient to characterize $\tilde{H}^m(\Pi)$ in terms of the decomposition (1.2.6).

**Lemma 1.2.2.** *Let $m \in \mathbb{N}$ and $u \in L_2(\Pi)$. Then $u \in \tilde{H}^m(\Pi)$ iff*

$$\begin{aligned} &\hat{u}_n \in H^m(\mathbb{R}_+), \qquad n \in \mathbb{Z}, \\ &\sum_{n\in\mathbb{Z}} n^{2\alpha_1} \|D^{\alpha_2}\hat{u}_n\|^2 < \infty, \qquad \alpha \in \mathbb{N}_0^2,\ |\alpha| \le m. \end{aligned} \tag{1.2.8}$$

*In this case, the series $u = \sum_{n\in\mathbb{Z}} e_n \otimes \hat{u}_n$ converges in $H^m(\Pi)$ and*

$$D^\alpha u = \sum_{n\in\mathbb{Z}} e_n \otimes n^{\alpha_1} D^{\alpha_2}\hat{u}_n, \qquad \alpha \in \mathbb{N}_0^2,\ |\alpha| \le m.$$

The proof is straightforward, so we can omit it. We remark that it is sufficient that the second condition in (1.2.8) holds only for the multi-indices $\alpha = (m, 0)$ and $\alpha = (0, m)$.

**3. Multiplication on the boundary.** In this subsection we assume that $\sigma : \mathbb{R} \to \mathbb{R}$ is a real-valued, ($2\pi$-)periodic function such that

$$\sigma \in L_q(\mathbb{T})$$

for some $1 < q \le \infty$. We denote by $M^{(\sigma)}$ the self-adjoint operator of multiplication by $\sigma$ in $L_2(\mathbb{T})$,

$$\begin{aligned} \mathcal{D}(M^{(\sigma)}) &:= \{f \in L_2(\mathbb{T}) :\ \sigma f \in L_2(\mathbb{T})\}, \\ (M^{(\sigma)} f)(x_1) &:= \sigma(x_1) f(x_1), \qquad x_1 \in \mathbb{T}. \end{aligned}$$

Moreover, we use the self-adjoint operator

$$\begin{aligned} \mathcal{D}(B) &:= H^1(\mathbb{T}), \\ Bf &:= \sum_{n\in\mathbb{Z}} \sqrt{1+n^2}\, \hat{f}_n e_n. \end{aligned}$$

We remark that $B^2 - I$ is the operator $-\frac{d^2}{dx_1^2}$ in $L_2(-\pi, \pi)$ with periodic boundary conditions, but we shall not use this fact. Our notation $B$ is motivated by its use in Chapter 3.
By Hölder's inequality and the Sobolev embedding $H^{1/2}(\mathbb{T}) \subset L_{2q'}(\mathbb{T})$ we have for $f \in H^{1/2}(\mathbb{T})$

$$\int_{-\pi}^{\pi} |\sigma(x_1)||f(x_1)|^2\, dx_1 \le \|\sigma\|_{L_q(\mathbb{T})} \|f\|^2_{L_{2q'}(\mathbb{T})} \le c\|\sigma\|_{L_q(\mathbb{T})} \|f\|^2_{H^{1/2}(\mathbb{T})}. \tag{1.2.9}$$

Since $B^{-1/2}$ maps $L_2(\mathbb{T})$ unitarily onto $H^{1/2}(\mathbb{T})$, it follows that $|M^{(\sigma)}|^{1/2} B^{-1/2}$ is a bounded operator on $L_2(\mathbb{T})$.
Indeed, this result can be considerably strenghened.

**Lemma 1.2.3.** *Under the above hypotheses,* $|M^{(\sigma)}|^{1/2}B^{-1/2} \in \Sigma_2(L_2(\mathbb{T}))$.

Recall (cf. Section 11.6 in [BS]) that $\Sigma_p(\mathfrak{H})$, $0 < p < \infty$, is the class of compact operators K on a Hilbert space $\mathfrak{H}$ for which $\sup_m(m^{1/p}\sqrt{\lambda_m(K^*K)}) < \infty$, where $\lambda_m(K^*K)$ denotes the $m$-th eigenvalue of $K^*K$ in decreasing order counting multiplicities.
It is easy to see that

$$\Sigma_r(\mathfrak{H}) \subset \mathfrak{S}_p(\mathfrak{H}) \subset \Sigma_p(\mathfrak{H}), \qquad r < p,$$

where $\mathfrak{S}_p(\mathfrak{H})$ is the usual Schatten class of order $p$ on $\mathfrak{H}$. The dependence on $\mathfrak{H}$ is usually dropped in the notation when the underlying Hilbert space is clear.
We will often use the fact that $K_1 \in \Sigma_{p_1}$, $K_2 \in \Sigma_{p_2}$ implies

$$K_1K_2 \in \Sigma_{(\frac{1}{p_1}+\frac{1}{p_2})^{-1}}. \tag{1.2.10}$$

*Proof.* Write $|M^{(\sigma)}|^{1/2}B^{-1/2} = \left(|M^{(\sigma)}|^{1/2}B^{-1/2q}\right)B^{-1/2q'}$ (without loss, $q < \infty$). Under Fourier transformation on $L_2(\mathbb{T})$ the last factor corresponds to multiplication by the sequence $\left((1+n^2)^{-1/4q'}\right)_{n\in\mathbb{Z}}$ and is therefore in the class $\Sigma_{2q'}$.
The product of the first two operators corresponds to multiplication by the sequence $\left((1+n^2)^{-1/4q}\right)_{n\in\mathbb{Z}} \in l_{2q,w}(\mathbb{Z})$ in Fourier space and by the function $\sqrt{|\sigma|} \in L_{2q}(\mathbb{T})$ in configuration space. Here $l_{p,w}(\mathbb{Z}) = \{(\alpha_n)_{n\in\mathbb{Z}} : \sup_m(m^{1/p}\alpha_m^*) < \infty\}$ where $(\alpha_m^*)_{m\in\mathbb{N}}$ denotes the non-increasing rearrangement of $(|\alpha_n|)_{n\in\mathbb{Z}}$. In [BKaS] a Cwikel-type estimate was obtained for operators of this form. This implies (cf. Theorem 4.8 in [BKaS]) $|M^{(\sigma)}|^{1/2}B^{-1/2q} \in \Sigma_{2q}$.
By use of (1.2.10) we can conclude $|M^{(\sigma)}|^{1/2}B^{-1/2} \in \Sigma_2$, as claimed. □

It follows in particular, that $\int_{-\pi}^{\pi}|\sigma(x_1)||f(x_1)|^2\,dx_1$, $f \in H^{1/2}(\mathbb{T})$, defines a compact form on $H^{1/2}(\mathbb{T})$. This implies

**Corollary 1.2.4.** *Let* $\epsilon > 0$, *then there exists a constant* $C_1(\epsilon,\sigma) > 0$ *such that*

$$\int_{-\pi}^{\pi}|\sigma(x_1)||f(x_1)|^2\,dx_1 \le \epsilon\|f\|^2_{H^{1/2}(\mathbb{T})} + C_1(\epsilon,\sigma)\|f\|^2, \qquad f \in H^{1/2}(\mathbb{T}).$$

Now let us pass to the situation on the halfstrip and on the halfplane. The trace operator $u \mapsto u(.,0)$ is bounded from $H^1(\Pi)$ to $H^{1/2}(\mathbb{T})$. Hence the form $\int_{-\pi}^{\pi}|\sigma(x_1)||u(x_1,0)|^2\,dx_1$, $u \in H^1(\Pi)$, is compact on $H^1(\Pi)$ and we obtain

**Corollary 1.2.5.** *Let* $\epsilon > 0$, *then there exists a constant* $C_2(\epsilon,\sigma) > 0$ *such that*

$$\int_{-\pi}^{\pi}|\sigma(x_1)||u(x_1,0)|^2\,dx_1 \le \epsilon\|u\|^2_{H^1(\Pi)} + C_2(\epsilon,\sigma)\|u\|^2, \qquad u \in H^1(\Pi).$$

Now let $u \in H^1(\mathbb{R}^2_+)$. For each $n \in \mathbb{Z}$ we apply Corollary 1.2.5 to the function $\Pi \ni x \mapsto u(x_1+2\pi n, x_2)$ and then sum over all $n \in \mathbb{Z}$. This yields

**Corollary 1.2.6.** *Let* $\epsilon > 0$, *then there exists a constant* $C_2(\epsilon,\sigma) > 0$ *such that*

$$\int_{\mathbb{R}}|\sigma(x_1)||u(x_1,0)|^2\,dx_1 \le \epsilon\|u\|^2_{H^1(\mathbb{R}^2_+)} + C_2(\epsilon,\sigma)\|u\|^2, \qquad u \in H^1(\mathbb{R}^2_+).$$

## 1.3 The Gelfand transformation

In this section we introduce the Gelfand transformation on the halfplane and discuss its basic properties. This unitary operator is the main tool in the Floquet-Bloch theory of periodic operators.
Because of the geometric structure of $\mathbb{R}^2_+$ we find it convenient to introduce first the Gelfand transformation on $\mathbb{R}$ in Subsection 1. Our exposition relies gratefully on [Su]. See also Section XIII.16 in [RS4] for a related construction. In Subsection 2 we will extend the operator trivially to $\mathbb{R}^2_+$. We prove the material needed for the direct integral decomposition in Section 2.2.

**1. The Gelfand transformation on $\mathbb{R}$.** For a function $f \in \mathcal{S}(\mathbb{R})$ we define

$$(\mathcal{U}_1 f)(k, x_1) := \sum_{n\in\mathbb{Z}} e^{-ik(x_1+2\pi n)} f(x_1 + 2\pi n), \qquad k \in [-\tfrac{1}{2}, \tfrac{1}{2}],\ x_1 \in [-\pi, \pi].$$

The inclusion $f \in \mathcal{S}(\mathbb{R})$ implies that the convergence is uniform, so we easily see that $\mathcal{U}_1 f \in C^\infty((-\frac{1}{2}, \frac{1}{2}) \times (-\pi, \pi))$ and $(\mathcal{U}_1 f)(k, .) \in C^\infty(\mathbb{T})$ for fixed $k \in [-\frac{1}{2}, \frac{1}{2}]$. Moreover, if we define

$$(EF)(k, x_1) := e^{ikx_1} F(k, x_1), \qquad k \in [-\tfrac{1}{2}, \tfrac{1}{2}],\ x_1 \in [-\pi, \pi],$$

for an arbitrary function $F$ on $[-\frac{1}{2}, \frac{1}{2}] \times [-\pi, \pi]$, we see that $(E\mathcal{U}_1 f)(., x_1) \in C^\infty(\mathbb{T}')$, which is by definition the space of functions in $C^\infty(-\frac{1}{2}, \frac{1}{2})$ which can be extended 1-periodically to a function in $C^\infty(\mathbb{R})$. This proves one inclusion of the following

**Lemma 1.3.1.**

$$\begin{aligned}\mathcal{U}_1(\mathcal{S}(\mathbb{R})) = \{F \in C^\infty((-\tfrac{1}{2}, \tfrac{1}{2}) \times (-\pi, \pi)) : F(k, .) \in C^\infty(\mathbb{T}),\ \ k \in [-\tfrac{1}{2}, \tfrac{1}{2}],\\ (EF)(., x_1) \in C^\infty(\mathbb{T}'),\ \ x_1 \in [-\pi, \pi]\}\end{aligned}$$

*Proof.* We only have to prove "$\supset$". Let $F$ be given with the above properties, then we have to find $f \in \mathcal{S}(\mathbb{R})$ such that

$$\sum_{n\in\mathbb{Z}} e^{-ik(x_1+2\pi n)} f(x_1 + 2\pi n) = F(k, x_1), \qquad k \in [-\tfrac{1}{2}, \tfrac{1}{2}],\ x_1 \in [-\pi, \pi].$$

Multiplying both sides by $e^{ikx_1}$ we see that $f(x_1 + 2\pi n)$ must be the $(-n)$-th Fourier coefficient of $(EF)(., x_1)$, whence

$$f(x_1 + 2\pi n) = \int_{-1/2}^{1/2} F(k, x_1) e^{ik(x_1+2\pi n)}\, dk, \qquad n \in \mathbb{Z}, \tag{1.3.1}$$

for every fixed $x_1 \in [-\pi, \pi]$. This equation uniquely defines a function $f$ on $\mathbb{R}$ and it suffices to prove that it belongs to the Schwartz class. This follows easily from the properties of $F$. $\square$

We have seen in the preceeding proof that for $f \in \mathcal{S}(\mathbb{R})$ and $x_1 \in [-\pi, \pi]$ (1.3.1) are the Fourier coefficients of $(EF)(., x_1) = (E\mathcal{U}_1 f)(., x_1)$, so by Parseval's equality

$$\int_{-1/2}^{1/2} |(\mathcal{U}_1 f)(k, x_1)|^2 \, dk = \sum_{n \in \mathbb{Z}} |f(x_1 + 2\pi n)|^2, \qquad x_1 \in [-\pi, \pi]. \tag{1.3.2}$$

Integration with respect to $x_1 \in [-\pi, \pi]$ yields

$$\int_{-1/2}^{1/2} \int_{-\pi}^{\pi} |(\mathcal{U}_1 f)(k, x_1)|^2 \, dx_1 \, dk = \sum_{n \in \mathbb{Z}} \int_{-\pi+2n\pi}^{\pi+2n\pi} |f(x_1)|^2 \, dx_1 = \int_{\mathbb{R}} |f(x_1)|^2 \, dx_1. \tag{1.3.3}$$

This allows us to extend $\mathcal{U}_1$ by continuity to an isometric operator

$$\mathcal{U}_1 : L_2(\mathbb{R}) \to \int_{-1/2}^{1/2} \bigoplus L_2(\mathbb{T}) \, dk,$$

the *Gelfand transformation on* $\mathbb{R}$.
Indeed, $\mathcal{U}_1$ is unitary: Being isometric, $\mathcal{R}(\mathcal{U}_1)$ is a subspace which, according to Lemma 1.3.1, contains a dense subset. Thus $\mathcal{R}(\mathcal{U}_1) = \int_{-1/2}^{1/2} \bigoplus L_2(-\pi, \pi) \, dk$.
Let us now investigate the Gelfand transformation on the Sobolev space $H^1(\mathbb{R})$. For $f \in \mathcal{S}(\mathbb{R})$ we have

$$(\mathcal{U}_1 D_1 f)(k, .) = (D_1 + k)(\mathcal{U}_1 f)(k, .), \tag{1.3.4}$$

and hence using (1.3.3)

$$\int_{-1/2}^{1/2} \|(\mathcal{U}_1 f)(k, .)\|^2_{H^1(-\pi,\pi),k} \, dk = \|f\|^2_{H^1(\mathbb{R})}.$$

Recall that the norms $\|.\|_{H^1(-\pi,\pi),k}$ were defined in (1.2.7) and are mutually equivalent, uniformly in $k \in [-\frac{1}{2}, \frac{1}{2}]$. Hence $\int_{-1/2}^{1/2} \|F(k, .)\|^2_{H^1(-\pi,\pi),k} \, dk$ defines an equivalent, non-standard norm in the space $\int_{-1/2}^{1/2} \bigoplus H^1(-\pi, \pi) \, dk$.
It follow from Lemma 1.3.1 that $(\mathcal{U}_1 f)(k, .) \in H^1(\mathbb{T})$, whence $\mathcal{U}_1 f$ belongs to the subspace

$$\int_{-1/2}^{1/2} \bigoplus H^1(\mathbb{T}) \, dk, \tag{1.3.5}$$

By continuity, $\mathcal{U}_1$ maps the Hilbert space $H^1(\mathbb{R})$ isometrically into the space (1.3.5), endowed with the non-standard norm. Since $\mathcal{U}_1(\mathcal{S}(\mathbb{R}))$ is dense in this space with respect to the non-standard norm (see Lemma 1.3.1), we obtain

$$\mathcal{U}_1(H^1(\mathbb{R})) = \int_{-1/2}^{1/2} \bigoplus H^1(\mathbb{T}) \, dk$$

and (1.3.4) holds for any $f \in H^1(\mathbb{R})$.

**2. The Gelfand transformation on $\mathbb{R}^2_+$.** We define the *Gelfand transformation on the halfplane* as operator from $L_2(\mathbb{R}^2_+) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ to $\int_{-1/2}^{1/2} \bigoplus L_2(\Pi)\, dk = \int_{-1/2}^{1/2} \bigoplus L_2(\mathbb{T})\, dk \otimes L_2(\mathbb{R}_+)$ by

$$\mathcal{U} := \mathcal{U}_1 \otimes I.$$

For $u \in \mathcal{S}(\mathbb{R}^2_+)$ we can rewrite this definition as

$$(\mathcal{U}u)(k, x) = \sum_{n\in\mathbb{Z}} e^{-ik(x_1+2\pi n)} u(x_1 + 2\pi n, x_2), \qquad k \in [-\tfrac{1}{2}, \tfrac{1}{2}],\ x \in \Pi.$$

It is clear that $\mathcal{U}$ is a unitary operator from $L_2(\mathbb{R}^2_+)$ to $\int_{-1/2}^{1/2} \bigoplus L_2(\Pi)\, dk$.
Let us investigate the Gelfand transformation on the Sobolev space $H^1(\mathbb{R}^2_+)$.

**Proposition 1.3.2.** *One has* $\mathcal{U}(H^1(\mathbb{R}^2_+)) = \int_{-1/2}^{1/2} \bigoplus \tilde{H}^1(\Pi)\, dk$ *and for* $u \in H^1(\mathbb{R}^2_+)$

$$(\mathcal{U}D_1 u)(k, .) = (D_1 + k)(\mathcal{U}u)(k, .) \qquad (\mathcal{U}D_2 u)(k, .) = D_2(\mathcal{U}u)(k, .). \tag{1.3.6}$$

*Proof.* For functions $u \in H^1(\mathbb{R}^2_+)$ of the form $u = f \otimes g$ with $f \in H^1(\mathbb{R})$, $g \in H^1(\mathbb{R}_+)$ we have $(\mathcal{U}u)(k, .) \in \tilde{H}^1(\Pi)$ and (1.3.6) holds. Moreover, for such $u$ it follows from (1.3.3) that

$$\int_{-1/2}^{1/2} \|(\mathcal{U}u)(k, .)\|^2_{H^1(\Pi),k}\, dk = \|u\|^2_{H^1(\mathbb{R}^2_+)}.$$

Recall that the norms $\|.\|_{H^1(\Pi),k}$ were defined in (1.2.7) and are mutually equivalent, uniformly in $k \in [-\frac{1}{2}, \frac{1}{2}]$. Hence $\int_{-1/2}^{1/2} \|F(k, .)\|^2_{H^1(\Pi),k}\, dk$ defines an equivalent, non-standard norm in the space $\int_{-1/2}^{1/2} \bigoplus H^1(\Pi)\, dk$.
Since every function $u \in H^1(\mathbb{R}^2_+)$ can be approximated in the $H^1(\mathbb{R}^2_+)$-norm by linear combinations of function of the above form, it follows by continuity that $\mathcal{U}$ maps the Hilbert space $H^1(\mathbb{R}^2_+)$ isometrically into the space

$$\int_{-1/2}^{1/2} \bigoplus \tilde{H}^1(\Pi)\, dk,$$

endowed with the non-standard norm, and that (1.3.6) holds for every $u \in H^1(\mathbb{R}^2_+)$. Finally, we conclude with Lemma 1.3.1 that $\mathcal{R}(\mathcal{U})$ contains a set which is dense in $\int_{-1/2}^{1/2} \bigoplus \tilde{H}^1(\Pi)\, dk$ with respect to the non-standard norm. By isometricity the assertion about $\mathcal{R}(\mathcal{U})$ follows, and the Proposition is proved. $\square$

The next assertion shows how the Gelfand transformation behaves under multiplication by a periodic function on the boundary.

**Lemma 1.3.3.** *Let $\sigma \in L_q(\mathbb{T})$ for some $1 < q \leq \infty$. Then for $u \in H^1(\mathbb{R}^2_+)$*

$$\int_{-1/2}^{1/2} \int_{-\pi}^{\pi} \sigma(x_1) |(\mathcal{U}u)(k, x_1, 0)|^2 \, dx_1 \, dk = \int_{\mathbb{R}} \sigma(x_1) |u(x_1, 0)|^2 \, dx_1.$$

*Proof.* By Proposition 1.3.2 and the results from Section 2 both sides are well-defined and bounded quadratic forms on $H^1(\mathbb{R}^2_+)$. Hence we may assume that $u$ is of the form $u = f \otimes g$ with $f \in \mathcal{S}(\mathbb{R})$, $g \in H^1(\mathbb{R}_+)$.
We multiply both sides in (1.3.2) by $\sigma(x_1)|g(0)|^2$ and integrate with respect to $x_1 \in [-\pi, \pi]$. This proves the Lemma. $\square$

## 1.4 Scattering theory

In this section we summarize the definitions and basic result on scattering theory. For proofs we refer to [Ya1].

**1. Existence and completeness of the wave operators.** Let $H_0$, $H$ be self-adjoint operators in a Hilbert space $\mathfrak{H}$. The projection onto the absolutely continuous part of the spectrum and the unitary group of $H_0$ are denoted by $P_0$ and $U_0(t) = \exp(-itH_0)$, respectively. Moreover, $\mathfrak{H}_0^{(ac)} := \mathcal{R}(P_0)$. For the similar objects related to the operator $H$ we omit the index "0".
In case of existence, the limit

$$W_\pm(H, H_0) := s - \lim_{t \to \pm\infty} U(-t) U_0(t) P_0$$

is called the *wave operator* for the pair $H$, $H_0$ and the sign $\pm$.
Thus the elements $u = W_\pm(H, H_0) u_0^\pm \in \mathcal{R}(W_\pm(H, H_0))$, $u_0^\pm \in \mathfrak{H}$, satisfy

$$\lim_{t \to \pm\infty} \|U(t)u - U_0(t)u_0^\pm\| = 0. \tag{1.4.1}$$

The wave operators are partial isometries with initial space $\mathfrak{H}_0^{(ac)}$. One easily establishes the *intertwining property*

$$W_\pm(H, H_0) \, H_0 = H \, W_\pm(H, H_0).$$

It follows that the subspace $\mathcal{R}(W_\pm(H, H_0))$ reduces $H$ and that the wave operator provides a unitary equivalence between the part of $H$ on this subspace and the absolutely continuous part of $H_0$. In particular,

$$\mathcal{R}(W_\pm(H, H_0)) \subset \mathfrak{H}^{(ac)}. \tag{1.4.2}$$

The wave operator $W_\pm(H, H_0)$ is said to be *complete* if equality holds in (1.4.2). It is easy to see that the completeness of $W_\pm(H, H_0)$ is equivalent to the existence

of $W_\pm(H_0, H)$. Thus, if the wave operator $W_\pm(H, H_0)$ exists and is complete, then the absolutely continuous parts of $H_0$ and $H$ are unitarily equivalent.
Let us conclude this brief overview with an important criterion for the existence and completeness of the wave operators, the Birman-Kreĭn theorem.

**Theorem 1.4.1.** *Let $H_0$, $H$ be self-adjoint operators on a Hilbert space $\mathfrak{H}$ such that $(H - z)^{-1} - (H_0 - z)^{-1}$ is trace class for some $z \in \rho(H_0) \cap \rho(H)$. Then the wave operators $W_\pm(H, H_0)$ exist and are complete.*

**2. The scattering operator.** We use the same notation as above and assume that both wave operators $W_\pm(H, H_0)$ exist. In this case the *scattering operator* is defined by

$$S(H, H_0) := W_+(H, H_0)^* \, W_-(H, H_0).$$

The scattering operator connects the asymptotics of $U(t)u$ as $t \to -\infty$ and as $t \to +\infty$ in terms of the free problem: If $u_0^\pm$ satisfy (1.4.1) and belong to $\mathfrak{H}_0^{(ac)}$, then $S(H, H_0)u_0^- = u_0^+$.
The scattering operator is a unitary operator on $\mathfrak{H}_0^{(ac)}$ provided $\mathcal{R}(W_+(H, H_0)) = \mathcal{R}(W_-(H, H_0))$.
By the intertwining property the scattering operator commutes with $H_0$ and hence decomposes into a one-parameter family of operators $S(\lambda; H, H_0)$, $\lambda \in \sigma(H_0)$, in a representation where $H_0$ acts diagonal. The operators $S(\lambda; H, H_0)$ are called *scattering matrix.*

# Chapter 2

# Definition of the operators. Direct integral decomposition

Throughout this chapter, we assume without further comment that $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is a periodic function such that for some $1 < q \leq \infty$

$$\sigma \in L_q(\mathbb{T}).$$

## 2.1 Definiton of the operators $H^{(\sigma)}$ and $H^{(\sigma)}(k)$

Our goal in this section is to define $H^{(\sigma)}$ as self-adjoint operator on the half-plane and to decompose it into a direct integral of self-adjoint operators $H^{(\sigma)}(k)$, $k \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, on the halfstrip.
The definition of $H^{(\sigma)}$ and $H^{(\sigma)}(k)$ in Subsection 1 and 2, respectively, will be given via quadratic forms. This allows us to include a wide class of functions $\sigma$. If $\sigma$ is smooth the functions in the domain of the operator $H^{(\sigma)}$ satisfy a boundary condition of third type.
Finally, in Subsection 3 we use the Bloch-Floquet theory (i.e., the Gelfand transformation from Section 1.3) to decompose $H^{(\sigma)}$ into the direct integral of the operators $H^{(\sigma)}(k)$, indexed by a parameter $k$ ("quasimomentum") from the intervall $\left[-\frac{1}{2}, \frac{1}{2}\right]$ ("Brillouin zone").

**1. Definition of the operators $H^{(\sigma)}$ on the halfplane.** In the Hilbert space $L_2(\mathbb{R}^2_+)$ consider the quadratic form

$$\begin{aligned} \mathcal{D}[h^{(\sigma)}] &:= H^1(\mathbb{R}^2_+), \\ h^{(\sigma)}[u] &:= \int_{\mathbb{R}^2_+} |Du(x)|^2\, dx + \int_{\mathbb{R}} \sigma(x_1)|u(x_1,0)|^2\, dx_1. \end{aligned} \tag{2.1.1}$$

According to Corollary 1.2.6 the integral $\int_{\mathbb{R}} \sigma(x_1)|u(x_1,0)|^2\, dx_1$ is well-defined for $u \in H^1(\mathbb{R}^2_+)$ and strongly subordinate to the closed and lower semibounded form

$h^{(0)}$. Hence the form $h^{(\sigma)}$ is closed and lower semibounded and there is a unique self-adjoint operator $H^{(\sigma)}$ with domain $\mathcal{D}(H^{(\sigma)})$ such that

$$\begin{aligned} &\mathcal{D}(H^{(\sigma)}) \subset H^1(\mathbb{R}^2_+), \\ &(H^{(\sigma)}u, v) = h^{(\sigma)}[u, v], \qquad u \in \mathcal{D}(H^{(\sigma)}),\ v \in H^1(\mathbb{R}^2_+). \end{aligned} \tag{2.1.2}$$

Taking $v \in C_0^\infty(\mathbb{R}^2_+)$ in (2.1.2) we see that the distributional Laplacian $\Delta u$ of $u \in \mathcal{D}(H^{(\sigma)})$ belongs to $L_2(\mathbb{R}^2_+)$ and that

$$H^{(\sigma)}u = -\Delta u.$$

The case $\sigma = 0$ corresponds to the *Neumann Laplacian* on the halfplane, whereas the case $\sigma \neq 0$ implements a (generalized) *boundary condition of the third type.* More precisely, we have

*Remark* 2.1.1. It is well-known that

$$\mathcal{D}(H^{(\sigma)}) = \left\{ u \in H^2(\mathbb{R}^2_+) : -\frac{\partial u}{\partial x_2} + \sigma u = 0 \text{ on } \mathbb{R} \times \{0\} \right\}$$

provided that $\sigma$ is smooth. We are able to prove this under the condition $\sigma \in W^1_q(\mathbb{T})$ for some $1 < q \leq \infty$, where $W^1_q(\mathbb{T})$ denotes the closure of $C^\infty(\mathbb{T})$ in $W^1_q(-\pi, \pi)$.

**2. Definition of the operators $H^{(\sigma)}(k)$ on the halfstrip.** Let $k \in [-\frac{1}{2}, \frac{1}{2}]$. In the Hilbert space $L_2(\Pi)$ consider the quadratic form

$$\begin{aligned} &\mathcal{D}[h^{(\sigma)}(k)] := \tilde{H}^1(\Pi), \\ &h^{(\sigma)}(k)[u] := \int_\Pi \left(|(D_1 + k)u(x)|^2 + |D_2 u(x)|^2\right)\, dx + \int_{-\pi}^{\pi} \sigma(x_1)|u(x_1, 0)|^2\, dx_1. \end{aligned} \tag{2.1.3}$$

According to Corollary 1.2.5 the integral $\int_{-\pi}^{\pi} \sigma(x_1)|u(x_1,0)|^2\, dx_1$ is well-defined for $u \in \tilde{H}^1(\Pi)$ and strongly subordinate to the closed and lower semibounded form $h^{(0)}(0)$. Because of the two-sided estimate in Lemma 1.2.1 between $h^{(0)}(0)$ and $h^{(0)}(k)$ the form $h^{(\sigma)}(k)$ is closed and lower semibounded and there is a unique self-adjoint operator $H^{(\sigma)}(k)$ with domain $\mathcal{D}(H^{(\sigma)}(k))$ such that

$$\begin{aligned} &\mathcal{D}(H^{(\sigma)}(k)) \subset \tilde{H}^1(\Pi), \\ &(H^{(\sigma)}(k)u, v) = h^{(\sigma)}(k)[u, v], \qquad u \in \mathcal{D}(H^{(\sigma)}(k)),\ v \in \tilde{H}^1(\Pi). \end{aligned} \tag{2.1.4}$$

Taking $v \in C_0^\infty(\Pi)$ in (2.1.4) we see that the distributional Laplacian $\Delta u$ of $u \in \mathcal{D}(H^{(\sigma)}(k))$ belongs to $L_2(\Pi)$ and that

$$H^{(\sigma)}u = (D_1 + k)^2 u + D_2^2 u = -\Delta u + 2kD_1 u + k^2 u.$$

In addition to the Neumann (if $\sigma = 0$) or third type (if $\sigma \neq 0$) boundary condition on $(-\pi, \pi) \times \{0\}$ we have periodic boundary conditions at $\{x_1 = \pm\pi\}$. More precisely, we have

*Remark* 2.1.2. As in the case of the halfplane (see Remark 2.1.1) one has

$$\mathcal{D}(H^{(\sigma)}(k)) = \left\{ u \in \tilde{H}^2(\Pi) : \ -\frac{\partial u}{\partial x_2} + \sigma u = 0 \text{ on } (-\pi, \pi) \times \{0\} \right\},$$

provided that $\sigma$ is smooth, e.g. $\sigma \in W_q^1(\mathbb{T})$ for some $1 < q \leq \infty$.

**3. Direct integral decomposition of the operator $H^{(\sigma)}$.** In the Hilbert space $\int_{-1/2}^{1/2} \bigoplus L_2(\Pi)\, dk$ we consider the self-adjoint operator

$$\tilde{H}^{(\sigma)} := \int_{-1/2}^{1/2} \bigoplus H^{(\sigma)}(k)\, dk,$$

i.e.,

$$\begin{aligned} \mathcal{D}(\tilde{H}^{(\sigma)}) = \Bigg\{ F \in \int_{-1/2}^{1/2} \bigoplus L_2(\Pi)\, dk : \ F(k) \in \mathcal{D}(H^{(\sigma)}(k)) \text{ for a.e. } k, \\ \int_{-1/2}^{1/2} \left\| H^{(\sigma)}(k) F(k) \right\|^2 dk < \infty \Bigg\}, \end{aligned}$$

$$\left( \tilde{H}^{(\sigma)} F \right)(k) = H^{(\sigma)}(k) F(k), \qquad k \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

The operators $H^{(\sigma)}(k)$ are measurable with respect to $k \in [-\frac{1}{2}, \frac{1}{2}]$.
Recall the definition of the Gelfand transformation $\mathcal{U}$ in Section 1.3.

**Proposition 2.1.3.**

$$\mathcal{U}\, H^{(\sigma)}\, \mathcal{U}^* = \int_{-1/2}^{1/2} \bigoplus H^{(\sigma)}(k)\, dk.$$

We prove this Proposition as a statement about quadratic forms. In the Hilbert space $\int_{-1/2}^{1/2} \bigoplus L_2(\Pi)\, dk$ consider the quadratic form

$$\tilde{h}^{(\sigma)} := \int_{-1/2}^{1/2} \bigoplus h^{(\sigma)}(k)\, dk,$$

i.e.,

$$\begin{aligned} \mathcal{D}[\tilde{h}^{(\sigma)}] = \Bigg\{ F \in \int_{-1/2}^{1/2} \bigoplus L_2(\Pi)\, dk : \ F(k) \in \tilde{H}^1(\Pi) \text{ for a.e. } k, \\ \int_{-1/2}^{1/2} \left| h^{(\sigma)}(k)[F(k)] \right| dk < \infty \Bigg\}, \end{aligned}$$

$$\tilde{h}^{(\sigma)}[F] = \int_{-1/2}^{1/2} h^{(\sigma)}(k)[F(k)]\, dk.$$

We remark that $h^{(\sigma)}(k)[u]$ is a measurable (indeed, quadratic) function of $k$ for each $u \in \tilde{H}^1(\Pi)$. Noting that the constants in the two sided estimate of Lemma 1.2.1 do not depend on $k$ we see that $\tilde{h}^{(\sigma)}$ is a closed and lower semibounded form. Since $\mathcal{D}(\tilde{H}^{(\sigma)}) \subset \mathcal{D}[\tilde{h}^{(\sigma)}]$ and $\left(\tilde{H}^{(\sigma)}F, G\right) = \tilde{h}^{(\sigma)}[F,G]$ for $F \in \mathcal{D}(\tilde{H}^{(\sigma)})$, $G \in \mathcal{D}(\tilde{H}^{(\sigma)})$, the correspondig self-adjoint operator coincides with $\tilde{H}^{(\sigma)}$.
The proof of Proposition 2.1.3 is now easy.

*Proof.* By the above it suffices to prove

$$\begin{aligned} \mathcal{U}\left(\mathcal{D}[h^{(\sigma)}]\right) &= \mathcal{D}[\tilde{h}^{(\sigma)}], \\ h^{(\sigma)}[u] &= \tilde{h}^{(\sigma)}[\mathcal{U}u], \qquad u \in H^1(\mathbb{R}^2_+). \end{aligned}$$

This follows immediately from Proposition 1.3.2 and Lemma 1.3.3. $\square$

Proposition 2.1.3 allows us to investigate the operator $H^{(\sigma)}$ by studying the fibers $H^{(\sigma)}(k)$. From the general theory of direct integral decompositions (cf. [RS, Thm. XIII.85]) we know

**Corollary 2.1.4.** *Let $\lambda \in \mathbb{R}$, then $\lambda \in \sigma\left(H^{(\sigma)}\right)$ iff for every $\epsilon > 0$*

$$\left|\left\{k \in \left[\tfrac{1}{2}, \tfrac{1}{2}\right] :\ \sigma\left(H^{(\sigma)}(k)\right) \cap (\lambda - \epsilon, \lambda + \epsilon) \neq 0\right\}\right| > 0,$$

*and $\lambda \in \sigma_p\left(H^{(\sigma)}\right)$ iff*

$$\left|\left\{k \in \left[\tfrac{1}{2}, \tfrac{1}{2}\right] :\ \lambda \in \sigma_p\left(H^{(\sigma)}(k)\right)\right\}\right| > 0.$$

*Moreover, if $H^{(\sigma)}(k)$ is absolutely continuous for a.e. $k \in [\frac{1}{2}, \frac{1}{2}]$, then $H^{(\sigma)}$ is absolutely continuous.*

**4. Comments and Remarks. Scaling behavior.** In the definition of the operator $H^{(\sigma)}$ in Subsection 1 it was not essential that the period of $\sigma$ is $2\pi$. Indeed, if $\tilde{\sigma} : \mathbb{R} \mapsto \mathbb{R}$ is $a$-periodic for some $a > 0$ and belongs to $L_{q,loc}(\mathbb{R})$ for some $1 < q \leq \infty$, then we can define a self-adjoint operator $H^{(\tilde{\sigma})}$ by the same quadratic form expression (2.1.1) with $\sigma$ replaced by $\tilde{\sigma}$. By a change of coordinates it is easy to see that $H^{(\tilde{\sigma})}$ is unitarily equivalent to $\left(\frac{2\pi}{a}\right)^2 H^{(\sigma)}$, where

$$\sigma(x_1) := \frac{a}{2\pi}\,\tilde{\sigma}\left(\frac{a x_1}{2\pi}\right), \qquad x_1 \in \mathbb{R},$$

is $2\pi$-periodic and belongs to $L_q(\mathbb{T})$. It is therefore no loss to assume that $\sigma$ has period $2\pi$.

## 2.2 The unperturbed operators $H^{(0)}$ and $H^{(0)}(k)$

Our goal in this section is to study the spectrum of the "unperturbed" (i.e., $\sigma = 0$) operators $H^{(0)}$ and $H^{(0)}(k)$ on the halfplane and on the halfstrip, as well as to give a rather explicit formula for the kernel function of the resolvent operator $\left(H^{(0)}(k) - z\right)^{-1}$.

The operators $H^{(0)}$ and $H^{(0)}(k)$ are diagonalized by variants of the Fourier transformation in Subsections 1 and 2, respectively. The formula for the kernel function of the resolvent operator $\left(H^{(0)}(k) - z\right)^{-1}$ ("free Green's function") will be derived by separation of variables from the one-dimensional case.

**1. The unperturbed operator $H^{(0)}$ on the halfplane.** Using the direct integral decomposition of Section 1 it is possible to carry out the spectral analysis of the operator $H^{(0)}$ in terms of the operators $H^{(0)}(k)$, $k \in [-\frac{1}{2}, \frac{1}{2}]$. However, in this subsection we choose a direct approach.

For $u \in \mathcal{S}(\mathbb{R}^2_+)$, the Schwartz class on $\mathbb{R}^2_+$ (see Section 1.1), let

$$(\Phi^{(0)}u)(p) := \frac{1}{\pi} \int_{\mathbb{R}^2_+} u(x) e^{-ip_1x_1} \cos(p_2x_2)\, dx, \qquad p \in \mathbb{R}^2_+.$$

Viewing $L_2(\mathbb{R}^2_+) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$, $\Phi^{(0)}$ is the tensor product of the Fourier transform on $L_2(\mathbb{R})$ and the Fourier cosine transform on $L_2(\mathbb{R}_+)$. Therefore it can be extended by continuity to a unitary operator

$$\Phi^{(0)} : L_2(\mathbb{R}^2_+) \to L_2(\mathbb{R}^2_+).$$

We remark that $\Phi^{(0)}u$ coincides with the restriction of the Fourier transform of $\tilde{u}$ to the upper halfplane. Here $\tilde{u} \in L_2(\mathbb{R}^2)$ denotes the extension of $u$ by reflection, i.e.,

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if} \quad x_2 \geq 0, \\ u(x_1, -x_2) & \text{if} \quad x_2 < 0. \end{cases}$$

For $u \in H^2(\mathbb{R}^2_+)$ one has $\tilde{u} \in H^2(\mathbb{R}^2)$ iff $\frac{\partial u}{\partial x_2} = 0$ on $\mathbb{R} \times \{0\}$. Combining this with well-known facts about the Fourier transformation on $\mathbb{R}^2$ we obtain

$$\Phi^{(0)} \left( \left\{ u \in H^2(\mathbb{R}^2_+) : \ \frac{\partial u}{\partial x_2} = 0 \text{ on } \mathbb{R} \times \{0\} \right\} \right) =$$
$$= \left\{ v \in L_2(\mathbb{R}^2_+) : \int_{\mathbb{R}^2_+} (1 + |p|^2)^2 |v(p)|^2\, dp < \infty \right\},$$
$$(\Phi^{(0)} H^{(0)} u)(p) = |p|^2 (\Phi^{(0)}u)(p), \qquad p \in \mathbb{R}^2_+.$$

This proves

**Proposition 2.2.1.** *The operator $H^{(0)}$ is unitarily equivalent to multiplication by $|p|^2$ on $L_2(\mathbb{R}_+)$. In particular, it is absolutely continuous with spectrum*

$$\sigma\left(H^{(0)}\right) = [0, +\infty)$$

*of infinite multiplicity.*

**2. The unperturbed operator $H^{(0)}(k)$ on the halfstrip.** Let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Recall the decomposition

$$L_2(\Pi) = \sum_{n\in\mathbb{Z}} \bigoplus \operatorname{span}\{e_n\} \otimes L_2(\mathbb{R}_+) \tag{2.2.1}$$

from Section 1.2. Take $u \in L_2(\Pi)$ and decompose it according to (2.2.1) as

$$u = \sum_{n\in\mathbb{Z}} e_n \otimes \hat{u}_n.$$

From Lemma 1.2.2 we know that $u \in \mathcal{D}\left(H^{(0)}(k)\right) = \left\{u \in \tilde{H}^2(\Pi) : \frac{\partial u}{\partial x_2}(.,0) = 0\right\}$ iff

$$\begin{array}{l} \hat{u}_n \in H^2(\mathbb{R}_+), \qquad \frac{d\hat{u}_n}{dx_2}(0) = 0, \\ \sum_{n\in\mathbb{Z}} \left((1+n^2)^2\|\hat{u}_n\|^2 + \|\frac{d^2\hat{u}_n}{dx_2^2}\|^2\right) < \infty, \end{array} \tag{2.2.2}$$

and in this case

$$H^{(0)}(k)u = \sum_{n\in\mathbb{Z}} e_n \otimes \left(-\frac{d^2\,\hat{u}_n}{dx_2^2} + (n+k)^2\,\hat{u}_n\right). \tag{2.2.3}$$

Consider the self-adjoint operator $H_{(0)}$ on $L_2(\mathbb{R}_+)$,

$$\begin{aligned} \mathcal{D}[H_{(0)}] &:= H^2(\mathbb{R}_+), \\ H_{(0)}f &:= -\frac{d^2 f}{dx_2^2}. \end{aligned}$$

It follows from (2.2.2) and (2.2.3) that the subspaces $\operatorname{span}\{e_n\} \otimes L_2(\mathbb{R}_+)$ reduce $H^{(0)}(k)$ and that

$$H^{(0)}(k) = \sum_{n\in\mathbb{Z}} \bigoplus \left(I \otimes (H_{(0)} + (n+k)^2)\right). \tag{2.2.4}$$

The Fourier cosine transformation on $L_2(\mathbb{R}_+^2)$ provides a unitary equivalence between $H_{(0)}$ and multiplication by $\xi^2$ on $\mathbb{R}_+$. Thus, the spectrum of $H_{(0)}$ is simple and absolutely continuous and coincides with $[0, +\infty)$. From (2.2.4) we obtain

**Proposition 2.2.2.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$, then the operator $H^{(0)}(k)$ is unitarily equivalent to multiplication by $\xi^2 + (n+k)^2$ in $\sum_{n\in\mathbb{Z}} \bigoplus L_2(\mathbb{R}_+)$. In particular, it is absolutely continuous with spectrum*

$$\sigma\left(H^{(0)}(k)\right) = [k^2, +\infty),$$

*and the spectral multiplicity of $\lambda \in [k^2, +\infty)$ is*

$$N(\lambda, k) := \sharp\{n \in \mathbb{Z} : \ (n+k)^2 \le \lambda\}.$$

We note that for $u \in \mathcal{S}(\Pi)$

$$(\mathcal{F}^{(0)}u)_n(\xi) := \frac{1}{\pi} \int_\Pi u(x) e^{-inx_1} \cos(\xi x_2)\, dx, \qquad \xi \in \mathbb{R}_+, \tag{2.2.5}$$

is the Fourier cosine transform of $\hat{u}_n$. By continuity we obtain the unitary operator

$$\mathcal{F}^{(0)} : L_2(\Pi) \to \sum_{n\in\mathbb{Z}} \bigoplus L_2(\mathbb{R}_+)$$

which provides the unitary equivalence between $H^{(0)}(k)$ and multiplication $\xi^2 + (n+k)^2$ in $\sum_{n\in\mathbb{Z}} \bigoplus L_2(\mathbb{R}_+)$.
Finally, we shall realize $L_2(\Pi)$ as direct integral $\mathfrak{h}(k)$ where $H^{(0)}(k)$ acts as multiplication by the independent variable. This will be needed in Section 3.4 for the construction of the scattering matrix. By Proposition 2.2.2 we take

$$\mathfrak{h}(k) := \int_{k^2}^{\infty} \bigoplus \mathbb{C}^{N(\lambda,k)}\, d\lambda.$$

We think of $\mathbb{C}^{N(\lambda,k)}$ as space of $N(\lambda, k)$-dimensional vectors $a$ whose entries $a_n$ have indices $n \in \mathbb{Z}$ such that $(n+k)^2 \le \lambda$. With this agreement we define an operator $\Gamma^{(0)}(k) : L_2(\Pi) \to \mathfrak{h}(k)$,

$$(\Gamma^{(0)}(k)u)_n(\lambda) := \frac{1}{\sqrt{2}\sqrt[4]{\lambda - (n+k)^2}} (\mathcal{F}^{(0)}u)_n(\sqrt{\lambda - (n+k)^2}), \qquad (n+k)^2 < \lambda.$$

(This definition is not complete if $\lambda = (n+k)^2$, $n \in \mathbb{Z}$. However, it suffices of course to define elements $\Gamma^{(0)}(k)u \in \mathfrak{h}(k)$ only almost everywhere with respect to $\lambda$.)
Since $\Gamma^{(0)}(k)$ differs from $\mathcal{F}^{(0)}$ only by the change of variables $\xi^2 + (n+k)^2 = \lambda$, we find that $\Gamma^{(0)}(k)$ is unitary and $\Gamma^{(0)}(k)\, H^{(0)}(k)\, \Gamma^{(0)}(k)^*$ is multiplication by the independent variable $\lambda$ in $\mathfrak{h}(k)$.
It follows from (2.2.5) that for $u \in \mathcal{S}(\Pi)$

$$(\Gamma^{(0)}(k)u)_n(\lambda) = \int_\Pi u(x) \overline{\psi_n^{(0)}(x; \lambda, k)}\, dx, \qquad (n+k)^2 < \lambda,$$

where

$$\psi_n^{(0)}(x; \lambda, k) := \frac{1}{\sqrt{2}\pi} \frac{e^{inx_1} \cos(\sqrt{\lambda - (n+k)^2} x_2)}{\sqrt[4]{\lambda - (n+k)^2}}, \qquad x \in \Pi.$$

These "generalized eigenfunctions" will be further discussed in Section 3.4.

**3. The free resolvent $R^{(0)}(z,k)$ on the halfstrip.** Let us derive a formula for the free resolvent kernel. We start with the operator $H_{(0)}$ on the halfline, as introduced in the previous subsection. Let $z \in \mathbb{C} \setminus [0,+\infty)$, then it is well-known (and easily verified by calculating the inverse Fourier cosine transform of the function $\xi \mapsto \frac{1}{\xi^2 - z}$) that $(H_{(0)} - z)^{-1}$ is an integral operator with kernel

$$r_{(0)}(x_2, y_2; z) := \frac{1}{2\sqrt{-z}} \left( e^{-\sqrt{-z}\,(x_2+y_2)} + e^{-\sqrt{-z}\,|x_2-y_2|} \right), \qquad x_2,\, y_2 \in \mathbb{R}_+. \quad (2.2.6)$$

Here and in the following we use the canonical branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ satisfying $\mathrm{Re}\sqrt{\cdot} > 0$.
Using the decomposition (2.2.4) we obtain for the resolvent operators

$$R^{(0)}(z,k) := \left(H^{(0)}(k) - z\right)^{-1}$$

on the halfstrip the following

**Proposition 2.2.3.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $z \in \mathbb{C} \setminus [k^2, +\infty)$. Then the resolvent $R^{(0)}(z,k)$ is an integral operator with kernel*

$$r^{(0)}(x,y;z,k) := \\ = \frac{1}{4\pi} \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{(n+k)^2 - z}} e^{in(x_1-y_1)} \left( e^{-\sqrt{(n+k)^2-z}\,(x_2+y_2)} + e^{-\sqrt{(n+k)^2-z}\,|x_2-y_2|} \right), \\ x,\, y \in \Pi,\ x_2 \neq y_2. \quad (2.2.7)$$

Note that the RHS of (2.2.7) converges absolutely and uniformly on compact subsets of $\{(x,y) \in \Pi \times \Pi : \ x_2 \neq y_2\}$.

*Proof.* Let $u \in L_2(\Pi)$, then (2.2.4) implies

$$R^{(0)}(z,k)u = \sum_{n\in\mathbb{Z}} e_n \otimes \left(H_{(0)} - (z - (n+k)^2\right)^{-1} \hat{u}_n.$$

Using (2.2.6) and the definition of $\hat{u}_n$ we obtain

$$\begin{aligned}(R^{(0)}(z,k)u)(x) &= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{inx_1} \int_0^\infty dy_2 \\ &\qquad r_{(0)}(x_2, y_2; z - (n+k)^2) \left( \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u(y) e^{-iny_1}\, dy_1 \right) = \\ &= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_\Pi e^{in(x_1-y_1)} r_{(0)}(x_2, y_2; z - (n+k)^2) u(y)\, dy = \\ &= \int_\Pi r^{(0)}(x,y;z,k) u(y)\, dy.\end{aligned}$$

The interchange of summation and integration in the last equality is justified since the RHS of (2.2.7) converges in $L_2(\Pi)$ for fixed $x$. This proves the Proposition. ☐

# Chapter 3

# Spectral analysis and scattering theory of the operators $H^{(\sigma)}(k)$ on the halfstrip

Throughout this chapter, we assume without further comment that $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is a periodic function such that for some $1 < q \leq \infty$

$$\sigma \in L_q(\mathbb{T}).$$

## 3.1 The absolutely continuous spectrum of the operators $H^{(\sigma)}(k)$

Our goal in this section is to prove that the absolutely continuous spectrum of $H^{(\sigma)}(k)$ coincides with $\sigma_{ac}\left(H^{(0)}(k)\right) = [k^2, +\infty)$. We will derive this by the Birman-Kreĭn theorem from the fact that the difference of resolvents of $H^{(\sigma)}(k)$ and $H^{(0)}(k)$ is of trace class.

Let us briefly sketch the problem we are facing when we consider the resolvent difference. For two self-adjoint operators $H$, $H_0$ in a Hilbert space $\mathfrak{H}$ satisfying

$$\mathcal{D}(H) = \mathcal{D}(H_0), \tag{3.1.1}$$

the following resolvent identity is well-known:

$$\begin{aligned} &(H - z)^{-1} - (H_0 - z)^{-1} = -(H_0 - z)^{-1}(H - H_0)(H - z)^{-1}, \\ &\qquad z \in \rho(H_0) \cap \rho(H). \end{aligned} \tag{3.1.2}$$

However, if the operators $H_0$, $H$ are defined via quadratic forms, the equality (3.1.1) may not hold, so the operator $H - H_0$ in (3.1.1) may not be well-defined. In Subsection 1 we will present a different approach on the resolvent identity which is valid under the weaker assumption

$$\mathcal{D}(|H|^{1/2}) = \mathcal{D}(|H_0|^{1/2}).$$

After having introduced some auxiliary operators in Subsection 2, we will derive a symmetric factorization of the resolvent difference in Subsection 3. The required compactness statement follows then immediately from Lemma 1.2.3.

**1. A general approach on the inversion of a perturbed operator.** This subsection presents the material from Section 1.9 in [Ya1]. We do not include proofs. For a similar construction see also Section 5 in [BSu].
We work in a general setting: Let $\mathfrak{H}$ be a Hilbert space and $H$, $H_0$ be self-adjoint operators satisfying

$$\mathcal{D}(|H|^{1/2}) = \mathcal{D}(|H_0|^{1/2}). \tag{3.1.3}$$

We note that $H$ and $H_0$ are not assumed to be semibounded. Denote their resolvents by

$$\begin{aligned} R_0(z) &:= (H_0 - z)^{-1} \qquad && z \in \rho(H_0), \\ R(z) &:= (H - z)^{-1} && z \in \rho(H). \end{aligned}$$

Suppose that there is an "auxiliary" Hilbert space $\mathfrak{G}$ and operators

$$G_0 : \mathfrak{H} \supset \mathcal{D}(G_0) \to \mathfrak{G}, \qquad G : \mathfrak{H} \supset \mathcal{D}(G) \to \mathfrak{G},$$

such that the following is true.

(H1) The operators $G_0$, $G$ are $|H_0|^{1/2}$-bounded, i.e.,

$$\begin{aligned} &\mathcal{D}(|H_0|^{1/2}) \subset \mathcal{D}(G_0), && \mathcal{D}(|H_0|^{1/2}) \subset \mathcal{D}(G), \\ &G_0(|H_0|^{1/2} + I)^{-1} \in \mathfrak{B}(\mathfrak{H}, \mathfrak{G}), \qquad && G(|H_0|^{1/2} + I)^{-1} \in \mathfrak{B}(\mathfrak{H}, \mathfrak{G}). \end{aligned} \tag{3.1.4}$$

(H2) The operators $G_0$, $G$ satisfy

$$(G_0 f, Gg) = (Gf, G_0 g), \qquad f,\, g \in \mathcal{D}(|H_0|^{1/2}). \tag{3.1.5}$$

(H3) The relation $H = H_0 + G^* G_0$ holds in the sense of forms, i.e.,

$$(Hf, f_0) = (f, H_0 f_0) + (Gf, G_0 f_0), \qquad f_0 \in \mathcal{D}(H_0),\ f \in \mathcal{D}(H). \tag{3.1.6}$$

The assumption (H1) guarantees that the operators $GR_0(z) : \mathfrak{H} \to \mathfrak{G}$ and $G_0(GR_0(\overline{z}))^* : \mathfrak{G} \to \mathfrak{G}$ are well-defined and bounded for $z \in \rho(H_0)$. Indeed, with $L_0 := (|H_0|^{1/2} + I)(|H_0| + I)^{-1/2} \in \mathfrak{B}(\mathfrak{H})$ we see that

$$GR_0(z) = \left(G(|H_0|^{1/2} + I)^{-1}\right) L_0 \left(\left(|H_0| + I\right) R_0(z)\right) (|H_0| + I)^{-1/2}$$

and

$$G_0(GR_0(\overline{z}))^* = \left(G_0(|H_0|^{1/2} + I)^{-1}\right) L_0 \left(\left(|H_0| + I\right) R_0(\overline{z})\right)^* L_0^* \left(G(|H_0|^{1/2} + I)^{-1}\right)^*$$

are products of bounded operators.
With slight abuse of notation we put

$$R_0(z) G^* := (GR_0(\overline{z}))^*, \qquad G_0 R_0(z) G^* := G_0(GR_0(\overline{z}))^*.$$

**Proposition 3.1.1.** *Let $H_0, H$ be self-adjoint operators satisfying* (3.1.3) *and assume that the operators $G_0, G$ satisfy (H1)-(H3). Let $z \in \rho(H_0)$, then $z \in \rho(H)$ iff $I + G_0 R_0(z) G^*$ is boundedly invertible, and in this case*

$$R(z) - R_0(z) = -R_0(z) G^* \left(I + G_0 R_0(z) G^*\right)^{-1} G_0 R_0(z). \tag{3.1.7}$$

For the *proof* we refer to Section 1.9 in [Ya1].

**2. Some auxiliary operators.** For $k \in [-\frac{1}{2}, \frac{1}{2}]$, $z \in \mathbb{C} \setminus [k^2, +\infty)$ we consider in $L_2(\mathbb{T})$ the operator

$$\begin{aligned} \mathcal{D}(B(z,k)) &:= H^1(\mathbb{T}), \\ B(z,k) f &:= \sum_{n \in \mathbb{Z}} \sqrt{(n+k)^2 - z}\, \hat{f}_n\, e_n. \end{aligned} \tag{3.1.8}$$

Recall that we have agreed to choose the canonical branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ satisfying $\operatorname{Re} \sqrt{\cdot} > 0$.
The operator $B(z,k)$ is invertible for $z \in \mathbb{C} \setminus [k^2, +\infty)$ and its square root is well-defined (it may be considered as the square root of an m-accretive operator). If $M^{(\sigma)}$ denotes as in Section 1.2 the operator of multiplication by $\sigma$ in $L_2(\mathbb{T})$, we define

$$\begin{aligned} T_0^{(\sigma)}(z,k) &:= (\operatorname{sgn} M^{(\sigma)}) |M^{(\sigma)}|^{1/2} B(z,k)^{-1/2}, \\ T^{(\sigma)}(z,k) &:= |M^{(\sigma)}|^{1/2} B(z,k)^{-1/2}. \end{aligned}$$

These are compact operators of class $\Sigma_2(L_2(\mathbb{T}))$. Indeed, let $B$ as in Section 1.2, then $B^{1/2} B(z,k)^{-1/2} \in \mathfrak{B}(L_2(\mathbb{T}))$ and hence

$$T^{(\sigma)}(z,k) = \left(|M^{(\sigma)}|^{1/2} B^{-1/2}\right) \left(B^{1/2} B(z,k)^{-1/2}\right) \in \Sigma_2$$

and $T_0^{(\sigma)}(z,k) = (\operatorname{sgn} M^{(\sigma)}) T^{(\sigma)}(z,k) \in \Sigma_2$.
Finally, for $z \in \mathbb{C} \setminus [k^2, +\infty)$ we consider the bounded operator $Y(z,k) : L_2(\Pi) \to L_2(\mathbb{T})$,

$$\begin{aligned} Y(z,k) u &:= \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt[4]{(n+k)^2 - z}} \left( \frac{1}{\sqrt{2\pi}} \int_\Pi u(x) e^{-inx_1} e^{-\sqrt{(n+k)^2 - z}\, x_2}\, dx \right) e_n = \\ &= \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt[4]{(n+k)^2 - z} \sqrt{2 \operatorname{Re} \sqrt{(n+k)^2 - z}}} \left( u, e_n \otimes \psi_{\sqrt{(n+k)^2 - \overline{z}}} \right) e_n, \end{aligned}$$

where we have put for $\operatorname{Re} \zeta > 0$

$$\psi_\zeta(x_2) := \sqrt{2 \operatorname{Re} \zeta}\, e^{-\zeta x_2}, \qquad x_2 \in \mathbb{R}_+.$$

Since the functions $e_n \otimes \psi_{\sqrt{(n+k)^2 - \overline{z}}}$ are normalized in $L_2(\Pi)$ it is easy to see that $Y(z,k)$ is a compact operator of class $\Sigma_1(L_2(\Pi), L_2(\mathbb{T}))$.

**3. The resolvent difference. The absolutely continuous spectrum of the operators $H^{(\sigma)}(k)$.** We are now ready to apply the general results from Subsection 1 to our situation. Denote the resolvents of the operators $H^{(0)}(k)$, $H^{(\sigma)}(k)$ by

$$\begin{aligned} R^{(0)}(z,k) &:= (H^{(0)}(k)-z)^{-1}, \qquad z \in \mathbb{C}\setminus[k^2,+\infty) = \rho\left(H^{(0)}(k)\right), \\ R^{(\sigma)}(z,k) &:= (H^{(\sigma)}(k)-z)^{-1}, \qquad z \in \rho\left(H^{(\sigma)}(k)\right). \end{aligned}$$

The following statement is of crucial importance.

**Proposition 3.1.2.** *Let $k \in [-\frac{1}{2},\frac{1}{2}]$ and $z \in \mathbb{C}\setminus[k^2,+\infty)$. Then $z \in \rho\left(H^{(\sigma)}(k)\right)$ iff the operator $I + T_0^{(\sigma)}(z,k)\,T^{(\sigma)}(\overline{z},k)^*$ is boundedly invertible, and in this case*

$$\begin{aligned} &R^{(\sigma)}(z,k) - R^{(0)}(z,k) = \\ &\quad = -Y(\overline{z},k)^*\,T^{(\sigma)}(\overline{z},k)^*\left(I + T_0^{(\sigma)}(z,k)\,T^{(\sigma)}(\overline{z},k)^*\right)^{-1} T_0^{(\sigma)}(z,k)\,Y(z,k). \end{aligned} \tag{3.1.9}$$

*Proof.* We want to apply the results of Subsection 1 to the case $\mathfrak{H} = L_2(\Pi)$, $\mathfrak{G} = L_2(\mathbb{T})$, $H_0 = H^{(0)}(k)$, $H = H^{(\sigma)}(k)$ and

$$\begin{aligned} \mathcal{D}(G_0) &:= \mathcal{D}(G) := \tilde{H}^1(\Pi), \\ (G_0 u)(x_1) &:= (\operatorname{sgn}\sigma(x_1))\sqrt{|\sigma(x_1)|}\,u(x_1,0), \qquad x_1 \in \mathbb{T}, \\ (Gu)(x_1) &:= \sqrt{|\sigma(x_1)|}\,u(x_1,0), \qquad x_1 \in \mathbb{T}. \end{aligned}$$

According to Corollary 1.2.5, the operators $G_0$, $G$ are well-defined and bounded from $\tilde{H}^1(\Pi)$ to $L_2(\mathbb{T})$. Since $(|H_0|^{1/2}+I)^{-1}$ is a bounded operator from $\mathfrak{H}$ to $\tilde{H}^1(\Pi)$, the assumption (H1) in Subsection 1 holds. The remaining (3.1.3), (H2) and (H3) are obvious, so that the statement of Proposition 3.1.1 holds. Let us determine the products $G_0R_0(z)$, $GR_0(z)$ and $G_0R_0(z)G^*$.
The explicit form (2.2.7) of the free resolvent implies that for $u \in \tilde{H}^1(\Pi)$

$$\begin{aligned} (R_0(z)u)(x_1,0) &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{(n+k)^2-z}} \int_\Pi u(y)\,e^{in(x_1-y_1)}\,e^{-\sqrt{(n+k)^2-z}\,y_2}\,dy = \\ &= \left(B(z,k)^{-1/2}\,Y(z,k)u\right)(x_1), \qquad x_1 \in \mathbb{T}, \end{aligned}$$

and hence

$$G_0R_0(z) = T_0^{(\sigma)}(z,k)\,Y(z,k), \qquad GR_0(z) = T^{(\sigma)}(z,k)\,Y(z,k).$$

Moreover, for $f \in L_2(\mathbb{T})$ we have

$$(Y(\overline{z},k)^*f)(x) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \frac{\hat{f}_n}{\sqrt[4]{(n+k)^2-z}}\,e^{inx_1}\,e^{-\sqrt{(n+k)^2-z}\,x_2}, \qquad x \in \Pi,$$

so that $Y(\overline{z},k)^*f \in \tilde{H}^1(\Pi)$ and $(Y(\overline{z},k)^*f)(.,0) = B(z,k)^{-1/2}f$. It follows that $G_0Y(\overline{z},k)^* = T_0^{(\sigma)}(z,k)$ and

$$G_0R_0(z)G^* = G_0\left(GR_0(\overline{z})\right)^* = T_0^{(\sigma)}(z,k)\,T^{(\sigma)}(\overline{z},k)^*.$$

This concludes the proof of the Proposition. □

As an easy consequence of (3.1.9) we obtain

**Theorem 3.1.3.** *Let $k \in [-\frac{1}{2},\frac{1}{2}]$, then*

$$R^{(\sigma)}(z,k) - R^{(0)}(z,k) \in \Sigma_{1/3}(L_2(\Pi)).$$

*In particular,*

$$\sigma_{ac}\left(H^{(\sigma)}(k)\right) = \sigma_{ess}\left(H^{(\sigma)}(k)\right) = [k^2,+\infty).$$

*Proof.* As we have remarked in Subsection 2, $T_0^{(\sigma)}(z,k)$, $T^{(\sigma)}(z,k) \in \Sigma_2(L_2(\mathbb{T}))$ and $Y(z,k) \in \Sigma_1(L_2(\Pi),L_2(\mathbb{T}))$, which implies (cf. (1.2.10)) $R^{(\sigma)}(z,k)-R^{(0)}(z,k) \in \Sigma_{1/3}(L_2(\Pi))$. Since $\sigma_{ac}\left(H^{(0)}(k)\right) = \sigma_{ess}\left(H^{(0)}(k)\right) = [k^2,+\infty)$, the statements about the essential and the absolutely continuous spectrum of the operators $H^{(\sigma)}(k)$ follow now from the Weyl and the Birman-Kreĭn theorem, respectively. □

**4. Comments and Remarks. Compactness of the resolvent difference.** In Theorem 3.1.3 we proved that

$$R^{(\sigma)}(z,k) - R^{(0)}(z,k) \in \Sigma_p \tag{3.1.10}$$

for $p = \frac{1}{3}$. Our proof used Lemma 1.2.3 for the compactness result on $T^{(\sigma)}(z,k)$ and thus did rely heavily on the explicit knowledge of the free resolvent kernel. A much more general approach, yielding only the slightly weaker index $p = \frac{1}{2}$ in (3.1.10), can be based on the method in [B2], see also [B1].

**Birman-Schwinger analysis.** Let us briefly explain the connection of the approach in Subsection 1 and what is known as Birman-Schwinger analysis. Suppose that the operators

$$G_0R_0(z)G, \qquad z \in \rho(H_0), \tag{3.1.11}$$

are compact. It can be shown (see Theorem 1.10.7 in [Ya1]) that then $\sigma_{ess}(H) = \sigma_{ess}(H_0)$. Hence by Proposition 3.1.1, a number $z \in \rho(H_0)$ belongs to $\sigma_{disc}(H)$ iff the operator (3.1.11) has eigenvalue $-1$.

This technique is frequently used when $H_0$ is a self-adjoint realization of $-\Delta$ and $G_0$, $G$ are multiplcation by real-valued functions $\sqrt{V}$, $\sqrt{|V|}$, respectivly. (Here $\sqrt{V} := (\operatorname{sgn} V)\sqrt{|V|}$.) Then the operator (3.1.11) may be written as

$$\sqrt{V}(-\Delta - z)^{-1}\sqrt{|V|} \tag{3.1.12}$$

and is usually called Birman-Schwinger operator.

Our operator $T_0^{(\sigma)}(z,0)T^{(\sigma)}(\overline{z},0)^*$ is formally obtained by setting $V = \sigma \otimes \delta$ in (3.1.12).

## 3.2 The absence of singular continuous spectrum of the operators $H^{(\sigma)}(k)$

Our goal in this section is to prove the absence of singular continuous spectrum. This will be achieved by controlling the behavior of the resolvent $R^{(\sigma)}(z,k)$ as the spectral parameter $z$ tends to the real axis. Indeed, for $u$ from a dense set in $L_2(\Pi)$ we will prove that $\lim_{\epsilon\to 0+} R^{(\sigma)}(\lambda \pm i\epsilon, k)$ exists in a suitable topology if $\lambda$ avoids an exceptional set. This assertion is usually called limiting absorption principle.
We will prove it first in Subsection 1 for the case $\sigma \equiv 0$ using the explicit form of the resolvent kernel. The role of the exceptional set in this case will be played by the points $z = (n+k)^2$, $n \in \mathbb{Z}$, where the spectral multiplicity of the operator $H^{(0)}(k)$ changes. Then in Subsection 2 we will carry over the result to the general case using the resolvent identity from Section 1. In addition to the points $z = (n+k)^2$, $n \in \mathbb{Z}$, the exceptional set will now also include the set of $\lambda$'s, for which one of the equations

$$f + T_0^{(\sigma)}(\lambda \pm i0, k)\, T^{(\sigma)}(\lambda \mp i0, k)^* f = 0$$

has a non-trivial solution. The Analytic Fredholm Alternative guarantees that this set is discrete in $\mathbb{R} \setminus \{(n+k)^2 \,:\, n \in \mathbb{Z}\}$ and can therefore not support a singular continuous spectrum.

**1. The limiting absorption principle for the unperturbed operator $H^{(0)}(k)$.** We want to extend the free resolvent $R^{(0)}(z,k)$ from $\mathbb{C} \setminus [k^2, +\infty)$ to the cut. Let us consider the explicit formula (2.2.7) for the resolvent kernel. Because of the poles at the points $z = (n+k)^2$, $n \in \mathbb{Z}$, we may not expect to extend it to the whole cut.
There is another phenomenon: Write $z = \lambda \pm i\epsilon$, where $\epsilon > 0$ and $\lambda > (n+k)^2$ for some $n \in \mathbb{Z}$. Then $\sqrt{(n+k)^2 - (\lambda \pm i\epsilon)} \to \mp i\sqrt{\lambda - (n+k)^2}$ as $\epsilon \to 0+$, so the factor $\left(e^{-\sqrt{(n+k)^2-(\lambda\pm i\epsilon)}\,(x_2+y_2)} + e^{-\sqrt{(n+k)^2-(\lambda\pm i\epsilon)}\,|x_2-y_2|}\right)$ in (2.2.7), as function of either $x_2$ or $y_2$, ceases to be square integrable in the limit $\epsilon = 0$. We remedy this problem by multiplying with a decaying function. Namely, for $s > 0$ consider the bounded operator $\Lambda_2^{-s}$ on $L_2(\Pi)$,

$$(\Lambda_2^{-s} u)(x) := \frac{1}{(1+x_2^2)^{s/2}} f(x), \qquad x \in \Pi.$$

The following result is called the limiting absorption principle for the free operator.

**Proposition 3.2.1.** *Let $s > \frac{1}{2}$ and $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then the limits*

$$\lim_{\epsilon \to 0+} \Lambda_2^{-s} R^{(0)}(\lambda \pm i\epsilon, k) \Lambda_2^{-s}, \qquad \lambda \neq (n+k)^2,\ n \in \mathbb{Z},$$

*exist in the operator norm topology and are uniform for $\lambda$ from compact intervals of $\mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$.*

*Proof.* Fix $\lambda \in \mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$, and define

$$r^{(0)}(x, y; \lambda \pm i0, k) := \lim_{\epsilon \to 0+} r^{(0)}(x, y; \lambda \pm i\epsilon, k), \qquad x \neq y \in \Pi,$$

as pointwise limit using formula (2.2.7). We will prove that

$$\int_\Pi \int_\Pi \left| r^{(0)}(x, y; \lambda \pm i\epsilon, k) - r^{(0)}(x, y; \lambda \pm i0, k) \right|^2 \frac{dx}{(1+x_2^2)^s} \frac{dy}{(1+y_2^2)^s} \longrightarrow 0 \quad (\epsilon \to 0+). \tag{3.2.1}$$

This means that $\Lambda_2^{-s} R^{(0)}(\lambda \pm i\epsilon, k) \Lambda_2^{-s}$ converges in (Hilbert-Schmidt) norm to an integral operator with kernel $\frac{1}{(1+x_2^2)^s} r^{(0)}(x, y; \lambda \pm i0, k) \frac{1}{(1+y_2^2)^s}$, so that the first part of the proposition follows.
For the proof of (3.2.1) we restrict ourselves to the "+"-case, the other being similar. For simplicity of notation, we put

$$c_n(\epsilon) := \sqrt{(n+k)^2 - \lambda - i\epsilon}, \qquad c_n := \lim_{\epsilon \to 0+} c_n(\epsilon).$$

Using Parseval's identity we can rewrite the LHS of (3.2.1) as

$$\begin{aligned}
&\int_\Pi \int_\Pi \left| r^{(0)}(x, y; \lambda \pm i\epsilon, k) - r^{(0)}(x, y; \lambda \pm i0, k) \right|^2 \frac{dx}{(1+x_2^2)^s} \frac{dy}{(1+y_2^2)^s} = \\
&= \int_{-\pi}^{\pi} dx_1 \int_{-\pi}^{\pi} dy_1 \int_0^\infty \frac{dx_2}{(1+x_2^2)^s} \int_0^\infty \frac{dy_2}{(1+y_2^2)^s} \Bigg| \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{inx_1} \\
&\qquad \frac{e^{-iny_1}}{\sqrt{2\pi}} \left( \frac{e^{-c_n(\epsilon)(x_2+y_2)} + e^{-c_n(\epsilon)|x_2-y_2|}}{2c_n(\epsilon)} - \frac{e^{-c_n(x_2+y_2)} + e^{-c_n|x_2-y_2|}}{2c_n} \right) \Bigg|^2 \\
&= \sum_{n \in \mathbb{Z}} \int_0^\infty \frac{dx_2}{(1+x_2^2)^s} \int_0^\infty \frac{dy_2}{(1+y_2^2)^s} \\
&\qquad \left| \frac{e^{-c_n(\epsilon)(x_2+y_2)} + e^{-c_n(\epsilon)|x_2-y_2|}}{2c_n(\epsilon)} - \frac{e^{-c_n(x_2+y_2)} + e^{-c_n|x_2-y_2|}}{2c_n} \right|^2 \leq \\
&\leq 2 \sum_{n \in \mathbb{Z}} \int_0^\infty \frac{dx_2}{(1+x_2^2)^s} \int_0^\infty \frac{dy_2}{(1+y_2^2)^s} \\
&\qquad \left( \left| \frac{e^{-c_n(\epsilon)(x_2+y_2)}}{2c_n(\epsilon)} - \frac{e^{-c_n(x_2+y_2)}}{2c_n} \right|^2 + \left| \frac{e^{-c_n(\epsilon)|x_2-y_2|}}{2c_n(\epsilon)} - \frac{e^{-c_n|x_2-y_2|}}{2c_n} \right|^2 \right) = \\
&= 2 \sum_{n \in \mathbb{Z}} \int_0^\infty \int_0^\infty \left( t_{n,\epsilon}(x_2+y_2) + t_{n,\epsilon}(x_2-y_2) \right) \frac{dx_2}{(1+x_2^2)^s} \frac{dy_2}{(1+y_2^2)^s},
\end{aligned}$$

where

$$t_{n,\epsilon}(a) := \left| \frac{e^{-c_n(\epsilon)|a|}}{2c_n(\epsilon)} - \frac{e^{-c_n|a|}}{2c_n} \right|^2, \qquad a \in \mathbb{R}.$$

If $(n+k)^2 < \lambda$ and $\epsilon \in (0,1]$, the functions $t_{n,\epsilon}$ are bounded uniformly in $\epsilon$ and converge pointwise to 0 as $\epsilon \to 0+$. It follows from Lebesgue's theorem that

$$\sum_{(n+k)^2<\lambda} \int_0^\infty \int_0^\infty (t_{n,\epsilon}(x_2+y_2) + t_{n,\epsilon}(x_2-y_2)) \frac{dx_2}{(1+x_2^2)^s} \frac{dy_2}{(1+y_2^2)^s} \longrightarrow 0,$$

$$(\epsilon \to 0+).$$

Suppose now that $(n+k)^2 > \lambda$. To control the convergence of the $t_{n,\epsilon}$ in terms of $n$, we need the following elementary estimates

$$|1 - e^{-\zeta}| \leq |\zeta|, \qquad \operatorname{Re} \zeta \geq 0,$$

$$|1 - \sqrt{1+it}| \leq \frac{1}{2}|t|, \qquad |1 - \frac{1}{\sqrt{1+it}}| \leq \frac{1}{2}|t|, \qquad t \in \mathbb{R}.$$

The last two estimates imply

$$|c_n - c_n(\epsilon)| \leq \frac{\epsilon}{2|c_n|}, \qquad |\frac{1}{c_n} - \frac{1}{c_n(\epsilon)}| \leq \frac{\epsilon}{2|c_n|^3}.$$

Thus

$$\begin{aligned}
t_{n,\epsilon}(a) &= \frac{e^{-2c_n|a|}}{4} \left| \frac{e^{c_n - c_n(\epsilon))|a|}}{c_n(\epsilon)} - \frac{1}{c_n} \right|^2 \leq \\
&\leq \frac{e^{-2c_n|a|}}{2} \left( \frac{1}{|c_n(\epsilon)|^2} \left| e^{c_n - c_n(\epsilon))|a|} - 1 \right|^2 + \left| \frac{1}{c_n(\epsilon)} - \frac{1}{c_n} \right|^2 \right) \leq \\
&\leq \frac{e^{-2c_n|a|}}{2} \left( \frac{\epsilon^2 |a|^2}{4|c_n|^4} + \frac{\epsilon^2}{4|c_n|^6} \right) \leq \\
&\leq \frac{1}{8} \epsilon^2 \frac{\beta+1}{|c_n|^6},
\end{aligned}$$

where

$$\beta := \max_{b \in \mathbb{R}} b^2 e^{-2|b|} < \infty.$$

It follows that

$$\begin{aligned}
&\sum_{(n+k)^2>\lambda} \int_0^\infty \int_0^\infty (t_{n,\epsilon}(x_2+y_2) + t_{n,\epsilon}(x_2-y_2)) \frac{dx_2}{(1+x_2^2)^s} \frac{dy_2}{(1+y_2^2)^s} \leq \\
&\leq 2 \sum_{(n+k)^2>\lambda} \left( \sup_{a \in \mathbb{R}} t_{n,\epsilon}(a) \right) \left( \int_0^\infty \frac{dx_2}{(1+x_2^2)^s} \right)^2 \leq \\
&\leq \frac{\epsilon^2}{4} \left( \int_0^\infty \frac{dx_2}{(1+x_2^2)^s} \right)^2 (\beta+1) \sum_{n \in \mathbb{Z}} \frac{1}{|c_n|^6}.
\end{aligned}$$

The RHS is finite and converges to 0 as $\epsilon \to 0+$, which completes the proof of (3.2.1).
Finally, we remark that the limit in (3.2.1) is uniform in $\lambda$ for $\lambda$ from a compact intervall not containing any of the points $(n+k)^2$, $n \in \mathbb{Z}$. This follows from the fact, that $c_n = c_n(\lambda)$ depends continuously on $\lambda$. $\square$

**2. The limiting absorption principle for the perturbed operator $H^{(\sigma)}(k)$.** Using the resolvent identity (3.1.9) we want to derive a limiting absorption principle for the operator $H^{(\sigma)}(k)$.
We start our considerations with the operators $T_0^{(\sigma)}(z,k)$, $T^{(\sigma)}(z,k)$.

**Lemma 3.2.2.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$, then the operator families*

$$T_0^{(\sigma)}(z,k),\, T^{(\sigma)}(z,k), \qquad z \in \mathbb{C} \setminus [k^2, \infty),$$

*can be extended norm-continuously to the cut from above and from below with the exception of the points $z = (n+k)^2$, $n \in \mathbb{Z}$. Denoting the (upper and lower) boundary values by*

$$T_0^{(\sigma)}(\lambda \pm i0, k), \qquad T^{(\sigma)}(\lambda \pm i0, k), \qquad \lambda \neq (n+k)^2,$$

*the sets*

$$\mathcal{N}_\pm(k) := \{\lambda \in \mathbb{R}\backslash\{(n+k)^2 : n \in \mathbb{Z}\} : \mathcal{N}(I + T_0^{(\sigma)}(\lambda \pm i0, k)\, T^{(\sigma)}(\lambda \mp i0, k)^*) \neq \{0\}\}$$

*are discrete in $\mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$.*

*Remark* 3.2.3. We will prove in Lemma 3.3.17 below that the sets $\mathcal{N}_+(k)$ and $\mathcal{N}_-(k)$ coincide.

*Proof.* Let

$$\tilde{B}(z,k), \qquad z \in \mathbb{C} \setminus \{(n+k)^2 + iy : n \in \mathbb{Z}, y \leq 0\} =: D_+(k)$$

be the analytic family of operators given by the same formal expression (3.1.8) as the operators $B(z,k)$, but with the branch of the square root defined on $\mathbb{C} \setminus \{iy : y \leq 0\}$. (Hence the square root $\sqrt{(n+k)^2 - z}$ in the $n$-th summand of (3.1.8) is well-defined for $z \in D_+(k)$.) Both branches coincide on the upper halfplane, so that

$$\tilde{B}(z,k) = B(z,k), \qquad z \in \mathbb{C}_+. \tag{3.2.2}$$

Let $B$ be as in Section 1.2, then $B^{1/2}B(z,k)^{-1/2}$, $z \in D_+(k)$, is an analytic family of bounded operators which are given by

$$B^{1/2}\tilde{B}(z,k)^{-1/2} f = \sum_{n\in\mathbb{Z}} \sqrt[4]{\frac{n^2+1}{(n+k)^2 - z}}\, \hat{f}_n\, e_n. \tag{3.2.3}$$

It follows from Lemma 1.2.3 that

$$\tilde{T}^{(\sigma)}(z,k) := |M^{(\sigma)}|^{1/2}\tilde{B}(z,k)^{-1/2} =$$
$$= \left(|M^{(\sigma)}|^{1/2}B^{-1/2}\right)\left(B^{1/2}\tilde{B}(z,k)^{-1/2}\right), \qquad z \in D_+(k),$$

is an analytic family of compact operators. Because of (3.2.2) it is a bounded analytic (and hence norm-continuous) extension of the family $T^{(\sigma)}(z,k)$ across the cut from above. We put

$$T^{(\sigma)}(\lambda+i0,k) := \tilde{T}^{(\sigma)}(\lambda,k), \qquad \lambda \neq (n+k)^2,\ n \in \mathbb{Z}.$$

The construction of the operators $T^{(\sigma)}(\lambda - i0,k)$ is similar, replacing $D_+(k)$ by $D_-(k) := \mathbb{C} \setminus \{(n+k)^2 + iy:\ n \in \mathbb{Z},\ y \geq 0\}$.
The statement about the operators $T_0^{(\sigma)}(z,k)$ follows by applying the bounded operator $\operatorname{sgn} M^{(\sigma)}$ to $T^{(\sigma)}(z,k)$.
Let us prove the statement about the sets $\mathcal{N}_\pm(k)$. It follows from (3.2.3) that $\|\tilde{T}^{(\sigma)}(z,k)\| < 1$ if $|\operatorname{Im} z|$ is large. Now the Analytic Fredholm Alternative (cf. Theorem VII.1.9 in [K]) applied to the operators $\tilde{T}_0^{(\sigma)}(z,k)\,\tilde{T}^{(\sigma)}(\overline{z},k)^*$ yields the discreteness of the sets $\mathcal{N}_\pm(k)$. This concludes the proof. ☐

**Proposition 3.2.4.** *Let $s > \frac{1}{2}$ and $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then the limits*

$$\lim_{\epsilon\to 0+} \Lambda_2^{-s} R^{(\sigma)}(\lambda \pm i\epsilon, k)\Lambda_2^{-s}, \qquad \lambda \notin \{(n+k)^2 :\ n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k),$$

*exist in the operator norm topology and are uniform for $\lambda$ from compact intervals of $\mathbb{R} \setminus (\{(n+k)^2 :\ n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k))$.*

*Proof.* We consider the resolvent identity (3.1.9). Since the norm-limits of the operators $\Lambda_2^{-s}R^{(0)}(\lambda \pm i\epsilon, k)\Lambda_2^{-s}$ and

$$T^{(\sigma)}(\lambda \mp i\epsilon,k)^* \left(I + T_0^{(\sigma)}(\lambda \pm i\epsilon,k)\, T^{(\sigma)}(\lambda \mp i\epsilon,k)^*\right)^{-1} T_0^{(\sigma)}(\lambda \pm i\epsilon,k)$$

exist and are uniform on compact intervals of $\mathbb{R} \setminus (\{(n+k)^2 : n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k))$, it suffices to prove that the limits

$$\lim_{\epsilon\to 0+} Y(\lambda \pm i\epsilon,k)\Lambda_2^{-s}, \qquad \lambda \neq (n+k)^2,\ n \in \mathbb{Z},$$

exist in the norm sense and are uniform for $\lambda$ from compact intervals of $\mathbb{R}\setminus\{(n+k)^2 : n \in \mathbb{Z}\}$.
We note that $Y(z,k)$ is an integral operator with kernel

$$k(x_1,y;z,k) := \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{1}{\sqrt[4]{(n+k)^2 - z}}\, e^{in(x_1-y_1)}\, e^{-\sqrt{(n+k)^2-z}\,y_2}, \qquad x_1 \in \mathbb{T},\ y \in \Pi.$$

Fix $\lambda \in \mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$ and define

$$k(x_1, y; \lambda \pm i0, k) := \lim_{\epsilon \to 0+} k(x_1, y; \lambda \pm i\epsilon, k), \qquad x \in \mathbb{T},\ y \in \Pi,$$

as pointwise limit. We claim that

$$\int_\Pi \int_\mathbb{T} |k(x_1, y; \lambda \pm i\epsilon, k) - k(x_1, y; \lambda \pm i0, k)|^2 \, dx_1 \frac{dy}{(1+y_2^2)^s} \to 0 \qquad (\epsilon \to 0+). \tag{3.2.4}$$

This means that $Y(\lambda \pm i\epsilon, k)\Lambda_2^{-s}$ converges in (Hilbert-Schmidt) norm to an integral operator with kernel $k(x_1, y; \lambda \pm i0, k)\frac{1}{(1+y_2^2)^s}$, which establishes the Proposition.
The proof of (3.2.4) uses the same techniques as the proof of (3.2.1), so we can omit it here. □

As an easy consequence of Proposition 3.2.4 we obtain

**Theorem 3.2.5.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$, then*

$$\sigma_{sc}\left(H^{(\sigma)}(k)\right) = \emptyset.$$

*Proof.* By Theorem 3.1.3, $\sigma_{sc}\left(H^{(\sigma)}(k)\right) \subset \sigma_{ess}\left(H^{(\sigma)}(k)\right) = [k^2, +\infty)$, so let $[a, b] \subset [k^2, +\infty) \setminus (\{(n+k)^2 : n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k))$ be a compact interval. Then for every $u$ from the dense set $\mathcal{R}(\Lambda_2^{-s})$ (with $s > \frac{1}{2}$ arbitrary) we have

$$\sup_{0<\epsilon<1} \int_a^b |(R^{(\sigma)}(\lambda \pm i\epsilon, k)u, u)|^2 \, d\lambda < \infty$$

by Proposition 3.2.4. It follows (cf. Theorem XIII.20 in [RS4] or Proposition 1.5.2 in [Ya1]) that $H^{(\sigma)}(k)$ is absolutely continuous on $\Lambda$. Therefore $\sigma_{sc}\left(H^{(\sigma)}(k)\right) \subset \{(n+k)^2 : n \in \mathbb{Z}\} \cup \mathcal{N}_+(k) \cup \mathcal{N}_-(k)$. Since the latter set accumulates only at $\{(n+k)^2 : n \in \mathbb{Z}\}$ and $+\infty$, we conclude $\sigma_{sc}\left(H^{(\sigma)}(k)\right) = \emptyset$, as claimed. □

## 3.3 The point spectrum of the operators $H^{(\sigma)}(k)$

Our goal in this section is to analyze the point spectrum of the operators $H^{(\sigma)}(k)$. We will prove that the eigenvalues have finite multiplicities and can accumulate at $+\infty$ only. Under the assumption $\sigma \geq 0$ we will see that there are no eigenvalues. Our method is more direct than in the previous sections of this chapter. In Subsection 1 we will introduce "boundary operators" $B^{(\sigma)}(\lambda \pm i0, k)$ on $L_2(\mathbb{T})$. These are operators of compact resolvent, and we will show in Subsection 2 that an eigenvalue 0 of $B^{(\sigma)}(\lambda \pm i0, k)$ corresponds to an eigenvalue $\lambda$ of $H^{(\sigma)}(k)$. (Again the situation is slightly more difficult at the points $\lambda = (n+k)^2$, $n \in \mathbb{Z}$.) Using this characterization we will derive the above stated properties of the eigenvalues from the Analytic Fredholm Alternative. We supplement this general result in Subsection 4 with conditions on $\sigma$ for the absence, location or existence of eigenvalues.

**1. Definition of the operators $B^{(\sigma)}(\lambda \pm i0, k)$.** Let $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $\lambda \in \mathbb{R}$. In the Hilbert space $L_2(\mathbb{T})$ we consider the quadratic forms

$$\begin{aligned} &\mathcal{D}[b^{(\sigma)}(\lambda \pm i0, k)] := H^{1/2}(\mathbb{T}), \\ &b^{(\sigma)}(\lambda \pm i0, k)[f] := \mp i \sum_{(n+k)^2<\lambda} \sqrt{\lambda - (n+k)^2}\, |\hat{f}_n|^2 + \\ &\qquad\qquad + \sum_{(n+k)^2>\lambda} \sqrt{(n+k)^2 - \lambda}\, |\hat{f}_n|^2 + \int_{-\pi}^{\pi} \sigma(x_1)|f(x_1)|^2\, dx_1. \end{aligned} \tag{3.3.1}$$

Of course, $b^{(0)}(\lambda \pm i0, k)$ are closed sectorial forms. According to Corollary 1.2.4 the integral $\int_{-\pi}^{\pi} \sigma(x_1)|f(x_1)|^2\, dx_1$ is well-defined for $f \in H^{1/2}(\mathbb{T})$ and strongly subordinate to $b^{(0)}(\lambda \pm i0, k)$. Hence $b^{(\sigma)}(\lambda \pm i0, k)$ are closed sectorial forms and there are unique m-sectorial operators $B^{(\sigma)}(\lambda \pm i0, k)$ with domains $\mathcal{D}(B^{(\sigma)}(\lambda \pm i0, k))$ such that

$$\begin{aligned} \mathcal{D}(B^{(\sigma)}(\lambda \pm i0, k)) &\subset H^{1/2}(\mathbb{T}), \\ (B^{(\sigma)}(\lambda \pm i0, k)f, g) &= b^{(\sigma)}(\lambda \pm i0, k)[f, g], \\ & \quad f \in \mathcal{D}(B^{(\sigma)}(\lambda \pm i0, k)),\ g \in H^{1/2}(\mathbb{T}). \end{aligned} \tag{3.3.2}$$

The operators $\mathrm{Re}\, B^{(\sigma)}(\lambda + i0, k)$ and $\mathrm{Re}\, B^{(\sigma)}(\lambda - i0, k)$, associated with the real parts of the quadratic forms $b^{(\sigma)}(\lambda \pm i0, k)$, are self-adjoint and coincide. We put

$$B^{(\sigma)}(\lambda, k) := \mathrm{Re}\, B^{(\sigma)}(\lambda + i0, k) = \mathrm{Re}\, B^{(\sigma)}(\lambda - i0, k). \tag{3.3.3}$$

We remark that $B^{(0)}(\lambda, k) = B(\lambda, k),\ \lambda < k^2$, in the notation of Section 1.
Of course, the operators $B^{(0)}(\lambda \pm i0, k)^{-1},\ \lambda \neq (n+k)^2,\ n \in \mathbb{Z}$, are compact. More generally, we have

**Lemma 3.3.1.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $\lambda \in \mathbb{R}$. Then the operators $B^{(\sigma)}(\lambda \pm i0, k)$, $B^{(\sigma)}(\lambda, k)$ have compact resolvent.*

*Proof.* It suffices (cf. Theorem VI.3.3 in [K]) to prove that $B^{(\sigma)}(\lambda, k)$ has compact resolvent. Since $b^{(\sigma)}(\lambda, k)$ differs from $b^{(0)}(\lambda, k)$ by a strongly subordinate perturbation, the assertion follows by the min-max-principle from the fact that $B^{(0)}(\lambda, k)$ has compact resolvent. □

**2. Characterization of eigenvalues of $H^{(\sigma)}(k)$.** We shall characterize the eigenvalues of the operator $H^{(\sigma)}(k)$ as the values $\lambda$ for which 0 is an eigenvalue of the operators $B^{(\sigma)}(\lambda \pm i0, k)$. More precisely, we have

**Proposition 3.3.2.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $\lambda \in \mathbb{R}$.*

1. *Suppose that* $u \in \mathcal{D}(H^{(\sigma)}(k))$ *satisfies* $H^{(\sigma)}(k)u = \lambda u$. *Define*

$$f(x_1) := u(x_1, 0), \qquad x_1 \in \mathbb{T}. \tag{3.3.4}$$

*Then* $\hat{f}_n = 0$ *if* $(n+k)^2 \le \lambda$, $f \in \mathcal{D}(B^{(\sigma)}(\lambda \pm i0, k))$ *and* $B^{(\sigma)}(\lambda \pm i0, k)f = 0$. *Moreover,*

$$u(x) = \frac{1}{\sqrt{2\pi}} \sum_{(n+k)^2 > \lambda} \hat{f}_n \, e^{inx_1} \, e^{-\sqrt{(n+k)^2 - \lambda}\, x_2}, \qquad x \in \Pi. \tag{3.3.5}$$

2. *Suppose that* $f \in \mathcal{D}(B^{(\sigma)}(\lambda \pm i0, k))$ *satisfies* $B^{(\sigma)}(\lambda \pm i0, k)f = 0$ *and* $\hat{f}_n = 0$ *if* $(n+k)^2 = \lambda$. *Define* $u$ *by* (3.3.5).
*Then* $u \in \mathcal{D}(H^{(\sigma)}(k))$ *and* $H^{(\sigma)}(k)u = \lambda u$. *Moreover,* $\hat{f}_n = 0$ *if* $(n+k)^2 < \lambda$ *and* (3.3.4) *holds.*

The proof relies on the following elementary observation.

**Lemma 3.3.3.** *Let* $k \in [-\frac{1}{2}, \frac{1}{2}]$ *and* $\lambda \in \mathbb{R}$.

1. *Let* $u \in \tilde{H}^1(\Pi)$, *then the following are equivalent:*

   *(i)* $u \in \mathcal{D}(H^{(\sigma)}(k))$ *and* $H^{(\sigma)}(k)u = \lambda u$,

   *(ii)*

$$\int_0^\infty D_2 \hat{u}_n \overline{D_2 \varphi} \, dx_2 + \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sigma(x_1) u(x_1, 0) e^{-inx_1} \, dx_1 \, \overline{\varphi(0)} =$$
$$= (\lambda - (n+k)^2) \int_0^\infty \hat{u}_n \overline{\varphi} \, dx_2, \qquad n \in \mathbb{Z}, \, \varphi \in H^1(\mathbb{R}_+).$$

2. *Let* $f \in H^{1/2}(\mathbb{T})$, *then the following are equivalent:*

   *(i)* $f \in \mathcal{D}(B^{(\sigma)}(\lambda \pm i0, k))$ *and* $B^{(\sigma)}(\lambda \pm i0, k)f = 0$,

   *(ii)*

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sigma(x_1) f(x_1) e^{-inx_1} \, dx_1 = 0 \quad \text{if } (n+k)^2 < \lambda,$$
$$\sqrt{(n+k)^2 - \lambda} \, \hat{f}_n + \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sigma(x_1) f(x_1) e^{-inx_1} \, dx_1 = 0 \quad \text{if } (n+k)^2 \ge \lambda.$$

We emphasize that, according to our convention, statements which contain the double index "$\pm$" are understood as two independent assertions. In particular, in the second part of the lemma the statements (i) corresponding to the upper and to the lower sign are equivalent, since the statement (ii) is independent of the sign.

*Proof.* 1) The assertion (1.i) is equivalent to

$$\int_{\Pi} \Big( (D_1+k)u\overline{(D_1+k)v} + D_2u\overline{D_2v} \Big)\, dx + \int_{-\pi}^{\pi} \sigma(x_1)u(x_1,0)\overline{v(x_1,0)}\, dx_1 = \\ = \lambda \int_{\Pi} u\overline{v}\, dx, \qquad v \in \tilde{H}^1(\Pi). \tag{3.3.6}$$

Taking $v = e_n \otimes \varphi,\ n \in \mathbb{Z}, \varphi \in H^1(\mathbb{R}_+)$, and integrating by parts we see that (1.i) implies (1.ii).
Conversely, since the functions of this special form are fundamental in $\tilde{H}^1(\Pi)$ (i.e., their linear span is dense, cf. Lemma 1.2.2), (3.3.6) holds for any $v \in \tilde{H}^1(\Pi)$ if it holds for all $v = e_n \otimes \varphi,\ n \in \mathbb{Z}, \varphi \in H^1(\mathbb{R}_+)$. This in turn follows from (1.ii).
2) The assertion (2.i) is equivalent to

$$\mp i \sum_{(n+k)^2<\lambda} \sqrt{\lambda-(n+k)^2}\, \hat{f}_n\, \overline{\hat{g}_n} + \sum_{(n+k)^2\geq\lambda} \sqrt{(n+k)^2-\lambda}\, \hat{f}_n\, \overline{\hat{g}_n} + \int_{-\pi}^{\pi} \sigma f \overline{g}\, dx_1 = 0, \\ g \in H^{1/2}(\mathbb{T}). \tag{3.3.7}$$

Taking $g = f$ and considering the imaginary part only we obtain $\hat{f}_n = 0$ if $(n+k)^2 < \lambda$, and taking $g = e_n,\ n \in \mathbb{Z}$, we see that (2.i) implies (2.ii).
Conversely, since $\{e_n,\ n \in \mathbb{Z}\}$ is fundamental in $H^{1/2}(\mathbb{T})$, (3.3.7) holds for any $g \in H^{1/2}(\mathbb{T})$ if it holds for all $g = e_n,\ n \in \mathbb{Z}$. This in turn follows from (2.ii). $\square$

*Proof (Proposition 3.3.2).* 1) Let $u \in \mathcal{D}(H^{(\sigma)}(k))$ such that $H^{(\sigma)}(k)u = \lambda u$. Then (1.i) in Lemma 3.3.3 holds. Taking $\varphi \in C_0^\infty(\mathbb{R}_+)$ we see that the generalized second derivative of $\hat{u}_n$ belongs to $L_2(\mathbb{R}_+)$ and satisfies $D_2^2\hat{u}_n = ((\lambda-(n+k)^2\hat{u}_n$. It follows that

$$\hat{u}_n(x_2) = \begin{cases} 0 & \text{if} \quad (n+k)^2 \leq \lambda, \\ \alpha_n e^{-\sqrt{(n+k)^2-\lambda}\, x_2} & \text{if} \quad (n+k)^2 > \lambda, \end{cases}$$

for some constants $\alpha_n \in \mathbb{C}$, and hence with $f$ defined by (3.3.4),

$$\hat{f}_n = \begin{cases} 0 & \text{if} \quad (n+k)^2 \leq \lambda, \\ \alpha_n & \text{if} \quad (n+k)^2 > \lambda. \end{cases}$$

It follows from $u \in \tilde{H}^1(\Pi)$, that

$$\sum_{(n+k)^2>\lambda} \big((n+k)^2-\lambda\big) \frac{|\hat{f}_n|^2}{2\sqrt{(n+k)^2-\lambda}} = \sum_{n\in\mathbb{Z}} \|D_2\hat{u}_n\|^2 < \infty,$$

so that $f \in H^{1/2}(\mathbb{T})$. Moreover, if we plug the identity

$$\int_0^\infty D_2\hat{u}_n\overline{D_2\varphi}\,dx_2 = \left(\lambda - (n+k)^2\right)\int_0^\infty \hat{u}_n\overline{\varphi}\,dx_2 + \sqrt{(n+k)^2-\lambda}\,\hat{f}_n\,\overline{\varphi(0)}$$

into (1.ii) of Lemma 3.3.3, we obtain (2.ii) and hence

$$f \in \mathcal{D}(B^{(\sigma)}(\lambda \pm i0, k)), \qquad B^{(\sigma)}(\lambda \pm i0, k)f = 0,$$

which concludes the proof of the first part.
2) Let $f \in \mathcal{D}(B^{(\sigma)}(\lambda \pm i0, k))$ such that $B^{(\sigma)}(\lambda \pm i0, k)u = 0$ and $\hat{f}_n = 0$ if $(n+k)^2 = \lambda$. Then (2.i) in Lemma 3.3.3 holds. With $u$ defined by (3.3.5) it follows from $f \in H^{1/2}(\mathbb{T})$ that

$$\sum_{n\in\mathbb{Z}}((1+n^2)\|\hat{u}_n\|^2 + \|D_2\hat{u}_n\|^2) = \sum_{(n+k)^2>\lambda} \frac{1+n^2+(n+k)^2-\lambda}{2\sqrt{(n+k)^2-\lambda}}\,|\hat{f}_n|^2 < \infty,$$

so that $u \in \tilde{H}^1(\Pi)$. Clearly, (3.3.4) holds. (Here we use the hypothesis $\hat{f}_n = 0$ if $(n+k)^2 = \lambda$.) It remains to verify (1.ii) in Lemma 3.3.3 for all $n \in \mathbb{Z}$, $\varphi \in H^1(\mathbb{R}_+)$. This in turn follows from (2.ii) by partial integration. □

Let us summarize the essential result of Proposition 3.3.2.

**Corollary 3.3.4.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $\lambda \in \mathbb{R}$, then*

$$\dim \mathcal{N}(H^{(\sigma)}(k) - \lambda) \le \dim \mathcal{N}(B^{(\sigma)}(\lambda + i0, k)) = \dim \mathcal{N}(B^{(\sigma)}(\lambda - i0, k)). \quad (3.3.8)$$

*If $\lambda \neq (n+k)^2$, $n \in \mathbb{Z}$, then equality holds.*
*In particular, the eigenvalues of $H^{(\sigma)}(k)$ have finite multiplicities.*

*Proof.* The operator $\mathcal{N}(H^{(\sigma)}(k) - \lambda) \ni u \mapsto f \in \mathcal{N}(B^{(\sigma)}(\lambda \pm i0, k))$ given by (3.3.4) is injective and, if $\lambda \neq (n+k)^2$, $n \in \mathbb{Z}$, also surjective. For the second statement note that $\mathcal{N}(B^{(\sigma)}(\lambda \pm i0, k))$ is finite dimensional because of Lemma 3.3.1. □

**3. The point spectrum of the operators $H^{(\sigma)}(k)$.** Using the characterization of the eigenvalues $\lambda$ of $H^{(\sigma)}(k)$ in terms of $B^{(\sigma)}(\lambda \pm i0, k)$ we shall prove now our main result on the point spectrum of the operators on the halfstrip.

**Theorem 3.3.5.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$, then $\sigma_p\left(H^{(\sigma)}(k)\right)$ consists of eigenvalues of finite multiplicities which may accumulate at $+\infty$ only.*

We will prove this Theorem in two steps. First (Lemma 3.3.6) we use the Analytic Fredholm Alternative to show that the only possible finite accumulation points of $\sigma_p\left(H^{(\sigma)}(k)\right)$ are the points $(n+k)^2$, $n \in \mathbb{Z}$. Then (Lemma 3.3.7) we use a continuity and monotonicity principle for eigenvalues to show that the

eigenvalues do actually not accumulate at these points.
We state (and prove) these statements in terms of the operators $B^{(\sigma)}(\lambda \pm i0, k)$ and their real parts $B^{(\sigma)}(\lambda, k)$. Note that

$$\mathcal{N}(B^{(\sigma)}(\lambda \pm i0, k)) \subset \mathcal{N}(B^{(\sigma)}(\lambda, k)). \tag{3.3.9}$$

Indeed, if $B^{(\sigma)}(\lambda \pm i0, k)f = 0$ we see from $\operatorname{Im} b^{(\sigma)}(\lambda \pm i0, k)[f] = 0$ that $\hat{f}_n = 0$ if $(n+k)^2 < \lambda$. Hence $(\operatorname{Re} b^{(\sigma)}(\lambda \pm i0, k))[f, g] = b^{(\sigma)}(\lambda \pm i0, k)[f, g] = 0$ for all $g \in H^{1/2}(\mathbb{T})$, i.e. $f \in \mathcal{N}(B^{(\sigma)}(\lambda, k))$.

**Lemma 3.3.6.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $\sigma \not\equiv 0$. Then the sets*

$$\mathcal{N}(k) := \{\lambda \in \mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\} : 0 \text{ is eigenvalue of } B^{(\sigma)}(\lambda \pm i0, k)\},$$
$$\tilde{\mathcal{N}}(k) := \{\lambda \in \mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\} : 0 \text{ is eigenvalue of } B^{(\sigma)}(\lambda, k)\}$$

*are discrete in $\mathbb{R} \setminus \{(n+k)^2,\ n \in \mathbb{Z}\}$.*

Note that the set $\mathcal{N}(k)$ does not depend on the choice of the $\pm$-sign. This follows from Proposition 3.3.2.
If $\sigma \equiv 0$, then $\mathcal{N}(k) = \emptyset$, but $\tilde{\mathcal{N}}(k) = [k^2, +\infty) \setminus \{(n+k)^2,\ n \in \mathbb{Z}\}$ (consider $f = e_0$).

*Proof.* Because of (3.3.9) it suffices to consider the set $\tilde{\mathcal{N}}(k)$. Let $\Lambda$ be a maximal open interval in $\mathbb{R} \setminus \{(n+k)^2,\ n \in \mathbb{Z}\}$ and $l := \sup \Lambda$. We prove that the points $\lambda$ for which $B^{(\sigma)}(\lambda, k)$ has eigenvalue 0 do not accumulate inside $\Lambda$.
For $z \in D := \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta \in \Lambda\}$ we consider the quadratic form

$$\begin{aligned} &\mathcal{D}[\tilde{b}^{(\sigma)}(z, k)] := H^{1/2}(\mathbb{T}), \\ &\tilde{b}^{(\sigma)}(z, k)[f] := \sum_{(n+k)^2 \geq l} \sqrt{(n+k)^2 - z}\, |\hat{f}_n|^2 + \int_{-\pi}^{\pi} \sigma(x_1) |f(x_1)|^2 \, dx_1. \end{aligned} \tag{3.3.10}$$

(As usual we choose the canonical branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ with $\operatorname{Re} \sqrt{\cdot} > 0$.)
Then for any $f \in H^{1/2}(\mathbb{T})$, $\tilde{b}^{(\sigma)}(z, k)[f]$ is an analytic function of $z \in D$ and coincides with $\operatorname{Re} b^{(\sigma)}(\lambda \pm i0, k)[f]$ if $z = \lambda \in \Lambda$.
It follows by arguments as in Subsection 1 that $\tilde{b}^{(\sigma)}(z, k)$ are closed sectorial forms. The correspondig operators $\tilde{B}^{(\sigma)}(z, k)$ form an analytic family of type (B) and coincide with $B^{(\sigma)}(\lambda, k)$ if $z = \lambda \in \Lambda$. It follows from Lemma 3.3.1 that $\tilde{B}^{(\sigma)}(z, k)$ has compact resolvent for any $z \in D$.
Moreover, 0 is not an eigenvalue of $\tilde{B}^{(\sigma)}(z, k)$ if $\operatorname{Im} z \neq 0$. Indeed, given $f \in \mathcal{N}(\tilde{B}^{(\sigma)}(\lambda, k))$ we see from $\operatorname{Im} \tilde{b}^{(\sigma)}(\lambda \pm i0, k)[f] = 0$ that $\hat{f}_n = 0$ if $(n+k)^2 \geq l$. Hence $\int_{-\pi}^{\pi} \sigma f \overline{g}\, dx_1 = \tilde{b}^{(\sigma)}(z, k)[f, g] = 0$ for all $g \in H^{1/2}(\mathbb{T})$, i.e. $\sigma f \equiv 0$. Since $\sigma \not\equiv 0$ and $f$ as trigonometric polynomial has only a finite number of zeroes, we conclude $f \equiv 0$.

Finally, it follows from the Analytic Fredholm Alternative (cf. Theorem VII.1.10 in [K]) applied to the operators $\tilde{B}^{(\sigma)}(z,k)$ that the set $\{\lambda \in \Lambda : 0$ is eigenvalue of $B^{(\sigma)}(\lambda,k)\}$ is a discrete subset of $\Lambda$. □

**Lemma 3.3.7.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $\sigma \not\equiv 0$. Then the sets $\mathcal{N}(k)$ and $\tilde{\mathcal{N}}(k)$ are discrete in $\mathbb{R}$.*

*Proof.* Because of (3.3.9) it suffices again to consider the set $\tilde{\mathcal{N}}(k)$. Let $\Lambda$ be a maximal open interval in $\mathbb{R} \setminus \{(n+k)^2,\ n \in \mathbb{Z}\}$, i.e. either

$$\Lambda = (-\infty, k^2), \tag{3.3.11}$$

or for some $n_1,\ n_2 \in \mathbb{Z}$

$$\Lambda = ((n_1+k)^2, (n_2+k)^2), \qquad (n+k)^2 \notin \Lambda,\ n \in \mathbb{Z}. \tag{3.3.12}$$

We prove that there are only finitely many points $\lambda \in \Lambda$ for which $B^{(\sigma)}(\lambda,k)$ has eigenvalue 0.
Recall that $B^{(\sigma)}(\lambda,k)$ is a self-adjoint operator with compact resolvent. We denote by $\mu_n(\lambda,k),\ \lambda \in \overline{\Lambda}$, its $n$-th eigenvalue, counting multiplicities. It follows from the min-max principle that $\mu_n(.,k)$ is a continuous function on $\overline{\Lambda}$, and from the inequality

$$B^{(\sigma)}(\lambda,k) \leq B^{(\sigma)}(\lambda',k), \qquad \lambda' \leq \lambda,$$

that it is non-increasing. Moreover, there is at most one $\lambda \in \overline{\Lambda}$ for which $\mu_n(\lambda,k) = 0$, since otherwise this would hold for a whole interval of $\lambda$'s in contradiction to Lemma 3.3.6.
We consider now the cases (3.3.11), (3.3.12) separately.
If $\Lambda$ is of the form (3.3.11), we note that the operator $B^{(\sigma)}(\lambda,k)$ is positive definite provided $-\lambda$ is large, hence the function $\mu_n(.,k)$ is positive for $-\lambda$ large. It follows that

$$\begin{aligned} &\sharp_{cm}\{\lambda \in (-\infty,k^2):\ 0 \text{ is eigenvalue of } B^{(\sigma)}(\lambda,k)\} = \\ &= \sharp\{n \in \mathbb{N}:\ \mu_n(\lambda,k) = 0 \text{ for some } \lambda \in (-\infty,k^2)\} = \\ &= \sharp\{n \in \mathbb{N}:\ \mu_n(k^2,k) < 0\} = \\ &= \sharp_{cm}\{\mu < 0:\ \mu \text{ is eigenvalue of } B^{(\sigma)}(k^2,k)\}, \end{aligned} \tag{3.3.13}$$

where $\sharp_{cm}\{...\}$ means that the cardinality of $\{...\}$ is determined according to multiplicities. The RHS of (3.3.13) is finite since $B^{(\sigma)}(k^2,k)$ has compact resolvent. This proves the assertion in the case (3.3.11).
If $\Lambda$ is of the form (3.3.12), we have

$$\begin{aligned} &\sharp_{cm}\{\lambda \in ((n_1+k)^2,(n_2+k)^2):\ 0 \text{ is eigenvalue of } B^{(\sigma)}(\lambda,k)\} = \\ &= \sharp\{n \in \mathbb{N}:\ \mu_n(\lambda,k) = 0 \text{ for some } \lambda \in ((n_1+k)^2,(n_2+k)^2)\} = \\ &= \sharp\{n \in \mathbb{N}:\ \mu_n((n_2+k)^2,k) < 0 \text{ and } \mu_n((n_1+k)^2,k) > 0\} = \\ &= \sharp_{cm}\{\mu < 0:\ \mu \text{ is eigenvalue of } B^{(\sigma)}((n_2+k)^2,k)\} - \\ &\quad -\sharp_{cm}\{\mu \leq 0:\ \mu \text{ is eigenvalue of } B^{(\sigma)}((n_1+k)^2,k)\}, \end{aligned}$$

and again the RHS is finite since $B^{(\sigma)}((n_1+k)^2,k)$, $B^{(\sigma)}((n_2+k)^2,k)$ have compact resolvent. This completes the proof of the Lemma. □

Theorem 3.3.5 follows now immediately.

*Proof (Theorem 3.3.5).* Combine Lemma 3.3.7 with Corollary 3.3.4. □

**4. Further results on the eigenvalues of the operators $H^{(\sigma)}(k)$.** Under stronger assumptions on $\sigma$ we obtain better results on the location (or absence) of eigenvalues of the operators on the halfstrip. Our main tool is Proposition 3.3.2.
We start with a simple example which shows that the case of an infinite sequence of eigenvalues accumulating at $+\infty$ can actually happen.

**Example 3.3.8.** Assume that $\sigma \in \mathbb{R}$ is constant and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then

$$\sigma_p\left(H^{(\sigma)}(k)\right) = \begin{cases} \emptyset & \text{if} \quad \sigma \geq 0, \\ \{-\sigma^2 + (n+k)^2 : \ n \in \mathbb{Z}\} & \text{if} \quad \sigma < 0. \end{cases}$$

In the case $\sigma < 0$ the eigenfunctions are given for $n \in \mathbb{Z}$ by

$$u_n(x) := \sqrt{\tfrac{|\sigma|}{\pi}}\, e^{inx_1}\, e^{\sigma x_2}, \qquad x \in \Pi.$$

By separation of variables (similarly as in Section 2.1) this is derived from the well-known properties of the operator $-\frac{d^2}{dx_2^2}$ on $L_2(\mathbb{R}_+)$ with boundary condition $f'(0) = \sigma f(0)$.

**Proposition 3.3.9.** *Assume that $\sigma$ is bounded from below and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then*

$$\begin{aligned} &\sigma_p\left(H^{(\sigma)}(k)\right) = \emptyset && \text{if } \sigma_0 \geq 0, \\ &\sigma_p\left(H^{(\sigma)}(k)\right) \subset \textstyle\bigcup_{n\in\mathbb{Z}}[-\sigma_0^2 + (n+k)^2, (n+k)^2) && \text{if } \sigma_0 < 0, \end{aligned}$$

*where $\sigma_0 := \operatorname{essinf}_{x_1\in\mathbb{T}} \sigma(x_1)$. Moreover, $-\sigma_0^2 + k^2 \in \sigma_p\left(H^{(\sigma)}(k)\right)$ iff $\sigma \equiv \sigma_0 < 0$ is (a.e.) a negative constant.*

*Proof.* Let $\lambda \notin \bigcup_{n\in\mathbb{Z}}[-\min\{\sigma_0, 0\}^2 + (n+k)^2, (n+k)^2)$ (i.e., $\lambda \in \mathbb{R}$ if $\sigma_0 \geq 0$). To arrive at a contradiction we assume that $\lambda$ is an eigenvalue of $H^{(\sigma)}(k)$ with correspondig eigenfunction $u \neq 0$. By Proposition 3.3.2, $f := u(.,0) \in \mathcal{N}(B^{(\sigma)}(\lambda \pm i0, k)) \setminus \{0\}$ and

$$\hat{f}_n = 0 \qquad \text{if } (n+k)^2 \leq \lambda.$$

Hence $b^{(0)}(\lambda \pm i0, k)[f] \geq \gamma \|f\|^2$ where

$$\gamma := \min\{\sqrt{(n+k)^2 - \lambda} : \ n \in \mathbb{Z},\ \hat{f}_n \neq 0\},$$

and

$$b^{(\sigma)}(\lambda \pm i0, k)[f] \geq (\gamma + \sigma_0)\|f\|^2.$$

Since $b^{(\sigma)}(\lambda \pm i0, k)[f] = 0$ and $f \neq 0$ it follows that $\gamma + \sigma_0 \leq 0$. However, from the choice of $\lambda$ it follows that $\gamma + \sigma_0 > 0$, so we arrive at a contradiction and have proven the first part of the Proposition.
Let us prove the second one. From Example 3.3.8 we know that $-\sigma_0^2 + k^2 \in \sigma_p(H^{(\sigma)}(k))$ if $\sigma$ is a negative constant. To prove the opposite implication let $\sigma$ be an arbitrary function bounded from below by $\sigma_0$ such that $\lambda := -\sigma_0^2 + k^2 \in \sigma_p(H^{(\sigma)}(k))$ with correspondig eigenfunction $u$. Then

$$\lambda\|u\|^2 = h^{(\sigma)}(k)[u] \geq h^{(\sigma_0)}(k)[u] \geq \lambda\|u\|^2,$$

so that

$$\int_{-\pi}^{\pi} \sigma(x_1)|u(x_1, 0)|^2 dx_1 = \int_{-\pi}^{\pi} \sigma_0|u(x_1, 0)|^2 dx_1 \tag{3.3.14}$$

and

$$h^{(\sigma_0)}(k)[u] = (\inf_{0 \neq v \in \tilde{H}^1(\Pi)} \frac{h^{(\sigma_0)}(k)[v]}{\|v\|^2})\|u\|^2.$$

It follows from general principles that $u \in \mathcal{D}(H^{(\sigma_0)}(k))$ and $H^{(\sigma_0)}(k)u = \lambda u$. From the explicit formulae for eigenfunctions of $H^{(\sigma_0)}(k)$ we know that $u(., 0)$ can have at most a finite number of zeroes, so $\sigma \equiv \sigma_0$ by (3.3.14). This completes the proof. $\square$

Combining this result with Theorems 3.1.3, 3.2.5 we obtain the important

**Theorem 3.3.10.** *Assume that $\sigma$ is non-negative and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then the operator $H^{(\sigma)}(k)$ is absolutely continuous.*

**Proposition 3.3.11.** *Assume that $\sigma$ is a trigonometric polynomial of degree $N > 0$ and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then with $\sigma_0 := \operatorname{essinf}_{x_1 \in \mathbb{T}} \sigma(x_1)$*

$$\sigma_p\left(H^{(\sigma)}(k)\right) \subset \left(-\sigma_0^2 + k^2, (N - |k|)^2\right).$$

*Proof.* The lower bound follows from Proposition 3.3.9. To arrive at a contradiction we assume that $\lambda \geq (N - |k|)^2$ is an eigenvalue of $H^{(\sigma)}(k)$ with corresponding eigenfunction $u \neq 0$. Then according to Proposition 3.3.2 and Lemma 3.3.3, $f := u(., 0) \neq 0$ and

$$\hat{f}_n = 0 \qquad \text{if } (n + k)^2 \leq \lambda, \tag{3.3.15}$$

$$\sqrt{(n + k)^2 - \lambda}\, \hat{f}_n + \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sigma(x_1) f(x_1) e^{-inx_1}\, dx_1 = 0 \qquad \text{if } (n + k)^2 \geq \lambda.$$

We rewrite the second equation as

$$\sqrt{(n + k)^2 - \lambda}\, \hat{f}_n + \frac{1}{\sqrt{2\pi}} \sum_{m=-N}^{N} \hat{\sigma}_m \hat{f}_{n-m} = 0 \qquad \text{if } (n + k)^2 \geq \lambda. \tag{3.3.16}$$

It follows from

$$\sharp\{n \in \mathbb{Z}: \ (n+k)^2 \le \lambda\} \ge \sharp\{n \in \mathbb{Z}: \ (n+k)^2 \le (N-|k|)^2\} \ge 2N$$

and (3.3.15), that $\hat{f}_n = 0$ for at least $2N$ consecutive $n$. Using $\hat{\sigma}_N = \overline{\hat{\sigma}_{-N}} \neq 0$ it is not hard to see from (3.3.16) that $\hat{f}_n = 0$ for all $n$. Hence $f = 0$, a contradiction. $\square$

*Remark* 3.3.12. The same idea allows one to estimate the multiplicity of an eigenvalue $\lambda \in [(n-1+|k|)^2, (n-|k|)^2)$ or $\lambda \in [(n-|k|)^2, (n+|k|)^2)$ in terms of $n \in \mathbb{N}$. It is necessary to study the number of linear independent solutions $(\hat{f}_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ (without requiring summability) of (3.3.16) subjected to the constraint (3.3.15). We do not go into details.

**Example 3.3.13.** Let $\alpha,\ \beta,\ \gamma \in \mathbb{R}$, $\beta^2+\gamma^2 \neq 0$ and

$$\sigma(x_1) := \alpha + \beta\sin(x_1) + \gamma\cos(x_1), \qquad x_1 \in \mathbb{T}.$$

Note that $\min_{x_1\in\mathbb{T}} \sigma(x_1) = \alpha - \sqrt{\beta^2+\gamma^2}$. Combining Propositions 3.3.9 and 3.3.11 we obtain

$$\begin{aligned} &\sigma_p\left(H^{(\sigma)}(k)\right) = \emptyset && \text{if } \alpha \ge \sqrt{\beta^2+\gamma^2} \ge 0,\\ &\sigma_p\left(H^{(\sigma)}(k)\right) \subset \left(-\left(\alpha - \sqrt{\beta^2+\gamma^2}\right)^2 + k^2, (1-|k|)^2\right) && \text{if } \alpha < \sqrt{\beta^2+\gamma^2} \neq 0. \end{aligned}$$

In particular, the operators $H^{(\sigma)}(\pm\frac{1}{2})$ do not have embedded eigenvalues.

In Proposition 3.3.9 we proved that $H^{(\sigma)}(k)$ has no eigenvalues if $\sigma$ is non-negative. The following two Propositions are in a certain sense a converse: It is shown that $H^{(\sigma)}(k)$ has an eigenvalue if $\sigma$ is "sufficiently negative". In Proposition 3.3.14 we understand this in a global sense (concerning the mean value of $\sigma$), in Proposition 3.3.15 we understand this in a local sense (concerning negative peaks of $\sigma$).

**Proposition 3.3.14.** *Assume that*

$$\hat{\sigma}_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sigma(x_1)\, dx_1 < 0$$

*and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then $H^{(\sigma)}(k)$ has an eigenvalue smaller or equal $k^2 - \frac{1}{2\pi}\hat{\sigma}_0^2$.*

*Proof.* Consider the trial function $u \in \tilde{H}^1(\Pi)$,

$$u(x) := e^{\hat{\sigma}_0 x_2/\sqrt{2\pi}}, \qquad x \in \Pi.$$

It is easy to see that

$$h^{(\sigma)}(k)[u] = (k^2 - \frac{1}{2\pi}\hat{\sigma}_0^2)\|u\|^2.$$

Since $\min \sigma_{ess}\left(H^{(\sigma)}(k)\right) = k^2$, the Proposition follows from the variational principle. $\square$

**Proposition 3.3.15.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$ and assume that there is a non-empty open connected subset $\Lambda \subset \mathbb{T}$ of the torus and a $\delta > 0$ such that*

$$\sigma(x_1) \leq -\sqrt{\left(\tfrac{\pi}{|\Lambda|}\right)^2 - k^2 + \delta}, \qquad x_1 \in \Lambda.$$

*Then $H^{(\sigma)}(k)$ has an eigenvalue smaller than $k^2 - \delta$.*

*Proof.* Identifying $\mathbb{T}$ with $(-\pi, \pi]$ and shifting the $x_1$-coordinate if necessary we may assume that $\Lambda = (-\pi, c)$ for some $-\pi < c \leq \pi$. (The case $\Lambda = \mathbb{T}$ can be included in the case $c = \pi$.) Define

$$\tilde{\sigma}(x_1) := \begin{cases} \sigma(x_1) & \text{if } x_1 \in \mathbb{T} \setminus \Lambda, \\ -\sqrt{\left(\tfrac{\pi}{|\Lambda|}\right)^2 - k^2 + \delta} & \text{if } x_1 \in \Lambda, \end{cases}$$

and denote by $g^{(\tilde{\sigma})}(k)$ the closed and lower semibounded quadratic form obtained as the restriction of $h^{(\sigma)}(k)$ to the set $\{u \in H^1(\Pi) :\ u(-\pi, .) = u(c, .) = u(\pi, .) = 0\}$. The corresponding self-adjoint operator $G^{(\tilde{\sigma})}(k)$ satisfies

$$H^{(\sigma)}(k) \leq G^{(\tilde{\sigma})}(k). \tag{3.3.17}$$

Because of the Dirichlet condition at $x_1 = c$ the operator $G^{(\tilde{\sigma})}(k)$ admits a decomposition $G^{(\tilde{\sigma})}(k) = G_1^{(\tilde{\sigma})}(k) \oplus G_2^{(\tilde{\sigma})}(k)$ according to $L_2(\Pi) = L_2((-\pi, c) \times \mathbb{R}_+) \oplus L_2((c, \pi) \times \mathbb{R}_+)$. The part $G_1^{(\tilde{\sigma})}(k)$ on $L_2((-\pi, c) \times \mathbb{R}_+)$ depends only on $\tilde{\sigma}|_\Lambda$, i.e. on the constant $-\sqrt{\left(\tfrac{\pi}{|\Lambda|}\right)^2 - k^2 + \delta}$. Using the decomposition $L_2((-\pi, c) \times \mathbb{R}_+) = \sum_{n \in \mathbb{N}} \bigoplus \{b_n(., k)\} \otimes L_2(\mathbb{R}_+)$ where

$$b_n(x_1, k) := \sqrt{\tfrac{2}{|\Lambda|}}\, e^{-ikx_1} \sin\left(\tfrac{n\pi(x_1 + \pi)}{|\Lambda|}\right), \qquad x_1 \in \Lambda,$$

and proceeding as in Section 2.1 (see also Example 3.3.8) we see that $G_1^{(\tilde{\sigma})}(k)$, and thus also $G^{(\tilde{\sigma})}(k)$, has an eigenvalue $k^2 - \delta$. Since $\min \sigma_{ess}\left(H^{(\sigma)}(k)\right) = k^2$, (3.3.17) together with the variational principle imply that $H^{(\sigma)}(k)$ has an eigenvalue smaller or equal $k^2 - \delta$.
To complete the proof we want to arrive at a contradiction by assuming that $k^2 - \delta$ is the lowest eigenvalue of $H^{(\sigma)}(k)$. Consider the eigenfunction $u$ of $G^{(\tilde{\sigma})}(k)$ corresponding to the eigenvalue $k^2 - \delta$,

$$u(x) := \begin{cases} \sqrt{2|a|}\, b_n(x_1, k)\, e^{-|a|x_2}, & x_1 \in [-\pi, c], \\ 0 & x_1 \in (c, \pi], \end{cases}$$

where $a := -\sqrt{\left(\tfrac{\pi}{|\Lambda|}\right)^2 - k^2 + \delta}$. Then $u \in \tilde{H}^1(\Pi)$, $\|u\| = 1$ and

$$h^{(\sigma)}(k)[u] = \inf\{h^{(\sigma)}(k)[v] :\ v \in \tilde{H}^1(\Pi),\ \|v\| = 1\}.$$

It follows from general principles that $u \in \mathcal{D}(H^{(\sigma)}(k))$ and $H^{(\sigma)}(k)u = \lambda u$. By Elliptic Regularity we must have $u \in H^2_{loc}(\Pi)$, which is obviously not true. This contradiction completes the proof of the Proposition. $\square$

*Remark* 3.3.16. Assume that $\sigma$ is fixed with $\hat{\sigma}_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sigma(x_1)dx_1 < 0$ and consider the operators $H^{(\alpha\sigma)}$ with "coupling constant" $\alpha > 0$. It follows from Proposition 3.3.14 that

$$\lambda_1^{(\sigma)}(\alpha, k) := \inf_{0 \neq u \in \tilde{H}^1(\Pi)} \frac{h^{(\alpha\sigma)}(k)[u]}{\|u\|^2}.$$

is an eigenvalue of $H^{(\alpha\sigma)}(k)$ and satisfies

$$\lambda_1^{(\sigma)}(\alpha, k) \leq k^2 - \frac{1}{2\pi}\hat{\sigma}_0^2\alpha^2.$$

This result can be strengthened: For sufficiently small $\alpha > 0$ and $k \in (-\frac{1}{2}, \frac{1}{2})$, $\lambda_1^{(\sigma)}(\alpha, k)$ is the only eigenvalue of $H^{(\alpha\sigma)}$ below $k^2$, it is simple and satisfies

$$\sqrt{k^2 - \lambda_1^{(\sigma)}(\alpha, k)} = -\frac{1}{\sqrt{2\pi}}\hat{\sigma}_0\alpha + C(\sigma, k)\alpha^2 + \mathcal{O}(\alpha^3) \qquad (\alpha \to 0+),$$

where $C(\sigma, k) \geq 0$ can be calculated explicitly. Moreover, it can be shown that $\lambda_1^{(\sigma)}(\alpha, k)$ is an eigenvalue below $k^2$ even if $\hat{\sigma}_0 = 0$, $\sigma \not\equiv 0$ (indeed, for any $\alpha \in \mathbb{R}$).

**5. Comments and Remarks. Dirichlet-to-Neumann operator.** For sufficiently regular $\sigma$, the operator $B^{(\sigma)}(\lambda \pm i0, k)$ is closely connected with the Dirichlet-to-Neumann operator from the theory of Boundary Value Problems. Indeed, let $\lambda \in \mathbb{R}$ and $f \in H^{3/2}(\mathbb{T})$ such that

$$\hat{f}_n = 0, \qquad (n + k)^2 \leq \lambda. \tag{3.3.18}$$

Then there is a function $u \in \tilde{H}^2(\Pi)$ satisfying

$$\begin{aligned} &\left((D_1 + k)^2 + D_2^2\right) u = \lambda u, \qquad \text{in } \Pi \\ &u(.,0) = f \qquad \text{in } [-\pi, \pi]. \end{aligned}$$

This can be seen by the same ideas as in the proof of Proposition 3.3.2. In particular, we find that $u$ is given by the formula (3.3.5). If $\sigma$ is sufficiently regular we have

$$g := -\frac{\partial u}{\partial x_2}(.,0) + \sigma u(.,0) \in L_2(\mathbb{T}).$$

The operator $f \mapsto g - \sigma f$ is usually called Dirichlet-to-Neumann operator. It follows from (3.3.5) that

$$g = \sum_{(n+k)^2 > \lambda} \sqrt{(n + k)^2 - \lambda}\hat{f}_n e_n + \sigma f = B^{(\sigma)}(\lambda \pm i0, k)f = B^{(\sigma)}(\lambda, k)f,$$

so our boundary operators $B^{(\sigma)}(\lambda \pm i0, k)$, $B^{(\sigma)}(\lambda, k)$ are m-sectorial extensions of the Dirichlet-to-Neumann operator differing only on the orthogonal complement of functions $f \in L_2(\mathbb{T})$ satisfying (3.3.18).

**Connection between $B^{(\sigma)}(\lambda \pm i0, k)$ and $T_0^{(\sigma)}(z, k)\, T^{(\sigma)}(\overline{z}, k)^*$.** Our analysis in Sections 1 and 2 was based on the resolvent identity (3.1.9) and its extension to real values of $z$. It can be proved in the general framework that $\lambda \in [k^2, +\infty) \setminus \{(n+k)^2 :\ n \in \mathbb{Z}\}$ is an eigenvalue of $H^{(\sigma)}(k)$ iff $-1$ is an eigenvalue of $T_0^{(\sigma)}(z, k)\, T^{(\sigma)}(\overline{z}, k)^*$, see Lemma 4.7.8 in [Ya1]. (The hypothesis of strong $H^{(0)}(k)$-smoothness is implicitly verified in Lemma 3.4.4 below.) However, the operators $T_0^{(\sigma)}(z, k)\, T^{(\sigma)}(\overline{z}, k)^*$ have no reasonable limits as $\lambda \to (n+k)^2$, $n \in \mathbb{Z}$, so it seems to be impossible to decide within this general approach whether these points are eigenvalues or accumulation points of eigenvalues. This is the reason why we had to change our point of view and introduce the operators $B^{(\sigma)}(\lambda \pm i0, k)$ in this section.

Let us discuss how they are connected. Let $\lambda \neq (n+k)^2$, $n \in \mathbb{Z}$, and $f,\ g \in L_2(\mathbb{T})$. Then $B^{(0)}(\lambda \pm i0, k)^{-1/2} f$, $B^{(0)}(\lambda \mp i0, k)^{-1/2} g \in H^{1/2}(\mathbb{T})$ and

$$\begin{aligned} b^{(\sigma)}(\lambda \pm i0, k)\left[B^{(0)}(\lambda \pm i0, k)^{-1/2} f,\ B^{(0)}(\lambda \mp i0, k)^{-1/2} g\right] = \\ = (f,\ g) + \left(T_0^{(\sigma)}(\lambda \pm i0, k) f,\ T^{(\sigma)}(\lambda \mp i0, k) g\right). \end{aligned}$$

Replacing $f$ by $T^{(\sigma)}(\lambda \mp i0, k)^* f$ we see that

$$\begin{aligned} b^{(\sigma)}(\lambda \pm i0, k)\left[B^{(0)}(\lambda \pm i0, k)^{-1/2} T^{(\sigma)}(\lambda \mp i0, k)^* f,\ B^{(0)}(\lambda \mp i0, k)^{-1/2} g\right] = \\ = \left(\left(I + T_0^{(\sigma)}(\lambda \pm i0, k)\, T^{(\sigma)}(\lambda \mp i0, k)^*\right) f,\ T^{(\sigma)}(\lambda \mp i0, k) g\right). \end{aligned}$$

Since $B^{(0)}(\lambda \mp i0, k)^{-1/2}$ maps $L_2(\mathbb{T})$ bijectively onto $H^{1/2}(\mathbb{T})$, it follows that $B^{(0)}(\lambda \pm i0, k)^{-1/2} T^{(\sigma)}(\lambda \mp i0, k)^* f \in \mathcal{D}(B^{(\sigma)}(\lambda \pm i0, k))$ and that

$$\begin{aligned} B^{(\sigma)}(\lambda \pm i0, k) B^{(0)}(\lambda \pm i0, k)^{-1/2} T^{(\sigma)}(\lambda \mp i0, k)^* f = \\ = B^{(0)}(\lambda \pm i0, k)^{1/2} T^{(\sigma)}(\lambda \mp i0, k)^* \left(I + T_0^{(\sigma)}(\lambda \pm i0, k)\, T^{(\sigma)}(\lambda \mp i0, k)^*\right) f. \end{aligned} \tag{3.3.19}$$

In particular, we can prove now Remark 3.2.3.

**Lemma 3.3.17.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$, then*

$$\mathcal{N}_+(k) = \mathcal{N}_-(k) = \mathcal{N}(k) = \sigma_p\left(H^{(\sigma)}(k)\right) \setminus \{(n+k)^2 :\ n \in \mathbb{Z}\}. \tag{3.3.20}$$

*Proof.* The last equality in (3.3.20) is clear from Proposition 3.3.2. Moreover, we saw in the proof of Theorem 3.2.5 that $H^{(\sigma)}(k)$ is absolutely continuous on $[k^2, +\infty) \setminus (\{(n+k)^2 :\ n \in \mathbb{Z}\} \cup (\mathcal{N}_+(k) \cap \mathcal{N}_-(k)))$. Hence $\mathcal{N}(k) \subset \mathcal{N}_+(k) \cap$

$\mathcal{N}_-(k)$.
Conversely, if $\lambda \in \mathcal{N}_\pm(k)$ and $f \neq 0$ such that

$$\Big(I + T_0^{(\sigma)}(\lambda \pm i0, k)\, T^{(\sigma)}(\lambda \mp i0, k)^*\Big)\, f = 0, \tag{3.3.21}$$

then because of (3.3.19) $B^{(\sigma)}(\lambda \pm i0, k)g = 0$ where $g := B^{(0)}(\lambda \pm i0, k)^{-1/2} T^{(\sigma)}(\lambda \mp i0, k)^* f$. Now (3.3.21) implies $T^{(\sigma)}(\lambda \mp i0, k)^* f \neq 0$, so $g \neq 0$ and $\lambda \in \mathcal{N}(k)$. $\square$

Together with Lemma 3.2.2, this is another proof of the first statement of Lemma 3.3.6. (However, we need the second statement of Lemma 3.3.6 for the proof of Theorem 3.3.5 via Lemma 3.3.7.)

## 3.4 Scattering theory for the operators $H^{(\sigma)}(k)$

Our goal in this section is to study the scattering properties of the operator $H^{(\sigma)}(k)$ on the halfstrip. The comparison operator is $H^{(0)}(k)$.
We have already proved the existence and completeness of the wave operators in Section 1. Here we briefly summarize the results in Subsection 1. In Subsection 2 we turn our attention to the scattering matrix. We derive a representation formula from the general Stationary Scattering Theory. Finally, in Subsection 3 we touch the subject of generalized eigenfunctions. In particular, we express the scattering matrix in terms of the asymptotics of these solutions of the stationary Schrödinger equation (with quasimomentum). This provides a graphic picture of scattering close to the physical intuition.

**1. Existence and completeness of the wave operators.** Recall the abstract definition of the wave operators from Section 1.4. In case of existence we will use the notation

$$W_\pm^{(\sigma)}(k) := W_\pm(H^{(\sigma)}(k), H^{(0)}(k)), \qquad k \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

Theorem 3.1.3 shows that the difference of resolvents $R^{(\sigma)}(z,k) - R^{(0)}(z,k)$ is trace class. Hence the Birman-Kreĭn theorem yields

**Theorem 3.4.1.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$, then the wave operators $W_\pm^{(\sigma)}(k)$ on the halfstrip exist and are complete.*

Under the additional assumption $\sigma \geq 0$ we can say more. Indeed, the operators $H^{(\sigma)}(k)$ are absolutely continuous by Theorem 3.3.10. We obtain

**Theorem 3.4.2.** *Assume that $\sigma$ is non-negative and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then the wave operators $W_\pm^{(\sigma)}(k)$ on the halfstrip exist and are unitary.*

**2. The scattering matrix.** Let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Recall from Section 2.2 the direct integral decomposition $\mathfrak{h}(k)$ of $L_2(\Pi)$ with fiber spaces $\mathbb{C}^{N(\lambda,k)}$ where $H^{(0)}(k)$ acts diagonal.
The completeness of the wave operators $W_{\pm}^{(\sigma)}(k)$ implies the unitarity of the *scattering operator*

$$S^{(\sigma)}(k) := W_{+}^{(\sigma)}(k)^* W_{-}^{(\sigma)}(k).$$

Because of the intertwining property of the wave operators the scattering operator commutes with $H^{(0)}(k)$ and can be decomposed in the space $\mathfrak{h}(k)$. Namely, for a.e. $\lambda \in [k^2, +\infty)$ there is an operator $S^{(\sigma)}(\lambda, k)$ on $\mathbb{C}^{N(\lambda,k)}$, the *scattering matrix*, such that

$$(\Gamma^{(0)}(k) S^{(\sigma)}(k) u)(\lambda) = S^{(\sigma)}(\lambda, k)(\Gamma^{(0)}(k) u)(\lambda), \qquad \text{a.e. } \lambda \in [k^2, +\infty),$$

or equivalently

$$\big(S^{(\sigma)}(k) u, v\big)_{L_2(\Pi)} = \int_{k^2}^{\infty} \big(S^{(\sigma)}(\lambda, k)(\Gamma^{(0)}(k) u)(\lambda), (\Gamma^{(0)}(k) v)(\lambda)\big)_{\mathbb{C}^{N(\lambda,k)}}\, d\lambda$$

for all $u$, $v \in L_2(\Pi)$. Since $S^{(\sigma)}(k)$ is unitary, the scattering matrix $S^{(\sigma)}(\lambda, k)$ is unitary for a.e. $\lambda$.
In this definition the scattering matrix is determined only almost everywhere. All the properties of $S^{(\sigma)}(\lambda, k)$ discussed below are also only satisfies for a.e. $\lambda$, which is not mentioned each time.
Let us introduce for $\lambda \in [k^2, +\infty) \setminus \{(n+k)^2 : \ n \in \mathbb{Z}\}$ two auxiliary operators $Z_0^{(\sigma)}(\lambda, k)$, $Z^{(\sigma)}(\lambda, k) : L_2(\mathbb{T}) \to \mathbb{C}^{N(\lambda,k)}$,

$$(Z_0^{(\sigma)}(\lambda, k) f)_n := \frac{1}{\sqrt{\pi}\sqrt[4]{\lambda - (n+k)^2}} \Big((\operatorname{sgn}\sigma)\sqrt{|\sigma|} f\Big)^{\widehat{}}_n, \qquad (n+k)^2 < \lambda,$$
$$(Z^{(\sigma)}(\lambda, k) f)_n := \frac{1}{\sqrt{\pi}\sqrt[4]{\lambda - (n+k)^2}} \Big(\sqrt{|\sigma|} f\Big)^{\widehat{}}_n, \qquad (n+k)^2 < \lambda.$$

The adjoint operators are given by

$$Z_0^{(\sigma)}(\lambda, k)^* a := (\operatorname{sgn}\sigma)\sqrt{|\sigma|} \sum_{(n+k)^2 < \lambda} \frac{a_n}{\sqrt{\pi}\sqrt[4]{\lambda - (n+k)^2}}\, e_n,$$
$$Z^{(\sigma)}(\lambda, k)^* a := \sqrt{|\sigma|} \sum_{(n+k)^2 < \lambda} \frac{a_n}{\sqrt{\pi}\sqrt[4]{\lambda - (n+k)^2}}\, e_n$$

for $a \in \mathbb{C}^{N(\lambda,k)}$.
We are now ready to state the main result of this section. The operators $T_0^{(\sigma)}(z, k)$, $T^{(\sigma)}(z, k)$ and their boundary values for $z = \lambda \pm i0$ have been defined in Sections 1 and 2.

**Theorem 3.4.3.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $\lambda \in [k^2, +\infty) \setminus (\{(n+k)^2 : n \in \mathbb{Z}\} \cup \mathcal{N}(k))$. Then*

$$S^{(\sigma)}(\lambda, k) = I - 2\pi i Z^{(\sigma)}(\lambda, k) \left( I + T_0^{(\sigma)}(\lambda + i0, k)\, T^{(\sigma)}(\lambda - i0, k)^* \right)^{-1} Z_0^{(\sigma)}(\lambda, k)^*. \tag{3.4.1}$$

We will derive this result from Theorem 5.7.1' in [Ya1]. We need some preparations.
Consider a closed interval

$$\Lambda = [a, b] \subset [k^2, +\infty) \setminus \{(n+k)^2 :\ n \in \mathbb{Z}\}, \tag{3.4.2}$$

and the spectral projection $E^{(0)}(\lambda, k)$ of $H^{(0)}(k)$ onto $\Lambda$. Put

$$\chi_n(\xi) := \begin{cases} 1 & \text{if } \xi^2 \in [a - (n+k)^2, b - (n+k)^2], \\ 0 & \text{otherwise.} \end{cases}$$

(The dependence on $\Lambda$, $k$ is not reflected in this notation.) Using the results from Section 2.2 we see that

$$E^{(0)}(\lambda, k)u = \sum_{n \in \mathbb{Z}} e_n \otimes \mathcal{F}_c^*(\chi_n \mathcal{F}_c \hat{u}_n), \tag{3.4.3}$$

where $\mathcal{F}_c$ denotes the Fourier cosine transformation on $L_2(\mathbb{R}_+)$. The sum in (3.4.3) is actually a finite sum since $\chi_n \equiv 0$ if $(n+k)^2 > b$. Moreover, $\chi_n \mathcal{F}_c \hat{u}_n$ has bounded support, so $\mathcal{F}_c^*(\chi_n \mathcal{F}_c \hat{u}_n)$ is a smooth function on $\overline{\mathbb{R}_+}$. In particular, we have $E^{(0)}(\lambda, k)u \in \tilde{H}^1(\Pi)$ for any $u \in L_2(\Pi)$. It is easy to see that $E^{(0)}(\lambda, k)$ is bounded as operator from $L_2(\Pi)$ to $\tilde{H}^1(\Pi)$.
Now consider the operators $G_0$, $G$ from $L_2(\Pi)$ to $L_2(\mathbb{T})$ as introduced in the proof of Proposition 3.1.2,

$$\begin{aligned} \mathcal{D}(G_0) &:= \mathcal{D}(G) := \tilde{H}^1(\Pi), \\ (G_0 u)(x_1) &:= (\operatorname{sgn} \sigma(x_1)) \sqrt{|\sigma(x_1)|}\, u(x_1, 0), \qquad x_1 \in \mathbb{T}, \\ (Gu)(x_1) &:= \sqrt{|\sigma(x_1)|}\, u(x_1, 0), \qquad x_1 \in \mathbb{T}. \end{aligned}$$

We have remarked there that $G_0$, $G$ are bounded operators from $\tilde{H}^1(\Pi)$ to $L_2(\mathbb{T})$. Hence $G_0 E^{(0)}(\lambda, k)$, $G E^{(0)}(\lambda, k)$ are bounded operators from $L_2(\Pi)$ to $L_2(\mathbb{T})$.
The next result shows that the operators $Z_0^{(\sigma)}(\lambda, k)$, $Z^{(\sigma)}(\lambda, k)$ coincide with the operators $G_0$, $G$ in the direct integral decomposition $\mathfrak{h}(k)$. More precisely, we have

**Lemma 3.4.4.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $\Lambda$ as in (3.4.2). Then for $f \in L_2(\mathbb{T})$*

$$\begin{aligned} Z_0^{(\sigma)}(\lambda, k) f &= (\Gamma^{(0)}(k)(G_0 E^{(0)}(\lambda, k))^* f)(\lambda), \qquad \lambda \in \Lambda, \\ Z^{(\sigma)}(\lambda, k) f &= (\Gamma^{(0)}(k)(G E^{(0)}(\lambda, k))^* f)(\lambda), \qquad \lambda \in \Lambda. \end{aligned} \tag{3.4.4}$$

*Proof.* We prove only the second relation. Let us first calculate the adjoint operator $(GE^{(0)}(\lambda,k))^*$. We have for $u \in L_2(\Pi)$

$$
\begin{aligned}
(e_n \otimes \mathcal{F}_c^* \chi_n, u)_{L_2(\Pi)} &= \int_0^\infty \mathcal{F}_c^* \chi_n(x_2) \overline{\left( \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u(x_1,x_2) e^{-inx_1}\, dx_1 \right)}\, dx_2 = \\
&= \int_0^\infty \chi_n(\xi) \overline{(\mathcal{F}_c \hat{u}_n)(\xi)}\, d\xi = \int_0^\infty \overline{\mathcal{F}_c (E^{(0)}(\Lambda,k)u)\widehat{_n}(\xi)}\, d\xi = \\
&= \sqrt{\frac{\pi}{2}}\, \overline{(E^{(0)}(\Lambda,k)u)\widehat{_n}(0)},
\end{aligned}
$$

where we have used (3.4.3) in the next to last step and the inversion theorem for $\mathcal{F}_c$ in the last step.
Now let $f \in L_2(\mathbb{T})$ be bounded (so that $\sqrt{|\sigma|} f \in L_2(\mathbb{T})$), then by Parseval's equality

$$
\begin{aligned}
\left(f, GE^{(0)}(\lambda,k)u\right)_{L_2(\mathbb{T})} &= \left(\sqrt{|\sigma|} f, E^{(0)}(\lambda,k)u(.,0)\right)_{L_2(\mathbb{T})} = \\
&= \sum_{n\in\mathbb{Z}} (\sqrt{|\sigma|} f)\widehat{_n}\, \overline{(E^{(0)}(\Lambda,k)u)\widehat{_n}(0)} = \left( \sqrt{\frac{2}{\pi}} \sum_{n\in\mathbb{Z}} (\sqrt{|\sigma|} f)\widehat{_n}\, e_n \otimes \mathcal{F}_c^* \chi_n, u \right)_{L_2(\Pi)}.
\end{aligned}
$$

We remark again that the last sum is a finite sum. Thus for any $f \in L_2(\mathbb{T})$

$$
(GE^{(0)}(\lambda,k))^* f = \sqrt{\frac{2}{\pi}} \sum_{n\in\mathbb{Z}} (\sqrt{|\sigma|} f)\widehat{_n}\, e_n \otimes \mathcal{F}_c^* \chi_n.
$$

Now it is easy to prove (3.4.4). Indeed,

$$
\mathcal{F}_c((GE^{(0)}(\lambda,k))^* f)\widehat{_n} = \sqrt{\frac{2}{\pi}} (\sqrt{|\sigma|} f)\widehat{_n}\, \chi_n,
$$

and thus for $\lambda \in \Lambda$ and $(n+k)^2 < \lambda$

$$
\begin{aligned}
&(\Gamma^{(0)}(k)(GE^{(0)}(\lambda,k))^* f)(\lambda) = \\
&\quad = \frac{1}{\sqrt{2}\sqrt[4]{\lambda-(n+k)^2}} \mathcal{F}_c((GE^{(0)}(\lambda,k))^* f)\widehat{_n}(\sqrt{\lambda-(n+k)^2}) = \\
&\quad = Z^{(\sigma)}(\lambda,k) f
\end{aligned}
$$

since $\chi_n(\sqrt{\lambda-(n+k)^2}) = 1$. This proves the second relation in (3.4.4). $\square$

We can rewrite (3.4.4) as

$$
\begin{aligned}
(G_0 u, f)_{L_2(\mathbb{T})} &= \int_{k^2}^\infty \left( (\Gamma^{(0)}(k)u)(\lambda), Z_0^{(\sigma)}(\lambda,k) f \right)_{\mathbb{C}^{N(\lambda,k)}}\, d\lambda, \\
(G u, f)_{L_2(\mathbb{T})} &= \int_{k^2}^\infty \left( (\Gamma^{(0)}(k)u)(\lambda), Z^{(\sigma)}(\lambda,k) f \right)_{\mathbb{C}^{N(\lambda,k)}}\, d\lambda,
\end{aligned}
\tag{3.4.5}
$$

for $u \in \mathcal{R}(E^{(0)}(\Lambda, k))$ and $f \in L_2(\mathbb{T})$. Indeed, both sides (e.g. of the second equation) are equal to

$$(u, (GE^{(0)}(\Lambda, k))^* f)_{L_2(\Pi)} =$$
$$= \int_{k^2}^{\infty} \left( (\Gamma^{(0)}(k)u)(\lambda), (\Gamma^{(0)}(k)(GE^{(0)}(\lambda, k))^* f)(\lambda) \right)_{\mathbb{C}^{N(\lambda,k)}} \, d\lambda.$$

Of course, (3.4.5) holds for all $u \in \mathcal{R}(E^{(0)}(\tilde{\Lambda}, k))$ where $\tilde{\Lambda}$ a finite union of sets $\Lambda$ as above.

Now the proof of Theorem 3.4.3 is immediate.

*Proof (Theorem 3.4.3).* We apply Theorem 5.7.1' in [Ya1]. Assumptions (5.7.5) and (5.5.6) in [Ya1] have already been checked in Section 2. Our formulae (3.4.5) verify the defining relation (5.4.2) in [Ya1] for the operators $Z_0^{(\sigma)}(\lambda, k)$, $Z^{(\sigma)}(\lambda, k)$ ($Z_0(\lambda, G_0)$, $Z_0(\lambda, G)$ in Yafaev's notation). The representation (5.7.6) in [Ya1] coincides with our equation (3.4.1). □

**3. Generalized eigenfunctions.** Recall from Section 2.2 the definition of the functions $\psi_n^{(0)}(.; \lambda, k)$. These are generalized eigenfunctions in the unperturbed case $\sigma = 0$. The result in the general case is

**Proposition 3.4.5.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$, $\lambda \in [k^2, +\infty) \setminus (\{(m+k)^2 : m \in \mathbb{Z}\} \cup \mathcal{N}(k))$ and $n \in \mathbb{Z}$ such that $(n+k)^2 < \lambda$. Then there are functions $\psi_n^{\pm}(.; \lambda, k) \in \tilde{C}^{\infty}(\Pi)$ satisfying*

$$\left( (D_1 + k)^2 + D_2^2 \right) \psi_n^{\pm}(.; \lambda, k) = \lambda \psi_n^{\pm}(.; \lambda, k) \qquad \text{in } \Pi$$

*and*

$$\lim_{\epsilon \to 0+} \frac{\partial \psi_n^{\pm}(.; \lambda, k)}{\partial x_2} \big|_{x_2 = \epsilon} = \sigma \psi_n^{\pm}(.; \lambda, k) \big|_{x_2 = 0} \qquad \text{in } L_1(\mathbb{T}).$$

*Moreover, let*

$$a_{mn}(\lambda, k) := i\sqrt{\pi} \int_{-\pi}^{\pi} \psi_n^{-}(x_1, 0; \lambda, k) \sigma(x_1) \overline{\psi_m^{(0)}(x_1, 0; \lambda, k)} \, dx_1, \tag{3.4.6}$$

*then for some $\alpha > 0$*

$$\psi_n^{-}(x; \lambda, k) = \psi_n^{(0)}(x; \lambda, k) -$$
$$- \frac{1}{\sqrt{2\pi}} \sum_{(m+k)^2 < \lambda} a_{mn}(\lambda, k) \frac{e^{imx_1} e^{i\sqrt{\lambda - (m+k)^2}\, x_2}}{\sqrt[4]{\lambda - (m+k)^2}} + o(e^{-\alpha x_2}) \qquad (x_2 \to \infty), \tag{3.4.7}$$

*uniformly in $x_1$.*

Of course, the functions $\psi_n^\pm(.;\lambda,k)$ depend on $\sigma$. This is not reflected in the notation. For $\sigma=0$ we obtain again the functions $\psi_n^{(0)}(.;\lambda,k)$.
We emphasize that the functions may not be continuous up to the boundary $[-\pi,\pi]\times\{0\}$.
Let us construct $\psi_n^\pm(.;\lambda,k)$ as solutions of an equation of Lippman-Schwinger type (see the remarks in Subsection 4).

**Lemma 3.4.6.** *Let $k$, $\lambda$, $n$ as above. Then the equations*

$$\begin{aligned}\psi_n^\pm(x;\lambda,k) &= \psi_n^{(0)}(x;\lambda,k)-\\ &-\int_{-\pi}^{\pi} r^{(0)}(x,y_1,0;\lambda\mp i0,k)\sigma(y_1)\psi_n^\pm(y_1,0;\lambda,k)\,dy_1, \qquad x\in\Pi,\end{aligned} \tag{3.4.8}$$

*have unique solutions $\psi_n^\pm(.;\lambda,k)$ satisfying*

$$\sqrt{|\sigma|}\,\psi_n^\pm(.,0;\lambda,k)\in L_2(\mathbb{T}). \tag{3.4.9}$$

*Proof.* With slight abuse of notation, we put $\sqrt{\sigma}:=(\operatorname{sgn}\sigma)\sqrt{|\sigma|}$. Since the function $\sqrt{\sigma}\,\psi_n^{(0)}(.,0;\lambda,k)$ belongs to $L_2(\mathbb{T})$ we can define for $\lambda$ as above

$$\varphi_n^\pm(.;\lambda,k):=\Big(I+T_0^{(\sigma)}(\lambda\mp i0,k)\,T^{(\sigma)}(\lambda\pm i0,k)^*\Big)^{-1}(\sqrt{\sigma}\,\psi_n^{(0)}(.,0;\lambda,k)).$$

Using that $T_0^{(\sigma)}(\lambda\mp i0,k)\,T^{(\sigma)}(\lambda\pm i0,k)^*$ is an integral operator with kernel

$$\sqrt{\sigma(x_1)}\,r^{(0)}(x_1,0,y_1,0;\lambda\mp i0,k)\sqrt{|\sigma(y_1)|},\qquad x_1,\ y_1\in\mathbb{T},$$

we see that

$$\begin{aligned}\varphi_n^\pm(x_1;\lambda,k) &= \sqrt{\sigma(x_1)}\,\psi_n^{(0)}(x_1,0;\lambda,k)-\\ &-\int_{-\pi}^{\pi}\sqrt{\sigma(x_1)}\,r^{(0)}(x_1,0,y_1,0;\lambda\mp i0,k)\sqrt{|\sigma(y_1)|}\,\varphi_n^\pm(y_1;\lambda,k)\,dy_1, \quad x_1\in\mathbb{T}.\end{aligned} \tag{3.4.10}$$

Now define

$$\begin{aligned}\psi_n^\pm(x;\lambda,k) &:= \psi_n^{(0)}(x;\lambda,k)-\\ &-\int_{-\pi}^{\pi} r^{(0)}(x,y_1,0;\lambda\mp i0,k)\sqrt{|\sigma(y_1)|}\,\varphi_n^\pm(y_1;\lambda,k)\,dy_1, \qquad x\in\Pi,\end{aligned}$$

It follows from (3.4.10) that

$$\varphi_n^\pm(x_1;\lambda,k)=\sqrt{\sigma(x_1)}\,\psi_n^\pm(x_1,0;\lambda,k),\qquad x_1\in\mathbb{T},$$

so $\psi_n^\pm(.;\lambda,k)$ satisfies (3.4.8) and (3.4.9).
The uniqueness of these solutions follows from the invertibility of $I+T_0^{(\sigma)}(\lambda\mp i0,k)\,T^{(\sigma)}(\lambda\pm i0,k)^*$. $\square$

To derive the properties of the functions $\psi_n^\pm(.;\lambda,k)$ we shall study the equation (3.4.8). We need the following result on the free resolvent kernel. Recall that for $\lambda \neq (n+k)^2$, $n \in \mathbb{Z}$,

$$r^{(0)}(x,y_1,0;\lambda\mp i0,k) = \mp\frac{i}{2\pi}\sum_{(n+k)^2<\lambda}\frac{1}{\sqrt{\lambda-(n+k)^2}}\,e^{in(x_1-y_1)}e^{\mp i\sqrt{\lambda-(n+k)^2}\,x_2}+$$
$$+\frac{1}{2\pi}\sum_{(n+k)^2>\lambda}\frac{1}{\sqrt{(n+k)^2-\lambda}}\,e^{in(x_1-y_1)}e^{-\sqrt{(n+k)^2-\lambda}\,x_2}, \qquad x\in\Pi,\ y_1\in\mathbb{T}. \tag{3.4.11}$$

**Lemma 3.4.7.** *Let $k\in[-\frac{1}{2},\frac{1}{2}]$, $\lambda\in[k^2,+\infty)\setminus\{(n+k)^2:\ n\in\mathbb{Z}\}$ and $f\in L_1(\mathbb{T})$. Let*

$$u^\pm(x) := \int_{-\pi}^{\pi} r^{(0)}(x,y_1,0;\lambda\mp i0,k)f(y_1)\,dy_1, \qquad x\in\Pi.$$

*Then $u^\pm\in\tilde{C}^\infty(\Pi)$,*

$$\left((D_1+k)^2+D_2^2\right)u^\pm=\lambda u^\pm \qquad \text{in } \Pi$$

*and*

$$\lim_{\epsilon\to 0+}\frac{\partial u^\pm}{\partial x_2}\Big|_{x_2=\epsilon} = -f \qquad \text{in } L_1(\mathbb{T}). \tag{3.4.12}$$

*Moreover, for some $\alpha>0$*

$$u^\pm(x) = \mp\frac{i}{\sqrt{2\pi}}\sum_{(n+k)^2<\lambda}\frac{\hat{f}_n}{\sqrt{\lambda-(n+k)^2}}\,e^{inx_1}e^{\mp i\sqrt{\lambda-(n+k)^2}\,x_2}+o(e^{-\alpha x_2}) \tag{3.4.13}$$
$$(x_2\to\infty),$$

*uniformly in $x_1$.*

The main technical difficulty lies in the verification of (3.4.12). It consists in making the formal identity $\frac{\partial r^{(0)}(x_1,0,y_1,0;\lambda\mp i0,k))}{\partial x_2} = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}e^{in(x_1-y_1)} = \delta(x_1-y_1)$ precise.

*Proof.* The series (3.4.11) converges uniformly and absolutely on $\{(x,y_1)\in\overline{\Pi}\times[-\pi,\pi]:\ x_2\geq\epsilon\}$ for every $\epsilon>0$. Thus we can interchange integration and summation, and obtain

$$u^\pm(x) = \mp\frac{i}{\sqrt{2\pi}}\sum_{(n+k)^2<\lambda}\frac{\hat{f}_n}{\sqrt{\lambda-(n+k)^2}}\,e^{inx_1}e^{\mp i\sqrt{\lambda-(n+k)^2}\,x_2}+$$
$$+\frac{1}{\sqrt{2\pi}}\sum_{(n+k)^2>\lambda}\frac{\hat{f}_n}{\sqrt{(n+k)^2-\lambda}}\,e^{inx_1}e^{-\sqrt{(n+k)^2-\lambda}\,x_2}, \qquad x_2\geq\epsilon.$$

It is easy to see from this that $u^\pm$ are infinitely smooth on $[-\pi,\pi]\times[\epsilon,+\infty)$ and, together with all its derivatives, periodic with respect to $x_1$. Also, the equations $((D_1+k)^2+D_2^2)\,u^\pm=\lambda u^\pm$ and the asymptotics (3.4.13) are clear.
Let us prove (3.4.12). We consider the "-"-case only, the other being similar. For $\epsilon>0$ define an operator $T_\epsilon: L_1(\mathbb{T})\to L_1(\mathbb{T})$,

$$T_\epsilon g := \sum_{(n+k)^2<\lambda} \hat{g}_n e^{i\epsilon\sqrt{\lambda-(n+k)^2}} e_n + \sum_{(n+k)^2>\lambda} \hat{g}_n e^{-\epsilon\sqrt{(n+k)^2-\lambda}} e_n,$$

so that

$$\frac{\partial u^-}{\partial x_2}|_{x_2=\epsilon} = -T_\epsilon f.$$

Of course, we have $T_\epsilon g\to g$ in $L_1(\mathbb{T})$ $(\epsilon\to 0)$ for every trigonometric polynomial $g$, so to conclude $T_\epsilon\to I$ strongly (i.e., (3.4.12)) it suffices to show

$$\sup_{\epsilon\in(0,1]} \|T_\epsilon\| < \infty. \tag{3.4.14}$$

Let $\psi_\epsilon := \sum_{(n+k)^2<\lambda} e^{i\epsilon\sqrt{\lambda-(n+k)^2}} e_n + \sum_{(n+k)^2>\lambda} e^{-\epsilon\sqrt{(n+k)^2-\lambda}} e_n$, then

$$(T_\epsilon g)\widehat{}_n = \hat{g}_n (\psi_\epsilon)\widehat{}_n = \frac{1}{\sqrt{2\pi}} (g*\psi_\epsilon)\widehat{}_n, \qquad n\in\mathbb{Z},$$

where $(g*\psi_\epsilon)(x_1) := \int_{-\pi}^{\pi} g(y_1)\psi_\epsilon(x_1-y_1)\,dy_1,\ x_1\in\mathbb{T}$. Hence

$$\|T_\epsilon g\|_{L_1(\mathbb{T})} = \frac{1}{\sqrt{2\pi}}\|g*\psi_\epsilon\|_{L_1(\mathbb{T})} \le \frac{1}{\sqrt{2\pi}}\|g\|_{L_1(\mathbb{T})}\|\psi_\epsilon\|_{L_1(\mathbb{T})}, \tag{3.4.15}$$

and so for (3.4.14) it suffices to show

$$\sup_{\epsilon\in(0,1]} \|\psi_\epsilon\|_{L_1(\mathbb{T})} < \infty. \tag{3.4.16}$$

Let us write

$$\begin{aligned}
\psi_\epsilon = &\sum_{(n+k)^2<\lambda} \left(e^{i\epsilon\sqrt{\lambda-(n+k)^2}} - e^{-\epsilon|n+k|}\right) e_n + \\
&+ \sum_{(n+k)^2>\lambda} \left(e^{-\epsilon\sqrt{\lambda-(n+k)^2}} - e^{-\epsilon|n+k|}\right) e_n + \\
&+ \sum_{n\in\mathbb{Z}} e^{-\epsilon|n+k|} e_n \\
=: &\ \psi_\epsilon^{(1)} + \psi_\epsilon^{(2)} + \psi_\epsilon^{(3)}.
\end{aligned}$$

Of course, $\|\psi_\epsilon^{(1)}\|_{L_1(\mathbb{T})}$ is bounded uniformly in $\epsilon\in(0,1]$. Using the estimate

$$\left|e^{-\epsilon\sqrt{\lambda-(n+k)^2}} - e^{-\epsilon|n+k|}\right| \le \frac{\epsilon\lambda}{2\sqrt{(n+k)^2-\lambda}}\, e^{-\epsilon\sqrt{(n+k)^2-\lambda}}$$

(which is established as in the proof of Proposition 2.2.1), we see that $\psi_\epsilon^{(2)} \to 0$ in $L_2(\mathbb{T})$ as $\epsilon \to 0$. In particular, $\|\psi_\epsilon^{(2)}\|_{L_1(\mathbb{T})}$ is bounded uniformly in $\epsilon \in (0,1]$. Thus, for (3.4.16) it suffices to show

$$\sup_{\epsilon\in(0,1]} \|\psi_\epsilon^{(3)}\|_{L_1(\mathbb{T})} < \infty. \tag{3.4.17}$$

It is convenient to transfer this problem from $\mathbb{T}$ to $\mathbb{R}$. Let $\varphi_\epsilon(x_1) := \sqrt{\frac{2}{\pi}}\frac{\epsilon}{\epsilon^2+x_1^2}\, e^{ikx_1}$, $x_1 \in \mathbb{R}$. Then

$$\|\varphi_\epsilon\|_{L_1(\mathbb{R})} = \|\varphi_1\|_{L_1(\mathbb{R})} < \infty \tag{3.4.18}$$

and

$$\hat{\varphi}_\epsilon(\xi) = e^{-\epsilon|\xi-k|}, \qquad \xi \in \mathbb{R}.$$

It follows from the Poisson summation formula that

$$\psi_\epsilon^{(3)}(x_1) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \hat{\varphi}_\epsilon(n) e^{inx_1} = \sum_{n\in\mathbb{Z}} \varphi_\epsilon(x_1 - 2\pi n), \qquad x_1 \in \mathbb{T},$$

so by (3.4.18) we conclude that

$$\|\psi_\epsilon^{(3)}\|_{L_1(\mathbb{T})} \le \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} |\varphi_\epsilon(x_1 - 2\pi n)|\, dx_1 = \|\varphi_1\|_{L_1(\mathbb{R})},$$

which proves (3.4.17) $\square$

*Remark* 3.4.8. We note that if $f \in L_p(\mathbb{T})$ for some $1 \le p < \infty$, then the convergence (3.4.12) holds in $L_p(\mathbb{T})$. Indeed, we only have to consider $T_\epsilon$ in the above proof as operator from $L_p(\mathbb{T})$ to $L_p(\mathbb{T})$ and use Young's inequality in (3.4.15).

Now Proposition 3.4.5 follows easily.

*Proof (Proposition 3.4.5).* Consider the functions $\psi_n^\pm(.;\lambda,k)$ as constructed in Lemma 3.4.6 and apply Lemma 3.4.7 with $f := \sigma\psi_n^\pm(.,0;\lambda,k) \in L_1(\mathbb{T})$. We note that in the "-"-case

$$\hat{f}_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \psi_n^-(x_1, 0; \lambda, k)\sigma(x_1) e^{-imx_1}\, dx_1 = -i\sqrt[4]{\lambda - (m+k)^2}\, a_{mn}(\lambda,k). \tag{3.4.19}$$

Now all the assertions follow immediately from the Lippmann-Schwinger type equation (3.4.8). $\square$

We can think of the numbers $a_{mn}(\lambda,k)$ as "scattering amplitudes". Their connection with the scattering matrix is given by the following

**Proposition 3.4.9.** *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$, $\lambda \in [k^2, +\infty)\backslash(\{(n+k)^2 : n \in \mathbb{Z}\} \cup \mathcal{N}(k))$. Then for $b \in \mathbb{C}^{N(\lambda,k)}$*

$$\left(S^{(\sigma)}(\lambda,k)\, b\right)_m = b_m - 2\sqrt{\pi} \sum_{(n+k)^2<\lambda} a_{mn}(\lambda,k)\, b_n, \qquad (m+k)^2 < \lambda. \tag{3.4.20}$$

*Proof.* We denote by $e^n$ the $n$-th canonical unit vector in $\mathbb{C}^{N(\lambda,k)}$. Then

$$Z_0^{(\sigma)}(\lambda,k)^* e^n = \sqrt{\sigma}\frac{1}{\sqrt{\pi}\sqrt[4]{\lambda-(n+k)^2}}\, e_n = \sqrt{\sigma}\,\psi_n^{(0)}(.,0;\lambda,k).$$

In the notation of Lemma 3.4.6 we have

$$\varphi_n^-(.;\lambda,k) = \Big(I + T_0^{(\sigma)}(\lambda+i0,k)\,T^{(\sigma)}(\lambda-i0,k)^*\Big)^{-1} Z_0^{(\sigma)}(\lambda,k)^* e^n,$$

and so because of the representation formula (3.4.1) for the scattering matrix

$$Z^{(\sigma)}(\lambda,k)\varphi_n^-(.;\lambda,k) = \frac{1}{2\pi i}(I - S^{(\sigma)}(\lambda,k))e^n.$$

Using $\varphi_n^-(.;\lambda,k) = \sqrt{\sigma}\psi_n^-(.,0;\lambda,k)$ and (3.4.19) we can rewrite the LHS as

$$\begin{aligned}\big(Z^{(\sigma)}(\lambda,k)\varphi_n^-(.;\lambda,k)\big)_m &= \frac{1}{\sqrt{\pi}\sqrt[4]{\lambda-(m+k)^2}}(\sigma\psi_n^-(.,0;\lambda,k))\widehat{\ }_m = \\ &= \frac{-i}{\sqrt{\pi}}\,a_{mn}(\lambda,k).\end{aligned}$$

This proves (3.4.20) in the case $b = e^n$. The general case follows by linearity. □

**Example 3.4.10.** Suppose that $\sigma \in \mathbb{R}$ is constant. Then the equations (3.4.8) of Lippmann-Schwinger-type can be solved explicitly and one obtains

$$\begin{aligned}\psi_n^{\pm}(x;\lambda,k) = {} & \frac{1}{\sqrt{2\pi}}e^{inx_1}\frac{1}{\sqrt{\pi}\sqrt[4]{\lambda-(n+k)^2}}\times \\ & \times\left(\cos(\sqrt{\lambda-(n+k)^2}\,x_2) \mp \frac{\sigma}{i\sqrt{\lambda-(n+k)^2}\pm\sigma}e^{\mp i\sqrt{\lambda-(n+k)^2}\,x_2}\right), \quad x \in \Pi.\end{aligned}$$

In particular, we find

$$a_{mn}(\lambda,k) = -\frac{1}{\sqrt{\pi}}\,\frac{\sigma}{i\sqrt{\lambda-(n+k)^2}-\sigma}\,\delta_{nm}$$

and for $b \in \mathbb{C}^{N(\lambda,k)}$

$$\big(S^{(\sigma)}(\lambda,k)\,b\big)_m = \sum_{(n+k)^2<\lambda}\frac{i\sqrt{\lambda-(n+k)^2}+\sigma}{i\sqrt{\lambda-(n+k)^2}-\sigma}\,b_n, \qquad (m+k)^2<\lambda.$$

**4. Comments and Remarks. Generalized eigenfunctions.** The idea of connecting scattering theory with generalized eigenfunctions goes back to A. Ya. Povzner and T. Ikebe, see Subsection 1.3 in [Ya2] for a (refined) presentation. We remark the analogy of the formulae (1.16), (1.17), (1.19), (1.24) in [Ya2] with our formulae (3.4.7), (3.4.6), (3.4.8), (3.4.20), respectively: Up to numerical constants it is only necessary to replace $V$ by $\sigma \otimes \delta$ and to think of the scattering amplitude not as a function of the directions $\theta$, $\omega \in \mathbb{S}^1$ of the incoming and outgoing wave, but as a function of the channel numbers $n$, $m \in \mathbb{Z}$ of the incoming and outgoing wave. For fixed energy $\lambda$ and quasimomentum $k$ only the open channels (i.e. the numbers $n \in \mathbb{Z}$ such that $(n+k)^2 \leq \lambda$) are involved in the scattering process. See also Chap. XVIII in [LL] for the physical interpretation.

We have touched the subject of generalized eigenfunctions only briefly. It is possible to prove that the definition

$$(\Gamma^{\pm}(k)u)_n(\lambda) := \int_{\Pi} u(x)\overline{\psi_n^{\pm}(x;\lambda,k)}\,dx, \qquad (n+k)^2 < \lambda,$$

for $u \in \mathcal{S}(\Pi)$ can be extended to a partially isometric, surjective operator $\Gamma^{\pm}(k) : L_2(\Pi) \to \mathfrak{h}(k)$ with initial space $\mathcal{R}(P_{ac}^{(\sigma)}(k))$, which diagonalizes $H^{(\sigma)}(k)$. Moreover, a proof for the existence and completeness of the wave operators can be obtained in this way without using the trace class property of the resolvent difference.

# Chapter 4

# Spectral analysis and scattering theory of the operators $H^{(\sigma)}$ on the halfplane

Throughout this chapter, we assume without further comment that $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is a periodic function such that for some $1 < q \leq \infty$

$$\sigma \in L_q(\mathbb{T}).$$

## 4.1 Scattering theory for the operators $H^{(\sigma)}$

Our goal in this section is to study the scattering properties of the operators $H^{(\sigma)}$ on the halfplane.
In Subsection 1 the comparison operator is $H^{(0)}$ and we will deduce existence of the wave operators from the results on the halfstrip using the Gelfand transformation. In Subsection 2 we will comment briefly on other natural choices of comparison operators.

**1. Existence of the wave operators.** We will denote the wave operators on the halfplane in case of existence by

$$W_{\pm}^{(\sigma)} := W_{\pm}(H^{(\sigma)}, H^{(0)}).$$

Moreover, $P_{ac}^{(\sigma)}(k)$ will denote the projection in $L_2(\Pi)$ onto the absolutely continuous subspace of $H^{(\sigma)}(k)$. Recall the definition of the Gelfand transformation $\mathcal{U}$ in Section 1.3.

**Theorem 4.1.1.** *The wave operators $W_{\pm}^{(\sigma)}$ on the halfplane exist and satisfy*

$$\mathcal{U}\, W_{\pm}^{(\sigma)}\, \mathcal{U}^* = \int_{-1/2}^{1/2} \bigoplus W_{\pm}^{(\sigma)}(k)\, dk. \tag{4.1.1}$$

*In particular,*

$$\mathcal{R}(W_+^{(\sigma)}) = \mathcal{R}(W_-^{(\sigma)}) = \int_{-1/2}^{1/2} \bigoplus \mathcal{R}(P_{ac}^{(\sigma)}(k))\, dk. \tag{4.1.2}$$

*Proof.* The RHS of (4.1.1) makes sense, since the family $W_\pm^{(\sigma)}(k)$, $k \in [-\frac{1}{2}, \frac{1}{2}]$, is measurable. Indeed, the unitary groups $U^{(\sigma)}(t,k) := \exp(-itH^{(\sigma)}(k))$ and $U^{(0)}(t,k) := \exp(-itH^{(0)}(k))$ are measurable in $k$ as bounded functions of the measurable $H^{(\sigma)}(k)$ and $H^{(0)}(k)$, and so the strong limit of their product is also measurable.
Put $U^{(\sigma)}(t) := \exp(-itH^{(\sigma)})$ and $U^{(0)}(t) := \exp(-itH^{(0)})$, then we know from Theorem XIII.85 in [RS4] that

$$\mathcal{U}\, U^{(\sigma)}(t)\, \mathcal{U}^* = \exp(-it\mathcal{U}\, H^{(\sigma)}\, \mathcal{U}^*) = \int_{-1/2}^{1/2} \bigoplus U^{(\sigma)}(t,k)\, dk,$$

$$\mathcal{U}\, U^{(0)}(t)\, \mathcal{U}^* = \exp(-it\mathcal{U}\, H^{(0)}\mathcal{U}^*) = \int_{-1/2}^{1/2} \bigoplus U^{(0)}(t,k)\, dk.$$

Hence for any $u \in L_2(\mathbb{R}^2_+)$

$$\left\| \mathcal{U}\, U^{(\sigma)}(-t) U^{(0)}(t) u - \left( \int_{-1/2}^{1/2} \bigoplus W_\pm^{(\sigma)}(k)\, dk \right) \mathcal{U} u \right\|^2 =$$

$$= \int_{-1/2}^{1/2} \left\| U^{(\sigma)}(-t,k) U^{(0)}(t,k)\, (\mathcal{U}u)(k,.) - W_\pm^{(\sigma)}(k)\, (\mathcal{U}u)(k,.) \right\|^2 dk \to 0$$

$$(t \to \pm\infty)$$

by Lebesgue's theorem. This means that the strong limits of $U^{(\sigma)}(-t)U^{(0)}(t)$ for $t \to \pm\infty$ exist and coincide with $\mathcal{U}^* \, (\int_{-1/2}^{1/2} \bigoplus W_\pm^{(\sigma)}(k)\, dk)\, \mathcal{U}$, which proves the Theorem. ☐

In the case $\sigma \geq 0$ we can say more. To avoid repetitions we postpone this to the next section.

**2. Scattering with other comparison operators.** In the previous subsection we have considered the scattering problem for $H^{(\sigma)}$ with respect to the comparison operator $H^{(0)}$, the Neumann Laplacian on the halfplane. Similar results hold also with other natural choices of comparison operators, namely the Dirichlet Laplacian on the halfplane and the Laplacian on the whole plane.
To see this it suffices to consider the scattering problem between these operators and $H^{(0)}$. This can studied in terms of the asymptotic behavior of the unitary groups, which are given explicitly after Fourier transformation. We do not go into details.

## 4.2 Spectral analysis of the operators $H^{(\sigma)}$

Our goal in this section is to analyze the spectrum of the operators $H^{(\sigma)}$ on the halfplane.
In Subsection 1 we prove a complete result in the case of non-negative $\sigma$, namely unitary equivalence of $H^{(\sigma)}$ and $H^{(0)}$. In the case of general $\sigma$, generically another part of $H^{(\sigma)}$ exists and is responsible for negative spectrum of $H^{(\sigma)}$. This is discussed in Subsection 2, where we establish a typical phenomenon of periodic operators, a gap in the spectrum.

**1. Main result in the case $\sigma \geq 0$.** Our main result is

**Theorem 4.2.1.** *Assume that $\sigma \geq 0$. Then the wave operators $W_\pm^{(\sigma)}$ exist, are unitary and satisfy*

$$H^{(\sigma)} = W_\pm^{(\sigma)} H^{(0)} W_\pm^{(\sigma)*}. \tag{4.2.1}$$

*In particular, $H^{(\sigma)}$ is absolutely continuous with spectrum $[0,+\infty)$.*

*Proof.* By Theorem 4.1.1, together with Theorem 3.3.10, the wave operators exist and are surjective, so they are unitary. Thus (4.2.1) follows from the intertwining property. ☐

It is clear that in the case of arbitrary $\sigma$ there is the part of $H^{(\sigma)}$ on $\mathcal{R}(W_\pm^{(\sigma)})$ which is unitarily equivalent to $H^{(0)}$, but there may be also a non-trivial part of $H^{(\sigma)}$ on $\mathcal{R}(W_\pm^{(\sigma)})^\perp$, which is due to the eigenvalues of the operators $H^{(\sigma)}(k)$ on the halfstrip. This part can be bounded (e.g., if $\sigma$ is a trigonometric polynomial of degree $N > 1$, see Corollary 3.3.11) or unbounded (e.g., if $\sigma$ is a negative constant, see Example 3.3.8). If it has non-empty absolutely continuous spectrum, then the wave operators are not complete. Usually, the part of $H^{(\sigma)}$ on $\mathcal{R}(W_\pm^{(\sigma)})^\perp$ will have a negative spectrum. This is studied in the next subsection.

**2. The negative spectrum of $H^{(\sigma)}$.** For $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $n \in \mathbb{N}$ we put

$$\begin{aligned} &\lambda_n(k) := \\ &\quad = \sup_{v_1,\dots,v_{n-1}\in L_2(\Pi)} \inf\left\{\tfrac{h^{(\sigma)}(k)[u]}{\|u\|^2} :\ u \in \tilde{H}^1(\Pi),\ (u, v_j) = 0 \text{ for } 1 \leq j < n\right\}. \end{aligned} \tag{4.2.2}$$

(The dependence on $\sigma$ is not reflected in this notation.)
The variational principle, together with Theorem 3.3.5, implies that for each $k$ only finitely many of the $\lambda_n(k)$, $n \in \mathbb{N}$, are different form $k^2$, and these are exactly the eigenvalues of $H^{(\sigma)}(k)$ below $k^2$. It is easy to see that $\lambda_n$ is a continuous function on $[-\frac{1}{2}, \frac{1}{2}]$ for each $n \in \mathbb{N}$. Combining this with Corollary 2.1.4 we see

$$\sigma\left(H^{(\sigma)}\right) = \bigcup_{n\in\mathbb{N}} \lambda_n([-\tfrac{1}{2}, \tfrac{1}{2}]) \cup [0,+\infty), \tag{4.2.3}$$

i.e., the spectrum of $H^{(\sigma)}$ has band structure.
Moreover, one easily sees that (4.2.3) coincides with the essential spectrum of $H^{(\sigma)}$.
The following result is in some sense a converse of our main Theorem 4.2.1. it says essentially that $H^{(\sigma)}$ is not unitarily equivalent to $H^{(0)}$ if $\sigma$ is "sufficiently negative".

**Proposition 4.2.2.** *Assume that either*

$$\int_{-\pi}^{\pi} \sigma(x_1)\, dx_1 < 0$$

*or*

$$\sigma(x_1) \leq -\frac{\pi}{|\Lambda|}, \qquad x_1 \in \Lambda,$$

*for a non-empty connected open subset $\Lambda \subset \mathbb{T}$ of the torus. Then*

$$\sigma_{ess}\left(H^{(\sigma)}\right) \supsetneq [0, +\infty).$$

*Proof.* Because of (4.2.3) it suffices to prove that there exists a $k \in [-\frac{1}{2}, \frac{1}{2}]$ such that $H^{(\sigma)}(k)$ has a negative eigenvalue. This follows from the assumptions and Propositions 3.3.14, 3.3.15 respectively. ☐

*Remark* 4.2.3. It follows from what was stated in Remark 3.3.16 that the conclusion of Proposition 4.2.2 remains valid under the assumption $\int_{-\pi}^{\pi} \sigma(x_1)\, dx_1 = 0$, $\sigma \not\equiv 0$.

We turn now to the question whether the negative spectrum of $H^{(\sigma)}$ has a gap. For $k \in [-\frac{1}{2}, \frac{1}{2}]$ we introduce self-adjoint operators $H_D^{(\sigma)}(k)$, $H_N^{(\sigma)}(k)$ in $L_2(\Pi)$ via the quadratic forms

$$\begin{aligned}
\mathcal{D}[h_D^{(\sigma)}(k)] &:= \{u \in H^1(\Pi) : \ u(.,\pi) = u(.,-\pi) = 0\}, \\
\mathcal{D}[h_N^{(\sigma)}(k)] &:= H^1(\Pi), \\
h_\nu^{(\sigma)}(k)[u] &:= \int_\Pi \left(|(D_1 + k)u(x)|^2 + |D_2 u(x)|^2\right)\, dx + \int_{-\pi}^{\pi} \sigma(x_1)|u(x_1,0)|^2\, dx_1, \\
&\quad \nu \in \{D, N\}.
\end{aligned}$$

Closedness and lower semiboundedness is established as in Section 2.1. The operators $H_D^{(\sigma)}(k)$, $H_N^{(\sigma)}(k)$ differ from $H^{(\sigma)}(k)$ by Dirichlet and natural boundary conditions, respectively, at $x_1 = \pm\pi$.
Define the numbers $\lambda_n^\nu$, $n \in \mathbb{N}$, by (4.2.2) with $k = 0$ and $\tilde{H}^1(\Pi)$ replaced by $\mathcal{D}[h_\nu^{(\sigma)}(0)]$ for $\nu \in \{D, N\}$. A sufficient condition for the existence of a gap is given by

**Proposition 4.2.4.** *Let $n \in \mathbb{N}$ such that $\lambda_n^D < \lambda_{n+1}^N$. Then*

$$\sigma\left(H^{(\sigma)}\right) \cap (\lambda_n^D, \lambda_{n+1}^N) = \emptyset.$$

*Proof.* By the inclusions of the form domains we have $H_N^{(\sigma)}(k) \leq H^{(\sigma)}(k) \leq H_D^{(\sigma)}(k)$, so by the variational principle

$$\lambda_n^N(k) \leq \lambda_n(k) \leq \lambda_n^D(k), \tag{4.2.4}$$

where $\lambda_n^\nu(k)$ are defined by (4.2.2) with $\tilde{H}^1(\Pi)$ replaced by $\mathcal{D}[h_\nu^{(\sigma)}(k)]$ for $\nu \in \{D, N\}$. The unitary operator $E_k$ on $L_2(\Pi)$, $(E_k u)(x) := e^{-ikx_1}u(x)$, $x \in \Pi$, satisfies $H_\nu^{(\sigma)}(k) = E_k H_\nu^{(\sigma)}(0)E_k^*$, $\nu \in \{D, N\}$, whence $\lambda_n^\nu(k) = \lambda_n^\nu$ for all $k \in [-\frac{1}{2}, \frac{1}{2}]$. Now (4.2.4) implies

$$\max_{k\in[-\frac{1}{2},\frac{1}{2}]} \lambda_n(k) \leq \lambda_n^D < \lambda_{n+1}^N \leq \min_{k\in[-\frac{1}{2},\frac{1}{2}]} \lambda_{n+1}(k),$$

which proves the Proposition. $\square$

We remark that in the case of constant $\sigma < 0$ it happens that

$$\max_{k\in[-\frac{1}{2},\frac{1}{2}]} \lambda_n(k) = \lambda_n^D = \lambda_{n+1}^N = \min_{k\in[-\frac{1}{2},\frac{1}{2}]} \lambda_{n+1}(k).$$

Let us conclude this subsection with an example where the first gap of $H^{(\sigma)}$ is open.

**Example 4.2.5.** Let $a$, $b \in \mathbb{R}$, $-\pi < c < \pi$ and

$$\sigma(x_1) := \begin{cases} a & \text{if } x_1 \in [-\pi, c), \\ b & \text{if } x_1 \in [c, \pi). \end{cases}$$

We claim that under the assumptions

$$a < -\tfrac{\pi}{\pi+c}, \qquad b \geq 0, \tag{4.2.5}$$

one has

$$\max_{k\in[-\frac{1}{2},\frac{1}{2}]} \lambda_1(k) < \min_{k\in[-\frac{1}{2},\frac{1}{2}]} \lambda_2(k), \tag{4.2.6}$$

with $\lambda_n(k)$ defined by (4.2.2).
Indeed, applying Proposition 3.3.15 with $\Lambda = (-\pi, c)$ and $\delta = k^2 + a^2 - (\frac{\pi}{\pi+c})^2$ (which is positive by (4.2.5)) we find that $\lambda_1(k)$ is an eigenvalue of $H^{(\sigma)}(k)$ and satisfies

$$\lambda_1(k) < -a^2 + (\tfrac{\pi}{\pi+c})^2$$

for all $k \in [-\frac{1}{2}, \frac{1}{2}]$. If we proceed similarly as in the proof of Proposition 3.3.15 with natural boundary conditions at $x_1 \in \{-\pi, c, \pi\}$ instead of Dirichlet boundary conditions (i.e., considering the form $g^{(\tilde{\sigma})}(k)$ on $H^1((-\pi, c)\times\mathbb{R}_+)\oplus H^1((c, \pi)\times\mathbb{R}_+)$) and use $b \geq 0$ we find the lower bounds $\lambda_1(k) \geq -a^2$ and

$$\lambda_2(k) \geq -a^2 + (\tfrac{\pi}{\pi+c})^2.$$

(Of course, $\lambda_2(k)$ need not be an eigenvalue.) This proves (4.2.6).

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